LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
- LIGO -
CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Document Type **LIGO-T960168-A - E** 10/18/96

Measurement of coating layer thickness

Hiroaki Yamamoto

Distribution of this document:

LIGO science

This is an internal working note
of the LIGO Project.

California Institute of Technology
LIGO Project - MS 51-33
Pasadena CA 91125
Phone (818) 395-2129
Fax (818) 304-9834
E-mail: info@ligo.caltech.edu

Massachusetts Institute of Technology
LIGO Project - MS 20B-145
Cambridge, MA 01239
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

WWW: http://www.ligo.caltech.edu/

1 INTRODUCTION

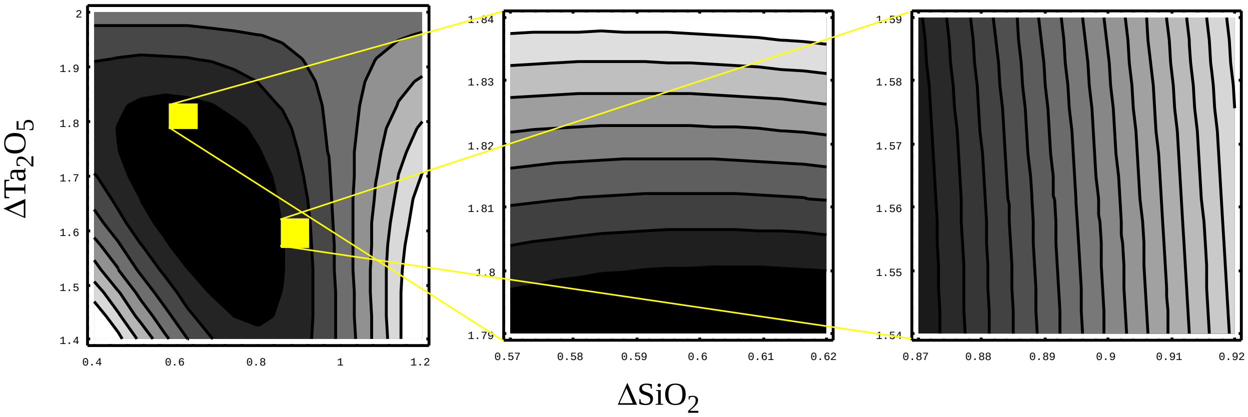
In order to achieve the required performance of LIGO, the phase disturbance caused by aberrations of the core optics mirrors must be very small (~ λ/800 rms). In order to study the uniformity of coatings of LIGO core optics, LIGO requested Research Electro-Optics, Inc. (REO) to make a special two layer AR (antireflaction) coating, using SiO_2 and Ta_2O_5 on a SiO_2 substrate. This note summarizes the analysis of this optic with the AR coating.

From the measurements of reflectance with different polarizations and incidence angles, the thickness of SiO_2 (ΔSiO_2) and that of Ta_2O_5 (ΔTa_2O_5) were calculated over a area within 4 inch radius. Both layers become thinner as the radius increases. The effect is 0.1% at 2 inch radius for the Ta_2O_5 layer and 0.4% for the SiO_2 layer. The ΔTa_2O_5 variation is smooth with no spatial high frequency components. The ΔSiO_2 distribution shows high frequency components, which are comparable to measurement errors.

Assuming a dual-accordion structure (see below) for the global variation of thickness, a phase map was predicted for a HR (high reflectance) coating of an input test mass (ITM, 16 layers) and end test mass (ETM, 40 layers coating). The RMS of the phase variation after subtracting the tilt and curvature was λ/560 for ITM and λ/270 for ETM within 4.5 cm radius.

2 DESIGN OF THE AR COATING

The AR coating was designed so that the reflectance is primarily sensitive to the ΔTa_2O_5. Figure 1 shows contour plots of the reflectance at the Ar^+ laser wavelength (λ_{Ar} is 514.5 nm) at normal incidence. The spacing of contour lines in (b) and (c) is 500 ppm.



(a) global view (b) ΔTa_2O_5 sensitive region (c) ΔSiO_2 sensitive region

Figure 1: Reflectance of Ar laser at normal incidence

The axes are optical thickness (physical thickness times refractive index) of each of the two materials in units of $\lambda_{Ar}/4$. Figure 1 (b) is an expanded view where the reflectance is sensitive to ΔTa_2O_5 and (c) is that sensitive to ΔSiO_2. The design coating thickness of the analyzed optic was $\Delta SiO_2 = 0.597$ and $\Delta Ta_2O_5 = 1.820$. The best fit values deduced from measurements are slightly different from these specification. The discrepancy appears to come from that the refractive indexes used in the analysis were not exactly the same those achieved in the coating. This does not affect the uniformity study. A discussion is given in Appendix 1 about the criteria to select the thickness of layers.

3 MEASUREMENT

For the calculation of the two coating layer thicknesses, 6 power reflectance measurements were performed using the Ar^+ laser, 3 incidence angles (12˚, 45˚ and 60˚) and both S and P polarization for each angle. These measurements were needed to obtain the necessary accuracy. For each measurement, a map was made along radial scans every 10˚, at 0.04 inch radial spacing. All the points were measured twice, the second one following the first set without modifying the setup. In other words, scans were performed for angles from 0˚ to 710˚. The parameters of the measurement set are summarized in the following table.

Table 1: Reflectance measurement parameters

Data set	P12	S12	P45	S45	P60	S60
Resolution (Eq. 1)	0.0018	0.0012	0.0018	0.0020	0.0015	0.0014
Polarization	P	S	P	S	P	S
Incident Angle	12˚		45˚		60˚	
Radius (inch)	4	4	4	4	4	2.4

All the data sets, except S60, show reasonably smooth behavior out to 4 inch radius and all data within 4 inch radius were used in the following analysis. The S60 data show very noisy structure at the peripheral region, possibly caused by dusts on the optic, and only data within 2.4 inch radius were used.

The resolution of the reflectance measurement was determined by comparing the reflectances of the same point in the first and second scan, i.e., the resolution in the above table is defined as

$$\sqrt{2} \bullet \frac{R(\theta + 360, r) - R(\theta, r)}{R(\theta + 360, r) + R(\theta, r)} \qquad \text{Eq. 1}$$

where R(θ,r) is the reflectance at angle θ and radius r.

The polarizations, incidence angles and overall normalizations are determined in the analysis explained below.

4 ANTIREFLECTION COATING

The detailed description of the antireflection coating may be found in “Thin film optical filters” by H. A. Macleod (published by Adam Hilger Ltd). The formula for the two layer coating is given in Eq. 2. In this equation, n_r is the refractive index, d_r is the physical thickness, θ_r is the angle in the material r, and material 0 stands for the air, S for the SiO_2 layer, T for the Ta_2O_5 layer, and m

$$\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} \cos\delta_S & \frac{(i\sin\delta_S)}{\eta_S} \\ i\eta_S\sin\delta_S & \cos\delta_S \end{bmatrix} \bullet \begin{bmatrix} \cos\delta_T & \frac{(i\sin\delta_T)}{\eta_T} \\ i\eta_T\sin\delta_T & \cos\delta_T \end{bmatrix} \bullet \begin{bmatrix} 1 \\ \eta_m \end{bmatrix}$$

$$Y = \frac{C}{B}$$

$$R = \left(\frac{\eta_0 - Y}{\eta_0 + Y}\right) \bullet \left(\frac{\eta_0 - Y}{\eta_0 + Y}\right)^* \qquad \text{Eq. 2}$$

$$\delta_r = \frac{2\pi n_r d_r \cos\theta_r}{\lambda}$$

$$\eta_r = n_r \cos\theta_r \qquad \text{for S polarization}$$

$$\eta_r = n_r / \cos\theta_r \qquad \text{for P polarization}$$

$$n_r \sin\theta_r = n_m \sin\theta_m$$

stands for the substrate SiO_2. In this analysis, the following refractive indices are used and assumed to be constant: $n_S = 1.4598$, $n_T = 2.1021$ and $n_m = 1.4578$.

The reflectance is a function of thicknesses, incidence angles and the polarization of the laser. Eq. 2 is the formula for the perfect S (electric field vector is parallel to the surface plane) and P polarization (magnetic field parallel to surface plane). When the field vector is not parallel to the surface, the reflectance is given by weighted sum of the reflectances for the S and P polarized cases:

$$R(\Theta_{pol}) = R_S \cdot \cos\Theta_{pol}^{\;2} + R_P \cdot \sin\Theta_{pol}^{\;2} \qquad \text{Eq. 3}$$

5 THICKNESS CALCULATION

There are 12 reflectance measurements for the same spot at angle θ and radius r: three angles, two polarizations and two scans, 0° to 350° and 360° to 710°. Two sets of measurements (6 in the first scan and 6 in the second scan) were analyzed separately, and the difference was taken as indicative of systematic error. For each spot, the two thicknesses of layers were calculated by minimizing the following quantity using the minimization program MINUIT (“MINUIT - Function Minimization and Error Analysis”, by F. James, CERN Program Library entry D506):

$$\sum_I \left\{ \left(\frac{\frac{R_I(\theta, r)}{(1+\delta_I)\Re(\Delta SiO_2, \Delta Ta_2O_5, \Theta_{inc}, \Theta_{pol})} - 1}{\varepsilon_I} \right)^2 + \left(\frac{\Theta_{inc} - \overline{\Theta_{inc}}}{\varepsilon_\Theta} \right)^2 \right\} \qquad \text{Eq. 4}$$

In this equation, I represents the data set, P12, S12, P45, S45, P60 and S60 and $R_I(\theta,r)$ is the reflectance measured at the point θ,r and ε_I is the resolution defined in Eq. 1. $\Re(\Delta SiO_2, \Delta Ta_2O_5, \Theta_{inc}, \Theta_{pol})$ is the reflectance calculated using Eq. 2 and Eq. 3, where Θ_{inc} is the incidence angle at the surface. The second term forces the incidence angle to be distributed around an average incident angle, $\overline{\Theta_{inc}}$, for the pairs of measurements, 12P and 12S, 45P and 45S and 60P and 60S.

Table 2 summarizes the unknowns which were fitted.

Table 2: Free Parameters in the minimization

Unknown	Meaning	Comment
ΔSiO_2	thickness of SiO_2 layer	one parameter for each spot
ΔTa_2O_5	thickness of Ta_2O_5 layer	one parameter for each spot
δ_I	overall normalization	one parameter for each set
Θ_{inc}	incident angle	one parameter along each radial direction for each data set
$\overline{\Theta_{inc}}$	average incident angle	one parameter for the pair of angles, 12P&S, 45P&S, 60P&S
Θ_{pol}	polarization	one parameter for each set

The calibration process introduces overall uncertainties of the order of 1% in power. The incidence angle has two uncertainties. The accuracy of the incident direction setup is of the order of 0.5˚. In addition to this uncertainty, there are various sources which can change the direction of the incidence angle, such as a change of orientation of the plate holding the optic due to the bearing motion. The accuracy of the polarization angle setup, determined by maximizing or minimizing the measured power, is of the order of 1˚. With this size of possible polarization admixture, the contamination of the two polarizations could possibly affect only P45 and P60.

The details of the minimization process and some of the results are given in Appendix 2.

The error in the extracted thicknesses was estimated by combining factors listed in the following table. MINUIT error is calculated after all iterations were done by fixing all other parameters. The comparison of the first set (angle 0˚-350˚) and the second set (360˚-710˚) gives an estimation of

Table 3: Error estimation contribution (unit is $\lambda/4$)

	ΔSiO_2	ΔTa_2O_5
thickness at r=0 (center)	0.624	1.872
MINUIT fit error	2.6×10^{-4}	0.9×10^{-4}
Set 0˚-350˚ vs. Set 360˚-710˚	3.5×10^{-4}	1.6×10^{-4}

Table 3: Error estimation contribution (unit is $\lambda/4$)

Convergence of repetition	1×10^{-4}	0.2×10^{-4}
Random error	5.5×10^{-4}	1.3×10^{-4}
Combined	10×10^{-3}	3.1×10^{-4}

the reproducibility of the measurement, including the change of environment during tests. A comparison of the results of different iteration levels gives an error estimation on how the minimization as a whole works. There is a time dependent systematic error which is not yet understood. This is estimated from the variation of the thickness at r = 0 calculated along different directions. The combined error was calculated by taking the quadrature sum of the first three errors and the random error was added linearly to it.

The variation of the layer thickness must be less than 0.02% to satisfy the $\lambda/800$ requirement when accordion structure is assumed ("Multi thin layer coating modeling", by H. Yamamoto, LIGO-T950008-A). The resolution of ΔTa_2O_5 is good enough to test this variation, but that of ΔSiO_2 is factor 10 worse than this requirement.

6 RESULTS - THICKNESS

The fit results are shown in the following figures. Figure 3 shows the contour plot of the two layers. As can be seen from this figure, the Ta_2O_5 layer is smooth and cylindrically symmetric, but the SiO_2 layer appears rough. To see this more quantitatively, the distributions of thickness along two diameters are shown in Figure 3. (a-1) and (b-1) show the thickness (solid line) and the 10th order Zernike polynomial fit (dashed line) and (a-2) and (b-2) show the residuals between measured thickness and the polynomial fit. From (a-1) and (b-1), again it is clear that the Ta_2O_5 layer is more smooth and symmetric, while the SiO_2 layer appears rough and asymmetric.

The standard deviation of the difference of the measured thickness from the polynomial fit ((a-2) and (b-2)) was 1.8×10^{-4} for the Ta_2O_5 layer and was 10.7×10^{-4} for the SiO_2 layer. Both are of comparable magnitude to the respective systematic uncertainties for each layer.

7 PHASEMAP

From the thickness variation measurements, a phase map for the LIGO HR mirror was calculated from the Zernike fit to the data assuming the dual accordion model (low frequency components of thickness variations accumulate linearly, "Multi thin layer coating modeling", by H. Yamamoto, LIGO-T950008-A). The phase map and cross sections along two diameters are shown in Figure 4. The phase map is given in nanometers. The RMS and P-V in the central 4.5 cm radius area, after subtracting tilt and curvature, was 4.0 nm ($\lambda/130$) and 17.8 nm for the 40 layer mirror, and 1.9 nm ($\lambda/270$) and 8.4 nm for the 16 layer mirror.

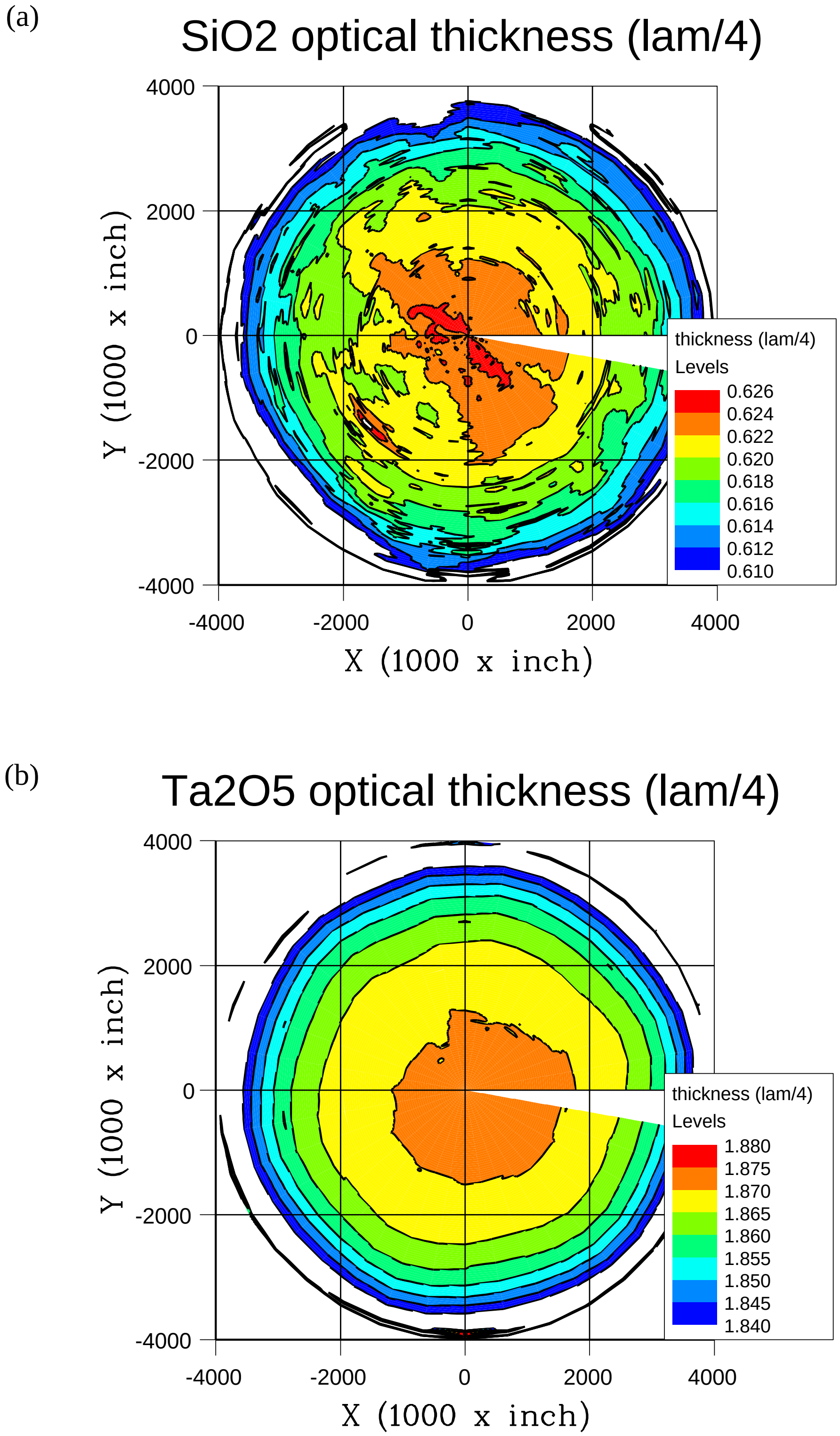


Figure 2: Contour Plot of layer thickness

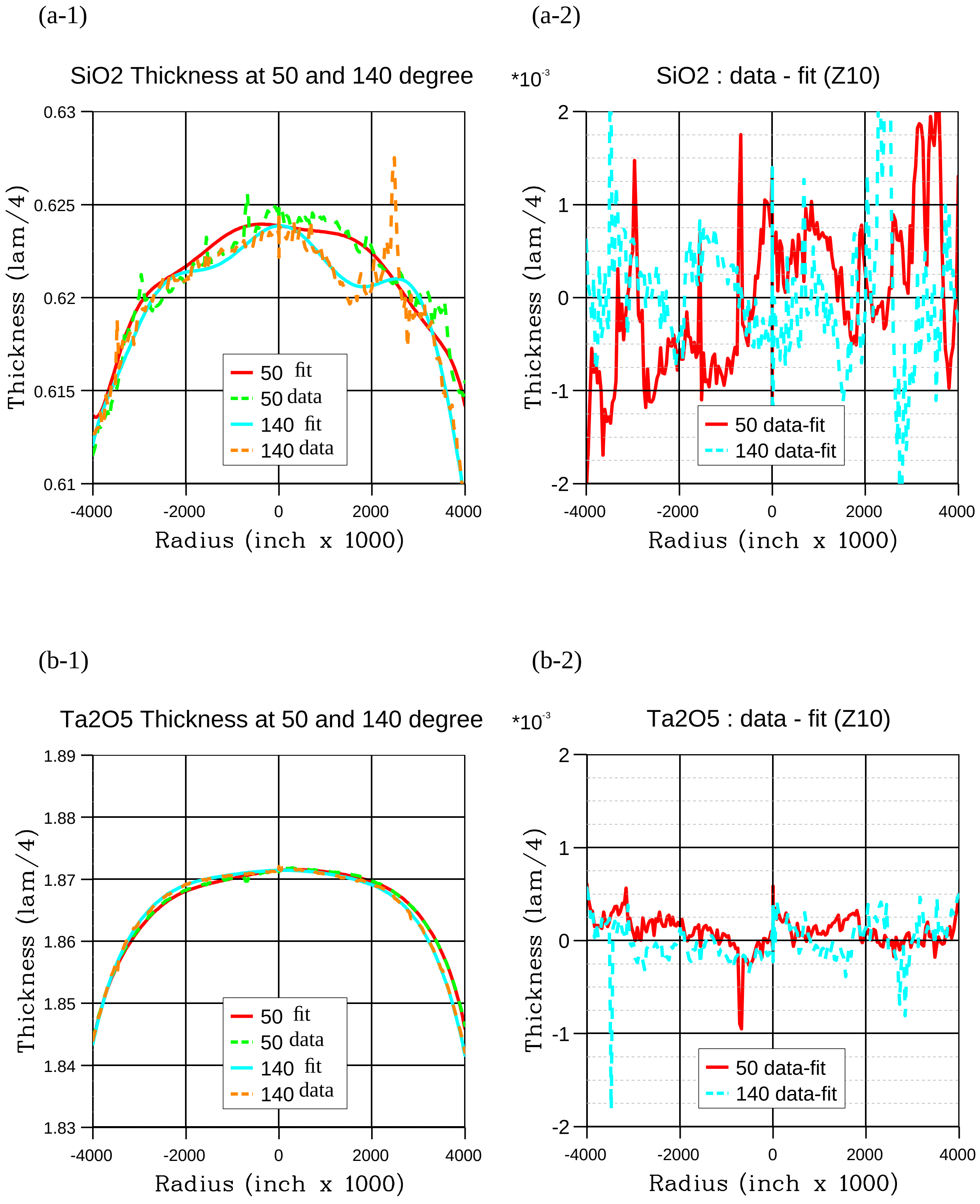


Figure 3: Thickness cross section

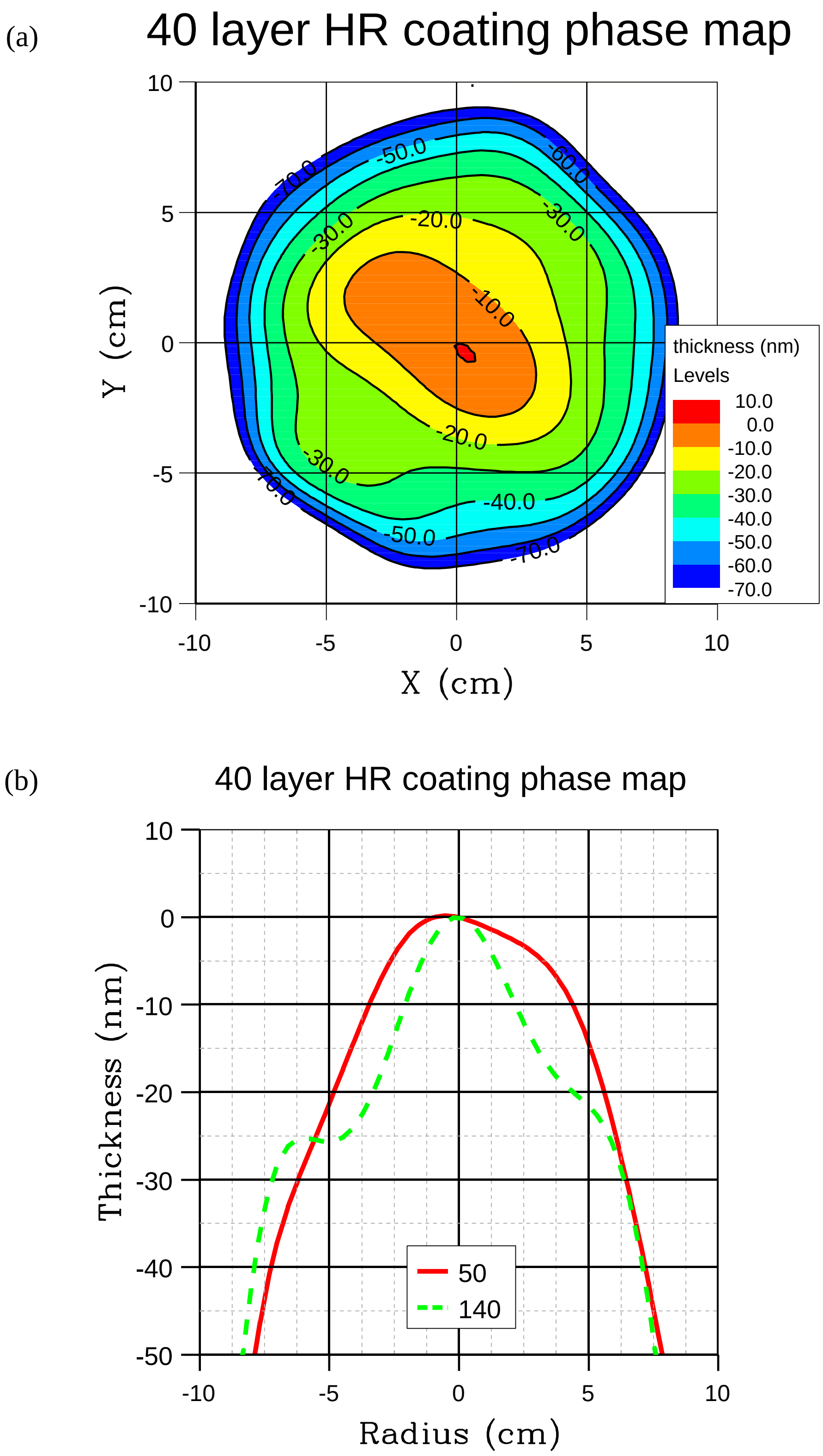


Figure 4: 40 layer HR mirror phasemap

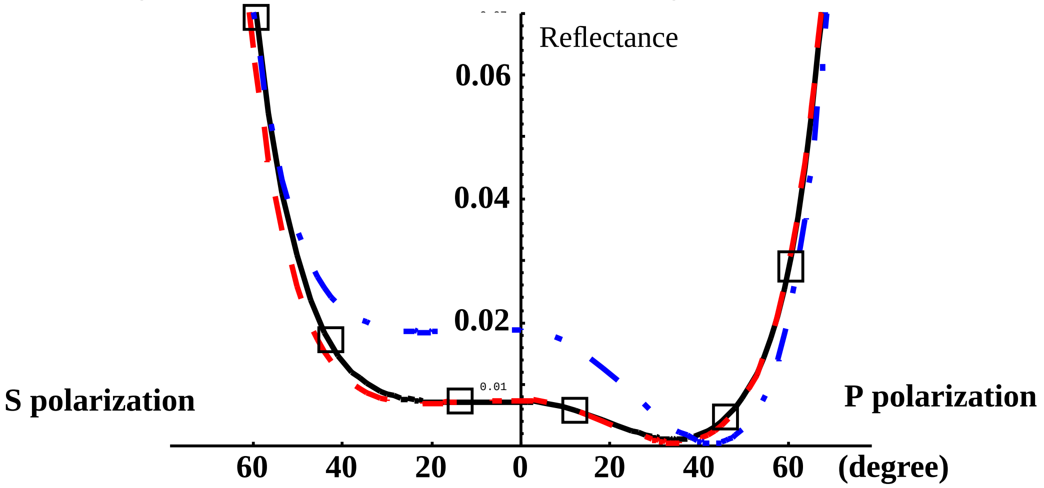
8 CONCLUSIONS

A noninterferometric method was developed to measure the coating layer thickness very accurately using special AR coatings. This is useful to predict the coating qualities of LIGO optics. Using a reasonable model relating one layer thickness to the entire HR coating, the phase maps of the input and output test masses were created. The rms of the variation of the phase map shows that the coating analyzed here is not good enough to meet the LIGO requirements. The analysis result was informed to REO to imrpove the coating for the next AR coatings.

APPENDIX 1 OPTIMAL THICKNESS

The thicknesses of the two coating layers were chosen so that the correlation of the two layers would not degrade the accuracy of each layer. Intuitively speaking, a point was chosen in Figure 1 (a) where the contour (equi--reflectance line) is horizontal (insensitive to ΔSiO_2) or vertical (insensitive to ΔTa_2O_5). In this appendix, further discussion is give about the choice and the thickness and incident angles of the layer.

(a) ΔTa_2O_5 sensitive coating ($\Delta SiO_2 = 0.597, \Delta Ta_2O_5 = 1.820$)



(b) ΔSiO_2 sensitive coating ($\Delta SiO_2 = 0.882, \Delta Ta_2O_5 = 1.560$)

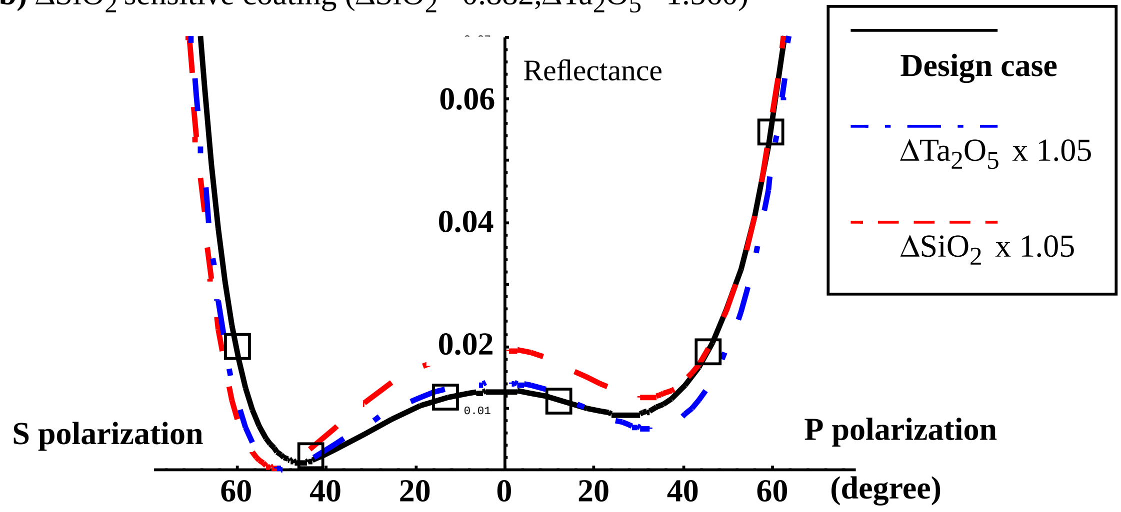


Figure 5: Incident angle and polarization dependence of Reflectance

Figure 5 shows the reflectance as a function of the incident angle for P and S polarized field. The right hand size of the reflectance axis represents the angle for the P polarization and the left for the S polarization. Figure (a) is a plot for the coating analyzed in this note. It was designed for the better measurement of ΔTa_2O_5. In this plot, the solid line is the reflectance with the designed thickness, $\Delta SiO_2 = 0.597$ and $\Delta Ta_2O_5 = 1.820$. The dashed line is that with the ΔSiO_2 5% thicker than the design value and the dash-dotted line is that with ΔTa_2O_5 5% thicker. Figure (b) is the case designed for the precise measurement of ΔSiO_2.

As is seen from these plots, the reflectance is very sensitive to one of the thicknesses in some regions, while it is sensitive to the other in other regions. This is because the contour of the reflectance, Figure 1, changes as the incident angle and the polarization changes. The following quantity can characterize the sensitivity of the reflectance to the change of thickness.

$$\Xi_m = \frac{\delta\Xi_m}{\delta R} = \xi \cdot \frac{\delta_m}{\varepsilon_R}$$

$$\delta\Xi_m = \frac{dR}{d\Delta_m} \cdot \delta\Delta_m, \quad \delta\Delta_m = \delta_m \cdot \Delta_m, \quad \delta R = R \cdot \varepsilon_R \qquad \text{Eq. 5}$$

$$\xi = \frac{dR}{d\Delta_m} \cdot \frac{\Delta_m}{R}$$

In this equation, R is the reflectance and Δ_m is the thickness of a material. The sensitivity of the reflectance to the change of the thickness is given by the ratio of $\delta\Xi_m$ and δR, where $\delta\Xi_m$ is the change of the reflectance when the thickness changed by $\delta\Delta_m$, and δR is the resolution of the reflectance. If Ξ_m is more than several times larger than 1, the change is observable, otherwise, the change is consistent with the resolution. The resolution δR was found to be roughly proportional to the reflectance, i.e., ε_R is around 0.1% for a wide range of the reflectance value, and fractional thickness change, $\delta_m=\delta\Delta_m/\Delta_m$, is the direct concern. E.g., if ξ is 10, the thickness can be measured to the accuracy of the order of 0.01%. This is a very crude argument, and the final accuracy depends on the details of the analysis, but ξ is a good measure of the sensitivity.

The following table shows the sensitivities for the two sets of coatings.

Table 4: Sensitivity ξ

angle (degree)	polar-ization	ΔTa_2O_5 sensitive coating (ΔSiO_2 =0.597,ΔTa_2O_5 =1.820)		ΔSiO_2 sensitive coating (ΔSiO_2 =0.882,ΔTa_2O_5 =1.560)	
		$\xi(SiO_2)$	$\xi(Ta_2O_5)$	$\xi(SiO_2)$	$\xi(Ta_2O_5)$
12	P	0.3	8.3	9.2	0.002
	S	0.3	7.7	9.8	0.5
45	P	3.6	8.5	2×10^{-4}	3.7
	S	3.9	0.2	10	26
60	P	0.4	1.8	0.2	1.4
	S	2.7	1.1	13	7.1

APPENDIX 2 MINIMIZATION PROCESS

In this appendix, the details of the minimization process is described. The minimization process goes as follows:

1. finding Θ_{inc} for 12P, 45P and 60P
    - •Fix all but ΔSiO_2, ΔTa_2O_5 and Θ_{inc} for 12P, 45P and 60P
    - •repeat minimization along one radial direction
    - •calculate the average of each Θ_{inc} as the improved estimation
    - •repeat this for all 36 directions
2. finding Θ_{inc} for 12S, 45S and 60S
    - •Fix all but ΔSiO_2, ΔTa_2O_5 and Θ_{inc} for 12S, 45S and 60S
    - •do the same as process 1
3. finding δ_I for 12P, 45P and 60P
    - •Fix all but ΔSiO_2, ΔTa_2O_5 and δ_I for 12P, 45P and 60P
    - •repeat minimization for all points
    - •calculate the average of each δ_I as the improved estimation
4. finding δ_I for 12S, 45S and 60S
    - •Fix all but ΔSiO_2, ΔTa_2O_5 and δ_I for 12S, 45S and 60S
    - •do the same as process 3
5. finding Θ_{pol} for 45P and 60P
    - •Fix all but ΔSiO_2, ΔTa_2O_5 and Θ_{pol} for 45P and 60P
    - •do the same as process 3
6. finding ΔSiO_2 and ΔTa_2O_5
    - •Fix all but ΔSiO_2 and ΔTa_2O_5
    - •do minimization for all points

These processes are repeated 50 times to obtain the final results. In the above description, “fix” means to use the best estimation, e.g., in the 9th iteration, the result of 8th iteration is used. Separate minimization of P and S cases (process 1 and 2, and process 3 and 4) was needed because the convergence of parameters for S polarization was slower than those for P. Polarization angles Θ_{pol} were fixed to be 0 for 12P, 12S, 45S and 60S, because the effect was estimated to be less than 0.1%. All steps except 6 (“finding ΔSiO_2 and ΔTa_2O_5") were done using points within 2.4 inch, and all points were used for step 6.

The parameters from this minimization is summarized in Table 5.

There are strong correlations between some of these parameters, which did not converge in the iteration process. The polarization angles were very uncertain. The current measurement accuracy

Table 5: Minimization result of parameters

Data set	P12	S12	P45	S45	P60	S60
δ_I	- 0.004	0.001	0.044	- 0.001	0.0003	- 0.006
$\overline{\Theta_{inc}}$	11.7°		44.8°		60.9°	
Θ_{pol}	0	0	2.6°	0	5°	0

prohibited resolving these correlated uncertainties. Fortunately, the thickness itself converged well.

APPENDIX 3 ZERNIKE FIT OF THICKNESS

The following are the coefficients of the Zernike polynomial fit to ΔSiO_2 and ΔTa_2O_5 up to 10th order. The thickness is measured in units of $\lambda_{Ar}/4$. The Zernike polynomial are normalized as

$$\int_0^1 \int_0^{2\pi} \left|Z_n^m\right|^2 \rho d\rho d\theta = \pi \tag*{Eq. 6}$$

The table `NMInd` is a Zernike polynomial index list corresponding to the coefficient tables, and one can calculate the thickness at (x,y) by the following formula, where x and y are measured in inch, R_0 = 4 inch and the formula is valid within 4 inch radius.

$$\sum_{i=0}^{65} Coeff[i] \bullet Z_{NMInd[i][0]}^{NMInd[i][1]}\left(\frac{x}{R_0}, \frac{y}{R_0}\right) \tag*{Eq. 7}$$

```
double SiO2_Coeff[66] = {
   0.619012333388677,      0.000156265256488991, -0.000321980508092953,
  -0.00282296170517754,  -5.45320688098129e-05, -0.000113109158825177,
  -2.85105721321284e-06, -8.80101932312538e-05,  2.9598739602242e-05,
   0.000212361124239879, -0.000518557632835147,  0.000113234537531658,
   8.74435184086548e-06,  3.90019171145128e-05, -5.03929076840117e-05,
   0.000143903003883856,  2.01517221627806e-05, -0.000108288208956386,
   6.82451773115462e-05,  6.5136127527658e-05,  -3.09191750504923e-05,
  -0.000246797186879683, -7.2103215671994e-05,  -0.000206920126111847,
   0.000168681160547177, -0.000102589224995085,  0.000111273096576501,
   0.000103575857393477,  2.29627121299996e-05,  8.75439746061231e-06,
  -4.52373718872082e-05,  5.88837812809871e-05, -8.07940881455159e-05,
  -0.000116593389391504, -9.24495766488402e-06, -1.83923068516447e-05,
   0.000125511532789133, -1.74555022680888e-05,  0.000142484444093809,
  -6.87640939986527e-05,  2.24743437208803e-05, -5.42896540159113e-05,
```

```
   2.02222730868513e-05, -2.30151081119354e-05,  4.66781195941343e-07,
  -0.000125612172644031,  4.09379677793508e-05,  5.80209319577049e-05,
  -3.0593070762432e-05,   4.51455778790111e-05, -9.9363258727246e-05,
   9.83868225054161e-05,  7.92701341978037e-06,  8.70374959132922e-05,
  -0.000105229968529457, -2.71582354047475e-05, -1.96598965836745e-06,
  -8.63135418964949e-05,  1.6419884695572e-05,  -2.36712468095223e-05,
  -1.68854004719202e-05, -2.67479447340342e-05,  4.42619138459018e-05,
  -8.56649909808693e-05, -0.000136297682436168, -2.12339569178882e-06
   };
double Ta2O5_Coeff[66] = {
   1.86234598256416,      0.000697474830465895, -0.00031057222570545,
  -0.00735367528389809,   6.22282310027014e-05,  0.000220082793750541,
   3.07495427456881e-05,  0.00012965451246771,   0.000157264008411776,
  -2.79873729618476e-05, -0.00208875048827788,  -3.71466754583935e-05,
  -4.6255600715418e-05,   2.47188798382459e-05,  5.53105055257562e-05,
   4.77074990926081e-06,  1.12829385936055e-05, -3.20585051803876e-05,
   3.32643822739324e-05, -4.54505334234589e-05,  0.000122583562132193,
  -0.000346744528526219,  1.12069618453088e-05, -3.66174628906737e-06,
  -1.32769724216994e-06, -1.92368957532105e-05,  6.68566311214454e-05,
   0.000526667028466874, -1.37092992069054e-05,  3.78744131761632e-05,
   9.5279303609876e-06,  -2.03822868063874e-05, -1.97506491783682e-05,
  -3.91736900159534e-05,  6.35108305215154e-05, -4.34210192884159e-05,
  -3.29524509797853e-05,  9.51865611105489e-06,  1.66297773030175e-05,
  -1.74231651459414e-06,  4.68859974450932e-06, -5.83544913795732e-06,
  -0.000107470210581129,  1.47631267322353e-05,  3.10872316314558e-05,
  -9.82525051039489e-06, -2.63889138162706e-05, -1.57729791157443e-05,
   9.40114084173473e-06,  1.33093452867471e-05,  3.06647317939234e-05,
  -1.33372697391081e-05, -7.40492733684629e-06, -3.32453400247268e-05,
   4.25031402438399e-05, -6.30133623123188e-05,  4.8638856544958e-06,
   2.0844279851528e-05,   6.10102849397591e-06, -1.24495435847617e-05,
  -1.79849472948392e-05,  2.07653328411359e-05, -1.05790051972946e-05,
  -1.1894692629988e-05,   5.43609418574919e-07, -6.46133195583986e-05
   };
int NMInd[66][2] = {
     0, 0, 1, 1, 1, -1, 2, 0, 2, 2, 2, -2,
     3, 1, 3, -1, 3, 3, 3, -3, 4, 0, 4, 2, 4, -2, 4, 4, 4, -4,
     5, 1, 5, -1, 5, 3, 5, -3, 5, 5, 5, -5,
     6, 0, 6, 2, 6, -2, 6, 4, 6, -4, 6, 6, 6, -6,
     7, 1, 7, -1, 7, 3, 7, -3, 7, 5, 7, -5, 7, 7, 7, -7,
     8, 0, 8, 2, 8, -2, 8, 4, 8, -4, 8, 6, 8, -6, 8, 8, 8, -8,
```

```
  9, 1, 9, -1, 9, 3, 9, -3, 9, 5, 9, -5, 9, 7, 9, -7, 9, 9, 9, -9,
  10, 0, 10, 2, 10, -2, 10, 4, 10, -4, 10, 6, 10, -6,
  10, 8, 10, -8, 10, 10, 10, -10
};
```