Uncertainty Estimation of the LIGO Response Function

Jeffrey S. Kissel
LIGO-T0900555

December 18, 2013

Contents

1 The Response Function Model, R

The LIGO interferometers' digital response to length change ΔL is

$$R_L(f,t) \equiv \frac{1+\gamma(t)G_L(f)}{\gamma(t)\; C_L(f)} \tag{1}$$

where

$$\Delta L_{ext}(f,t) = R_L(f,t)\; e_D(f) \tag{2}$$

with $e_D(f)$ as the digital readout of light captured at the anti-symmetric port and $\Delta L_{ext}(f,t)$ is the external length disturbance composed of both signal and noise.

The terms in the response function are the following: the interferometer's response to DARM length changes known as the length sensing function $C_L(f)$ (its slow time dependence is tracked by the real, positive coefficient $\gamma(t)$), and the open loop gain of the differential arm (DARM) length control loop $G_L(f) = A(f)\; D(f)\; C_L(f)$, whose components (besides $C_L(f)$) are the actuation function of the test masses which define the DARM length degree of freedom $A(f)$, the digital control filters $D(f)$. The loop is drawn schematically in Figure 1.

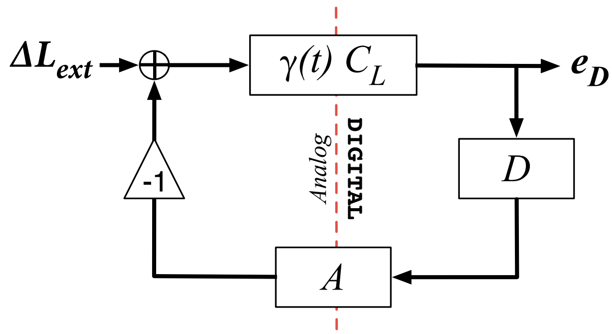

Figure 1: Diagram of the differential arm length control loop. External changes in differential arm length ΔL_{ext} are sensed by the interferometer and digitized according to the length sensing function $C_L(f)$, from which the differential arm length error signal $e_D(f)$ is derived. The loop is closed by a set of digital filters $D(f)$, whose control signal $s_D(f)$ is transformed by the actuation function of the interferometer's end test masses $A(f)$ into physical control of the differential arm length.

This document serves to define the assumptions made about, and derive equations for, the dominant uncertainty terms in the magnitude and phase of the response function the error budget,

$$\left(\frac{\sigma_{|R|}}{|R|}\right)^2 = \left(\frac{\sigma_{|A|}}{|A|}\right)^2 + \Re e\{W\}^2 \left(\frac{\sigma_{|G|}}{|G|}\right)^2 + \Im m\{W\}\sigma^2_{\phi_G} + \Re e\{W\}^2 \left(\frac{\sigma_\gamma}{\gamma}\right)^2 \quad (3)$$

$$\sigma^2_{\phi_R} = \sigma^2_{\phi_A} + \Im m\{W\}^2 \left(\frac{\sigma_{|G|}}{|G|}\right)^2 + \Re e\{W\}^2 \, \sigma^2_{\phi_G} + \Im m\{W\}^2 \left(\frac{\sigma_\gamma}{\gamma}\right)^2 \quad (4)$$

For convenience and legibility, we drop all function dependence and subscripts in our notation in the remainder of this document, such that $R_L(f,t) = R$, $G_L(f) = G$, $A(f) = A$, $D(f) = D$, $C_L(f)$, and $\gamma(t) = \gamma$.

2 Assumptions of the Uncertainty in the Response Function

Standard analysis techniques [1] tell us that if we approximate a non-linear function $f(x_i)$ using a taylor expansion then to first order,

$$f \simeq f_0 + \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right) x_i \quad (5)$$

then the variance or "uncertainty" on $f(x_i)$ for N variables x_i is

$$\sigma^2_f = \sum_{i,j}^{N} \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right)\sigma^2_{x_i x_j} = \sum_{i}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma^2_{x_i} + \sum_{i \neq j}^{N} \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right)\sigma^2_{x_i x_j} \quad (6)$$

where we have divided the uncertainty into uncorrelated terms and correlated terms containing

$$\sigma^2_{x_i} = \lim_{n\to\infty}\left[\frac{1}{n}\sum_k^n\left[(x_i)_k - \overline{x_i}\right]^2\right], \tag{7}$$

and

$$\sigma^2_{x_i x_j} = \lim_{n\to\infty}\left[\frac{1}{n}\sum_k^n\left[(x_i)_k - \overline{x_i}\right]\left[(x_j)_k - \overline{x_j}\right]\right] \tag{8}$$

where n is the number the measurements of a given set of N variables.

In principle, we must therefore find the uncertainty in all components of the response function (Eq. 1). However, in practice we model the sensing function C as it is intrinsically difficult at best to measure independently i.e. without the interferometer under control of the closed loop. Hence, in order to estimate the errors of the response function, we re-cast it in terms of quantities that can be easily measured, (remembering that $C = G/AD$)

$$R = A\ D\ \frac{(1+\gamma\ G)}{\gamma\ G} \tag{9}$$

Ignoring that the functions A, D and G are complex for the time being, we find the uncertainty on Eq. 9 to be

$$\begin{aligned}\sigma^2_R = {} & \sigma^2_A\left(\frac{\partial R}{\partial A}\right)^2 + \sigma^2_D\left(\frac{\partial R}{\partial D}\right)^2 + \sigma^2_G\left(\frac{\partial R}{\partial G}\right)^2 + \sigma^2_\gamma\left(\frac{\partial R}{\partial \gamma}\right)^2 \\ & + 2\ \sigma^2_{AD}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial D}\right) + 2\ \sigma^2_{AG}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial G}\right) + 2\ \sigma^2_{A\gamma}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial \gamma}\right) \\ & + 2\ \sigma^2_{DG}\left(\frac{\partial R}{\partial D}\right)\left(\frac{\partial R}{\partial G}\right) + 2\ \sigma^2_{D\gamma}\left(\frac{\partial R}{\partial D}\right)\left(\frac{\partial R}{\partial \gamma}\right) + 2\ \sigma^2_{G\gamma}\left(\frac{\partial R}{\partial G}\right)\left(\frac{\partial R}{\partial \gamma}\right).\end{aligned} \tag{10}$$

We do not assign any error to the digital filters D. This function is a well known digital quantity and any errors are negligible compared with all other error terms in the response function. Hence, for the purposes of error propagation we assume $D_k - \overline{D} = 0$ for all n measurements, drop it from the error budget, and

$$\begin{aligned}\sigma^2_R = {} & \sigma^2_A\left(\frac{\partial R}{\partial A}\right)^2 + \sigma^2_G\left(\frac{\partial R}{\partial G}\right)^2 + \sigma^2_\gamma\left(\frac{\partial R}{\partial \gamma}\right)^2 \\ & + 2\ \sigma^2_{AG}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial G}\right) + 2\ \sigma^2_{A\gamma}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial \gamma}\right) \\ & + 2\ \sigma^2_{G\gamma}\left(\frac{\partial R}{\partial G}\right)\left(\frac{\partial R}{\partial \gamma}\right).\end{aligned} \tag{11}$$

As γ is a function of time alone, and A and G are functions of frequency alone, **we assume** γ **is by definition are uncorrelated with** A **and** G: on average, we expect to find equal distributions of positive and negative values for their respective covariant terms such that they vanish in the limit of a large random selection of observations. This leaves only the actuation function A and open loop gain G to have some possible correlation, and the relative uncertainty becomes

$$\frac{\sigma_R^2}{R^2} = \left[\frac{\sigma_A^2}{R^2}\left(\frac{\partial R}{\partial A}\right)^2 + \frac{\sigma_G^2}{R^2}\left(\frac{\partial R}{\partial G}\right)^2 + 2\,\frac{\sigma_{AG}^2}{R^2}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial G}\right)\right] + \frac{\sigma_\gamma^2}{R^2}\left(\frac{\partial R}{\partial \gamma}\right)^2. \tag{12}$$

We can write out each weighting coefficient explicitly using Eq. 9

$$\left(\frac{\partial R}{\partial A}\right) = D\frac{1+\gamma G}{\gamma G} \qquad \left(\frac{\partial R}{\partial G}\right) = -AD\frac{1}{\gamma G^2} \qquad \left(\frac{\partial R}{\partial \gamma}\right) = -AD\frac{1}{\gamma^2 G}$$

$$\begin{aligned}
\frac{1}{R^2}\left(\frac{\partial R}{\partial A}\right)^2 &= \frac{1}{A^2} \\
\frac{1}{R^2}\left(\frac{\partial R}{\partial G}\right)^2 &= \frac{1}{G^2}\left(\frac{1}{1+\gamma G}\right)^2 = W^2\frac{1}{G^2} \\
\frac{1}{R^2}\left(\frac{\partial R}{\partial \gamma}\right)^2 &= \frac{1}{\gamma^2}\left(\frac{1}{1+\gamma G}\right)^2 = W^2\frac{1}{\gamma^2} \\
\frac{1}{R^2}\left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial R}{\partial G}\right) &= -\frac{1}{A}\frac{1}{G}\left(\frac{1}{1+\gamma G}\right) = -W\frac{1}{A}\frac{1}{G}
\end{aligned} \tag{13}$$

where we've defined the weighting function

$$W \equiv \frac{1}{1+\gamma G}. \tag{14}$$

In doing so, we find that assuming A and G are correlated *reduces* the estimate of the total response function error,

$$\frac{\sigma_R^2}{R^2} = \left[\frac{\sigma_A^2}{A^2} + W^2\,\frac{\sigma_G^2}{G^2} - 2W\,\frac{\sigma_{AG}^2}{AG}\right] + W^2\frac{\sigma_\gamma^2}{\gamma^2} \tag{15}$$

as opposed to if we assume they are uncorrelated,

$$\frac{\sigma_R^2}{R^2} = \left[\frac{\sigma_A^2}{A^2} + W^2\,\frac{\sigma_G^2}{G^2}\right] + W^2\frac{\sigma_\gamma^2}{\gamma^2} \tag{16}$$

In principle, there is a third, "worst case" scenario, in which σ_{AG}^2 is negative implying that A and G are *anti*-correlated. In this case, the uncertainty is inflated to

$$\frac{\sigma_R^2}{R^2} = \left[\frac{\sigma_A^2}{A^2} + W^2\,\frac{\sigma_G^2}{G^2} + 2W\,\frac{|\sigma_{AG}^2|}{AG}\right] + W^2\frac{\sigma_\gamma^2}{\gamma^2}. \tag{17}$$

However, we can imagine no physical grounds for this case to occur.

We adopt the conservative assumption that $A(f)$ and $G(f)$, and therefore all terms in response function are uncorrelated,

$$\frac{\sigma_R^2}{R^2} = \frac{\sigma_A^2}{R^2}\left(\frac{\partial R}{\partial A}\right)^2 + \frac{\sigma_G^2}{R^2}\left(\frac{\partial R}{\partial G}\right)^2 + \frac{\sigma_\gamma^2}{R^2}\left(\frac{\partial R}{\partial \gamma}\right)^2. \tag{18}$$

Though we suspect that the uncertainty in G comes from our uncertainty in A, we have no direct evidence that this is the case. It is conceivable (in fact, probable) that there is uncertainty in our model of C. Hence, we adopt this conservative assumption, which implies that we treat the uncertainty in G as though it were composed entirely of the uncertainty in C.

Though we take A and G to be uncorrelated in the overall response function uncertainty estimate, we assign the systematic uncertainty in A to be the larger of the frequency dependent error in the measurements of A and the residuals between model and measurement of G. In practice

3 The Complex Response Function Uncertainty Estimation

We choose to report the variance separated into the magnitude and phase of the complex response function. If the magnitude and phase of a given complex function X are defined as usual,

$$\begin{aligned} |X| &= \sqrt{(X\ X^*)} \\ \phi_X = \arctan\left(\frac{\Im m\{X\}}{\Re e\{X\}}\right) &= \arctan\left(\frac{1}{i}\ \frac{X - X^*}{X + X^*}\right) \end{aligned}$$

then the magnitude and phase of Eq. (9) is

$$|R| = \sqrt{\left(\frac{|A||D|}{\gamma|G|}\right)^2 (1 + (\gamma|G|)^2 + 2\gamma|G|\cos(\phi_G))} \tag{19}$$

$$\phi_R = \arctan\left(\frac{\gamma|G|\sin(\phi_A + \phi_D) + \sin(\phi_A + \phi_D - \phi_G)}{\gamma|G|\cos(\phi_A + \phi_D) + \cos(\phi_A + \phi_D - \phi_G)}\right) \tag{20}$$

The remainder of this document will focus on the details of calculating the (relative) error in magnitude $(\sigma_{|R|}/|R|)^2$ and (absolute) error in phase $\sigma^2_{\phi_R}$ of our response function model R.

3.1 Relative Magnitude Uncertainty, $\left(\frac{\sigma_{|R|}}{|R|}\right)^2$

We know $|R| = f(|A|, |D|, |G|, \phi_G, \gamma)$. As described in §2, we ignore terms involving uncertainty in $\sigma_{|D|}$, and we treat all remaining variables as uncorrelated, such that

$$\sigma^2_{|R|} = \left(\frac{\partial|R|}{\partial|A|}\right)^2 \sigma^2_{|A|} + \left(\frac{\partial|R|}{\partial|G|}\right)^2 \sigma^2_{|G|} + \left(\frac{\partial|R|}{\partial\phi_G}\right)^2 \sigma^2_{\phi_G} + \left(\frac{\partial|R|}{\partial\gamma}\right)^2 \sigma^2_{\gamma} \tag{21}$$

and therefore the relative variance is

$$\begin{aligned} \frac{\sigma^2_{|R|}}{|R|^2} &= \left(\frac{\partial|R|}{\partial|A|}\right)^2 \frac{\sigma^2_{|A|}}{|R|^2} + \left(\frac{\partial|R|}{\partial|G|}\right)^2 \frac{\sigma^2_{|G|}}{|R|^2} + \left(\frac{\partial|R|}{\partial\phi_G}\right)^2 \frac{\sigma^2_{\phi_G}}{|R|^2} + \left(\frac{\partial|R|}{\partial\gamma}\right)^2 \frac{\sigma^2_{\gamma}}{|R|^2} \\ &= \left(\frac{(\partial_{|A|}|R|)^2}{|R|^2}\right)\sigma^2_{|A|} + \left(\frac{(\partial_{|G|}|R|)^2}{|R|^2}\right)\sigma^2_{|G|} + \left(\frac{(\partial_{\phi_G}|R|)^2}{|R|^2}\right)\sigma^2_{\phi_G} + \left(\frac{(\partial_{\gamma}|R|)^2}{|R|^2}\right)\sigma^2_{\gamma} \\ \left(\frac{\sigma_{|R|}}{|R|}\right)^2 &= \left(\frac{\partial_{|A|}|R|}{|R|}\right)^2 \sigma^2_{|A|} + \left(\frac{\partial_{|G|}|R|}{|R|}\right)^2 \sigma^2_{|G|} + \left(\frac{\partial_{\phi_G}|R|}{|R|}\right)^2 \sigma^2_{\phi_G} + \left(\frac{\partial_{\gamma}|R|}{|R|}\right)^2 \sigma^2_{\gamma} \qquad (22) \end{aligned}$$

The weighting coefficients for each variance term in Eq. (22) are

$$\left(\frac{\partial_{|A|}|R|}{|R|}\right)^2 = \frac{1}{|A|^2} \tag{23}$$

$$\left(\frac{\partial_{|G|}|R|}{|R|}\right)^2 = \frac{1}{|G|^2}\frac{(1+\gamma|G|\cos\phi_G)^2}{(1+(\gamma|G|)^2+2\gamma|G|\cos\phi_G)^2} \tag{24}$$

$$\left(\frac{\partial_{\phi_G}|R|}{|R|}\right)^2 = \frac{(\gamma|G|)^2\sin^2(\phi_G)}{(1+(\gamma|G|)^2+2\gamma|G|\cos\phi_G)^2} \tag{25}$$

$$\left(\frac{\partial_{\gamma}|R|}{|R|}\right)^2 = \frac{1}{\gamma^2}\frac{(1+\gamma|G|\cos\phi_G)^2}{(1+(\gamma|G|)^2+2\gamma|G|\cos\phi_G)^2} \tag{26}$$

Eqs. (24), (25), and (26), can be cleaned up a bit if we note that the real and imaginary parts of the weighting function W are

$$\begin{aligned}
W &\equiv \frac{1}{1+\gamma G} \\
\Re e\left\{W\right\} &= \Re e\left\{\frac{1}{1+\gamma G}\right\} \\
&= \frac{1}{2}\left(\frac{1}{1+\gamma G}+\frac{1}{1+\gamma G^*}\right) \\
&= \frac{1}{2}\left(\frac{1}{1+\gamma|G|e^{i\phi_G}}+\frac{1}{1+\gamma|G|e^{-i\phi_G}}\right) \\
&= \frac{1}{2}\left(\frac{(1+\gamma|G|e^{-i\phi_G})+(1+\gamma|G|e^{i\phi_G})}{(1+\gamma|G|e^{i\phi_G})(1+\gamma|G|e^{-i\phi_G})}\right) \\
&= \frac{1}{2}\left(\frac{2+\gamma|G|(e^{i\phi_G}+e^{-i\phi_G})}{(1+(\gamma|G|)^2)+\gamma|G|(e^{i\phi}+e^{-i\phi})}\right) \\
&\quad \left(\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta\right) \\
\Re e\left\{W\right\} &= \frac{1+\gamma|G|\cos(\phi_G)}{(1+(\gamma|G|)^2+2\gamma|G|\cos(\phi_G))} \\
\Rightarrow \Re e\left\{W\right\}^2 &= \frac{(1+\gamma|G|\cos(\phi_G))^2}{(1+(\gamma|G|)^2+2\gamma|G|\cos(\phi_G))^2} \qquad (27) \\
\Im m\left\{W\right\} &= \Im m\left\{\frac{1}{1+\gamma G}\right\} \\
&= \frac{1}{2i}\left(\frac{1}{1+\gamma G}-\frac{1}{1+\gamma G^*}\right) \\
&= \frac{1}{2i}\left(\frac{1}{1+\gamma|G|e^{i\phi_G}}-\frac{1}{1+\gamma|G|e^{-i\phi_G}}\right) \\
&= \frac{1}{2i}\left(\frac{(1+\gamma|G|e^{-i\phi_G})-(1+\gamma|G|e^{i\phi_G})}{(1+\gamma|G|e^{i\phi_G})(1+\gamma|G|e^{-i\phi_G})}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2i}\left(\frac{-\gamma|G|(e^{i\phi_G}-e^{-i\phi_G})}{(1+(\gamma|G|)^2)+\gamma|G|(e^{i\phi}+e^{-i\phi})}\right) \\
&\left(\frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta\right) \\
\Im m\,\{W\} &= \frac{\gamma|G|\sin(\phi_G)}{(1+(\gamma|G|)^2+2\gamma|G|\cos(\phi_G))} \\
\Rightarrow \Im m\,\{W\}^2 &= \frac{(\gamma|G|)^2\sin^2(\phi_G)}{(1+(\gamma|G|)^2+2\gamma|G|\cos(\phi_G))^2} \qquad (28)
\end{aligned}$$

Thus, we combine Eqs. (22) through (28) to arrive at our final expression for the relative magnitude error on the response function,

$$\left(\frac{\sigma_{|R|}}{|R|}\right)^2 = \left(\frac{\sigma_{|A|}}{|A|}\right)^2 + \Re e\{W\}^2\left(\frac{\sigma_{|G|}}{|G|}\right)^2 + \Im m\{W\}^2\,\sigma^2_{\phi_G} + \Re e\{W\}^2\left(\frac{\sigma_\gamma}{\gamma}\right)^2 \qquad (29)$$

3.2 Phase Uncertainty, $\sigma^2_{\phi_R}$

The calculation of the phase error is quite similar. We know $\phi_R = f(\phi_A, \phi_D, |G|, \phi_G, \gamma)$, but we ignore uncertainty in ϕ_D and take the conservative estimate that all other uncertainties are uncorrelated such that the absolute variance in phase is

$$\sigma^2_{\phi_R} = \left(\frac{\partial\phi_R}{\partial\phi_A}\right)^2\sigma^2_{\phi_A} + \left(\frac{\partial\phi_R}{\partial|G|}\right)^2\sigma^2_{|G|} + \left(\frac{\partial\phi_R}{\partial\phi_G}\right)^2\sigma^2_{\phi_G} + \left(\frac{\partial\phi_R}{\partial\gamma}\right)^2\sigma^2_\gamma. \qquad (30)$$

Thus, as before we calculate the weighting factors,

$$\begin{aligned}
\left(\frac{\partial\phi_R}{\partial\phi_A}\right)^2 &= 1 \qquad (31) \\
\left(\frac{\partial\phi_R}{\partial|G|}\right)^2 &= \frac{\gamma^2\sin^2(\phi_G)}{(1+(\gamma|G|)^2+2\gamma|G|\cos\phi_G)^2} = \Im m\{W\}^2\frac{1}{|G|^2} \qquad (32) \\
\left(\frac{\partial\phi_R}{\partial\phi_G}\right)^2 &= \frac{(1+\gamma|G|\cos\phi_G)^2}{(1+(\gamma|G|)^2+2\gamma|G|\cos\phi_G)^2} = \Re e\{W\}^2 \qquad (33) \\
\left(\frac{\partial\phi_R}{\partial\gamma}\right)^2 &= \frac{|G|^2\sin^2(\phi_G)}{(1+(\gamma|G|)^2+2\gamma|G|\cos\phi_G)^2} = \Im m\{W\}^2\frac{1}{\gamma^2} \qquad (34)
\end{aligned}$$

and substitute back into Eq. (30),

$$\sigma^2_{\phi_R} = \sigma^2_{\phi_A} + \Im m\{W\}^2\left(\frac{\sigma_{|G|}}{|G|}\right)^2 + \Re e\{W\}^2\,\sigma^2_{\phi_G} + \Im m\{W\}^2\left(\frac{\sigma_\gamma}{\gamma}\right)^2 \qquad (35)$$

4 Summary

The model of the LIGO interferometer's response to gravitational wave strain from an optimally-oriented source is

$$R \equiv \frac{1+\gamma G}{\gamma C} \qquad (36)$$

which, for the purposes of error estimation can be re-written in terms of easily measurable quantities,

$$R = A\ D\ \frac{(1+\gamma\ G)}{\gamma\ G} \tag{37}$$

Where $G = ADC$, and we assume that the digital filter function D has negligible error.

We report the errors of our model in terms or relative magnitude and absolute phase error, which are comprised of the measured error in each model component,

$$\left(\frac{\sigma_{|R|}}{|R|}\right)^2 = \left(\frac{\sigma_{|A|}}{|A|}\right)^2 + \Re e\{W\}^2 \left(\frac{\sigma_{|G|}}{|G|}\right)^2 + \Im m\{W\}\sigma_{\phi_G}^2 + \Re e\{W\}^2 \left(\frac{\sigma_\gamma}{\gamma}\right)^2 \tag{38}$$

$$\sigma_{\phi_R}^2 = \sigma_{\phi_A}^2 + \Im m\{W\}^2 \left(\frac{\sigma_{|G|}}{|G|}\right)^2 + \Re e\{W\}^2\ \sigma_{\phi_G}^2 + \Im m\{W\}^2 \left(\frac{\sigma_\gamma}{\gamma}\right)^2 \tag{39}$$

where $W = 1/(1+G)$ is the weighting function of the DARM control loop. In these definitions is the implicit assumption that the uncertainties in the actuation function A and open loop gain G are treated conservatively as uncorrelated.

References

[1] P. R. Bevington, D. K. Robinson. **Data Reduction and Error Analysis for the Physical Sciences**. *McGraw Hill*, p40-41 (2003)