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Hankel Cavity Simulation Package

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## 1 Introduction

The cavity field calculation using Discrete Hankel Transformation<sup>1)</sup> is derived from the FFT formulation under the condition that the system is axially symmetric. Under this condition, fields can be calculated by a matrix inversion, in the same way as fields are calculated using modal model.

In order to calculate fields with the same accuracy as the FFT simulation using Wfft and Nfft, the Hankel method calculation is done in a window of Wfft/2 with sampling points of Nfft/2. The FFT method uses Nfft x Nfft 2 dimensional matrix, while the Hankel method uses Nfft/2 1 dimensional vector. In order to calculate fields explicitly, the FFT method needs to calculate an inverse of Nfft\*Nfft x Nfft\*Nfft matrix, which is practically impossible, and it uses an alteration method to calculate the stationary state. The Hankel method needs to calculate an inverse of Nfft/2 x Nfft/2 matrix, and there is no need to do an iterative calculation to search for the stationary state.

Because the Hankel method is based on the same foundation as the FFT method, it could be used to calculate the finite aperture effect<sup>3)</sup>, and the fields in a degenerate cavity<sup>4)</sup>.

A simple matlab code package has been prepared to make this technique easier to use. HankelTK is a class to define underlying mathematical tools and CavityTK is a class to built necessary cavity objects, like reflections or propagations. The adaptive coordinate scaling is support. A few examples are provided.

## 2 Calculation using discrete Hankel transform

Essentially, all calculations are done in the radial space at the following points:

$$r_k = a \zeta_k / \zeta_N$$

where are  $\zeta_k$  zeros of J1 bessel function and a is the spatial window, corresponding to Wfft/2 in the FFT method, and N is the number of sampling points. Fields are defined by a vector whose elements are field values at these radial points.

Most of the operators are diagonal.

Mirror reflection by front size

$$M_\alpha = -r \cdot \exp(i2k \frac{r_\alpha^2}{2R} + 2ik\delta(r_\alpha))$$

Mirror reflection by back size

$$M_\alpha = r \cdot \exp(-i2k \frac{n \cdot r_\alpha^2}{2R} + 2ik\phi(r_\alpha)) = r \cdot \exp(-i2k(\frac{n}{2R} - \frac{1}{2f})r_\alpha^2)$$

Transmission

$$T_\alpha = t(r_\alpha) \cdot \exp(i\phi(r_\alpha)) = t(r_\alpha) \cdot \exp(-ik(\frac{(n-1)}{2R} - \frac{1}{2f})r_\alpha^2)$$

The propagator is the only non diagonal operator, which are given as follows.

$$P_{\alpha\beta}(z) = \sum_{\sigma=0}^N H_{\alpha\sigma}^{(-)} \tilde{G}_{\sigma}(z) H_{\sigma\beta}^{(+)}$$

$$\tilde{G}_{\alpha}(z) = \exp\left(i \frac{z}{2k} \rho_{\alpha}^2\right)$$

$\rho_k = \zeta_k / a$  : spatial frequency sampling

$$H_{\alpha\beta}^{(+)} = \frac{2a^2 J_0(\zeta_{\alpha}\zeta_{\beta} / \zeta_N)}{\zeta_N^2 \cdot J_0^2(\zeta_{\beta})} : f \leftarrow x$$

$$H_{\alpha\beta}^{(-)} = \frac{2J_0(\zeta_{\alpha}\zeta_{\beta} / \zeta_N)}{a^2 \cdot J_0^2(\zeta_{\beta})} : x \leftarrow f$$

In order to remedy the violation of the unitality due to the use of finite N, inverse of  $H^{(+)}$  should be used instead of  $H^{(-)}$ .

When the beam size changes during the propagation, the propagator is changed as follows<sup>5)</sup>.

$$\tilde{P}(z, \gamma) = C_2 \cdot P(z / \gamma) \cdot C_1$$

$$\psi \equiv k \frac{(\gamma - 1)}{2z}$$

$$C_1^{\alpha\alpha} = \exp(-i\psi \cdot r_{\alpha}^2)$$

$$C_2^{\alpha\alpha} = \frac{1}{\gamma} \exp\left(-i \frac{1}{\gamma} \psi \cdot r_{\alpha}^2\right)$$

The parameter  $\gamma$  is the ratio of the window size, window size at the end / window size at the source.  $C_1$  and  $C_2$  are diagonal matrixes evaluated at the coordinated of the source and at the end respectively.

### 3 Field examples

Two examples are shown here. As is clear that the calculation procedure is the same as that used in the modal model based calculation.

Symbols used

fA : reflection by he front side of mirror A

bA : reflection by the back side of mirror A

tA : transmission through mirror A

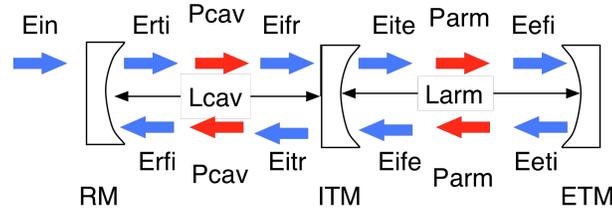
P : propagator

FP cavity

$$E_{cav} = \frac{1}{1 - e^{i\phi} C} \cdot tITM \cdot E_{in}$$

$$C = fITM \cdot P \cdot fETM \cdot P$$

Coupled Cavity



$$E_{rti} = (1 - C_{cav} - f_{RM} \cdot P_{cav} \cdot t_{ITM} \cdot P_{arm} \cdot f_{ETM} \cdot P_{arm} \frac{1}{1 - C_{arm}} t_{ITM} \cdot P_{cav})^{-1} \cdot t_{RM} \cdot E_{in}$$

$$E_{ite} = (1 - C_{arm})^{-1} \cdot t_{ITM} \cdot P_{cav} \cdot E_{rti}$$

$$C_{cav} = f_{RM} \cdot P_{cav} \cdot b_{ITM} \cdot P_{cav}$$

$$C_{arm} = f_{ITM} \cdot P_{arm} \cdot f_{ETM} \cdot P_{arm}$$

## 4 Matlab codes

The main codes are HankelTK.m and CavityTK.m. These codes use new object definitions, and old versions of matlab (7.4 and older, I think) may not work.

Three example files are provided. These are intended to show how to use these classes. But they serve for some practical studied as is. Each example has an example use at the top.

HankelProp : calculate field propagation to test the adaptive propagator

```
[F1, F2] = HankelProp( ROC1, ROC2, dist, a1, N )
```

HankelFP : FP cavity with thermal effects. del1 and del2 can be surface deformation map.

```
[fromITM, toETM, fromETM, toITM, roundLoss, Ein, rList] =  
HankelFP( N, a, T1, T2, ROC1, ROC2, del1, del2, invF, R, Lcav, phi )
```

HankelCC : coupled cavity with a simple thermal lens in ITM.

```
[Fcav, Farm, Ein, rList] =  
HankelCC( N, a, Trm, Titm, Tetm, ROCrm, ROCitm, ROCetm, invF, del,  
Lcav, Larm, phiCav, phiArm )
```

## 5 References

1. J-Y Viney and P. Hello, J. Mod. Opt **40** 1981-1993 (1993)
2. H. Yamamoto T070039, "SIS (Stationary Interferometer Simulation) Manual"
3. H. Yamamoto T0900306, "Mode matching and diffraction loss of FP cavity with thermal deformations and TCS"
4. H. Yamamoto T1000226, "Marginally stable cavity simulation : FFT vs Modal Model"
5. H. Yamamoto T0900640, "Acceleration of the FFT simulation of stable cavity"