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Considerations on Parametric Instability in Fabry-Perot Interferometer

William Kells, Erika D'Ambrosio
Caltech M.C.18-34
Pasadena 91125

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Abstract

We evaluate the parametric phenomenon for the main optical mode coupled to other resonant waves excited by acoustic interaction with the mirrors of a Fabry-Perot. We apply this to the arm cavity of a gravitational wave antenna. Under certain assumptions we find that the amplitude of these parasitic modes is expressed by analytic solutions that are always damped. We analyze both the zero detuning and the detuned case and solve the equations. The form of the solution shows that for equally spaced and excited cavity modes the instability is expressed by a threshold condition, which is well approximated for Ligo arm resonator parameters.

1 Introduction

Recently it has been proposed that coherent light stored in the Fabry-Perot arm cavities of gravitational wave interferometers may experience an instability with respect to its ponderomotive interaction with acoustic modes of the cavity mirrors [1]. The basic phenomenon is portrayed as quite universal. However the perhaps unique conditions attained in such interferometers (high acoustic and optical mode Q's, very high CW optical intensities and extraordinarily low free spectral range $\simeq$ acoustic mode frequencies) conspire to lower the instability threshold to within design parameters.

Here we demonstrate some restrictions to this phenomenon, which ameliorate potential seriousness. In particular we apply a more complete analysis to the *one dimensional model* of a cavity (corresponding to sec. II in [1]) showing that no significant instability occurs. For the more general case where the interaction mixes transverse optical modes (corresponding to sec. III of [1]) instabilities are indeed possible. For such possibilities we argue that the typical interferometer design precludes any problematically large number of "parametric resonances".

Finally we discuss the nature of instabilities from a different (time domain) point of view. This helps to clarify, from an overall cavity perspective, why some modes interact to produce instability and others do not.

2 Condition of stability in the frequency domain for the resonant case

Following the formalism and notations of [1] we write for the full parametric interaction of the one-dimensional cavity model

$$\mathcal{L}_{int} \simeq -\int \frac{x u_z < H_0 + H_1 + H_2 >^2}{8\pi}\bigg|_{z=0} \mathrm{d}r_\perp = -2\omega_0 q_0 B \frac{x}{L}(\omega_1 q_1 + \omega_2 q_2) \qquad (1)$$

(neglecting terms $\sim q_1 q_2$ assuming that they can be completely ignored) where a contribution H_2 for an anti-Stokes mode has not been previously taken into account. The elastic oscillation mode $x(t)$ couples the pump optical mode q_0 with q_1 and q_2. The frequency

$$\omega_m = \omega_2 - \omega_0 = \omega_0 - \omega_1$$

corresponds to the mechanical mode described in [1]. Because the optical modes we are considering are characterized by practically the same parameters, due to the very small free spectral range compared to $\frac{c}{\lambda}$ (where λ is the wavelength of the field and c the speed of light) we assume

$$B_1 = B_2 \equiv B \qquad \delta_1 = \delta_2 \equiv \delta$$

being B a coupling constant and δ the bandwidth of the optical modes.

The Lagrangian coordinates are $q_i(t) = D_i(t)e^{-i\omega_i t} + h.c.$. The degree of freedom representing the acoustic mode has a similar expression. Because of the spatial distribution of these variables, the mirror excited mode must have the correct shape in order to couple the optical modes. The efficiency of such overlap is included in the constant $0 < B < 1$ representing the spatial matching.

If there were no coupling q_i and x would behave as free oscillators. When the term (1) is considered the perturbation drives the dynamics. The fast variation is separated from the slowly changing amplitudes D_i and X.

The equations of motion

$$\begin{aligned}
\partial_t D_1 + \delta D_1 &= \frac{iB\omega_0}{L} X^* D_0 e^{-i\Delta_1 t} \\
\partial_t D_2 + \delta D_2 &= \frac{iB\omega_0}{L} X D_0 e^{-i\Delta_2 t} \\
\partial_t X + \delta_m X &= \frac{iB\omega_0}{ML\omega_m} \left\{ D_0 D_1^* \omega_1 e^{-i\Delta_1 t} + D_0^* D_2 \omega_2 e^{i\Delta_2 t} \right\}
\end{aligned}$$

correspond to the equations (A1) and (A2) in [1]. We have introduced the two small detuning frequencies

$$\Delta_1 = \omega_0 - \omega_1 - \omega_m \qquad \Delta_2 = \omega_0 - \omega_2 + \omega_m \tag{2}$$

and this notation is shown in Fig.1. In order to obtain a condition of parametric instability we seek solutions $X = \chi e^{-i\Omega t}$. If we assume $\Delta_1 = \Delta_2 = 0$ we can rewrite the previous equations as follows

$$\begin{aligned}
D_1 &= \frac{iB\omega_0}{L(\delta + i\Omega^*)} \chi^* e^{i\Omega^* t} D_0 \\
D_2 &= \frac{iB\omega_0}{L(\delta - i\Omega)} \chi e^{-i\Omega t} D_0
\end{aligned}$$

and substitute them in

$$\chi(\delta_m - i\Omega) = \frac{\mathcal{R}'_0 \Lambda \delta \delta_m}{\omega_0(\delta - i\Omega)} \left\{ \chi\omega_1 e^{-i\Omega t} - \chi\omega_2 e^{-i\Omega t} \right\} e^{i\Omega t} \tag{3}$$

where we have introduced

$$\Lambda \mathcal{R}'_0 = \frac{B^2|D_0|^2\omega_0^2}{M\omega_m L^2}\frac{\omega_0}{\delta_m \delta} \tag{4}$$

using almost the same notations as in [1]. The only slight difference is

$$\mathcal{R}'_0 = \frac{\omega_0}{\omega_1}\mathcal{R}_0$$

where $\mathcal{R}_0$ was defined in (4) of [1]. Solving for Ω gives

$$\Omega = \frac{-i(\delta+\delta_m)}{2} \pm i\sqrt{\frac{(\delta-\delta_m)^2}{4} + \frac{(\omega_1-\omega_2)}{\omega_0}\mathcal{R}'_0\Lambda\delta\delta_m}$$

that implies the solution

$$\begin{aligned} X &= \chi e^{-i\Omega t} \\ &= \chi \exp\left\{-t\left[\frac{\delta+\delta_m}{2} \pm \sqrt{\frac{(\delta-\delta_m)^2}{4} + \frac{(\omega_1-\omega_2)}{\omega_0}\mathcal{R}'\Lambda\delta\delta_m}\right]\right\} \end{aligned} \tag{5}$$

for the slowly varying amplitude X. The stability condition is fulfilled for

$$\mathcal{R}'_0\Lambda\frac{\omega_1-\omega_2}{\omega_0} < 1$$

that is always satisfied since numerically $(\omega_1 - \omega_2) < 0$.

The contribution containing ω_2 is not taken into account in [1], where the condition for instability, in this no detuning situation is given as

$$\mathcal{R}'_0\Lambda\frac{\omega_1}{\omega_0} > 1 \qquad .$$

This is then exactly recovered by dropping the term in ω_2 leading to (5). Keeping the contribution in ω_2 gives a solution that is always damped as we can verify by inspection of the equation (5).

3 Condition of stability in the frequency domain with detuning

We seek a solution in the case $\Delta_1 = -\Delta_2 = \Delta$. Using the same notation of [1], the definition (4) and Δ_i as prescribed in (2), we look for solutions of the following form:

$$D_1(t) = D_1 \exp[i(\Omega^* - \frac{\Delta}{2})t] = D_1 e^{\lambda_- t} \qquad X^*(t) = \chi^* \exp[i(\Omega^* + \frac{\Delta}{2})t] = \chi^* e^{\lambda_+ t}$$

$$D_2(t) = D_2 \exp[-i(\Omega - \frac{\Delta}{2})t] = D_2 e^{\lambda_-^* t} \qquad X(t) = \chi \exp[-i(\Omega + \frac{\Delta}{2})t] = \chi e^{\lambda_+^* t}$$

where no assumption on the reality of $\lambda = i\Omega^*$ has been made. If we substitute the previous expressions in the equation for $D_1(t)$ and $D_2(t)$ we find

$$\begin{aligned} D_1 &= \frac{iB\omega_0}{L(\delta + i(\Omega^* - \frac{\Delta}{2}))} \chi^* e^{i(\Omega^* - \frac{\Delta}{2})t} D_0 \\ D_2 &= \frac{iB\omega_0}{L(\delta - i(\Omega - \frac{\Delta}{2})} \chi e^{-i(\Omega - \frac{\Delta}{2})t} D_0 \end{aligned}$$

implying a solution

$$\begin{aligned} &\chi(\delta_m - i(\Omega + \frac{\Delta}{2})) = \\ &\frac{\mathcal{R}'_0 \Lambda \delta \delta_m}{\omega_0(\delta - i(\Omega - \frac{\Delta}{2}))} \left\{ \chi \omega_1 e^{-i(\Omega - \frac{\Delta}{2})t} e^{-i\Delta t} - \chi \omega_2 e^{-i(\Omega - \frac{\Delta}{2})t} e^{-i\Delta t} \right\} e^{i(\Omega + \frac{\Delta}{2})t} \end{aligned} \tag{6}$$

that corresponds to (3) for $\Delta \neq 0$. Solving (6) for the unknown variable Ω gives two solutions. Using the formal expression

$$\Omega = \frac{-i(\delta + \delta_m)}{2} \pm i\sqrt{(\frac{\delta - \delta_m}{2} - \frac{i\Delta}{2})^2 + \frac{(\omega_1 - \omega_2)}{\omega_0} \mathcal{R}'_0 \Lambda \delta \delta_m}$$

we find a condition for stable solutions $\sim e^{-i\Omega t}$. The condition for stability is

$$-\frac{\mathcal{R}'_0 \Lambda \delta \delta_m}{\omega_0}(\omega_1 - \omega_2) + \delta \delta_m [1 + \frac{\Delta^2}{(\delta + \delta_m)^2}] > 0 \tag{7}$$

that is always fulfilled.

Formula (7) has been obtained with no approximation. If ω_2 is dropped and $>$ is inverted we recover the condition (A5) in [1] for the instability threshold.

4 Interpretation in the time-domain

Stability is evidently preserved when *symmetric* scattering to optical modes resonating at $\omega_0 \pm \omega_m$ is possible. Therefore we anticipate that instability may occur with the inclusion of scattering from the pump mode to different *higher transverse modes* (HTMs). In this situation if $\omega_1 = \omega_0 \mp \omega_m$ holds for some HTM, then the symmetry is broken in that no *resonant* mode exists at $\omega_1 = \omega_0 \pm \omega_m$. Then the two modes analysis of [1] pertains.

The mechanisms at work here can be, perhaps more clearly, viewed in time domain. For net *work* to be done on the acoustic mode the ponderomotive force and hence intensity of light impinging upon it must acquire an acoustic component.

It is well known that intra-cavity *phase* modulation (as in Fig.2 the physical effect of one dimensional mirror dithering) induces just such *amplitude* modulation [2]. Limiting to the exact parametric resonant scattering case, the fields generated upon reflection by an incipient acoustically (amplitude $X = \chi \cos \omega_m t$) mirror are:

$$\begin{array}{lll} \psi_0 e^{i\omega_0 t} & \overset{pump}{\rightarrow} & \psi_0 e^{i\omega_0 t} \\ & \overset{scatter}{\searrow} & -i\Gamma \psi_1 \cos \omega_m t \, e^{i\omega_0 t} \qquad \Gamma = \frac{4\pi}{\lambda}\chi \end{array}$$

with $\Gamma << 2\omega_0 \chi / c$. That is, to first order, the pump is unaffected and a pair of scattering sidebands is generated. Generally only one sideband will be resonant in the cavity, say at $\omega_0 - \omega_m$. In this case the Poynting vector into the acoustic surface is

$$P_{em} \propto |\psi_0|^2 - \Gamma \psi_1 \psi_0 \sin \omega_m t$$

at first order in Γ. This beat amplitude modulation then is just correct (phase and transverse form) to do positive work on the acoustic mode. If the upper $(\omega_0 + \omega_m)$ sideband were resonant then work would be extracted from the acoustic mode since

$$P_{em} \propto |\psi_0|^2 + \Gamma \psi_1 \psi_0 \sin \omega_m t \qquad .$$

The pure one dimensional situation $(\psi_1 = \psi_0)$ is special, in that both $\omega_0 \pm \omega_m$ resonate. Then no amplitude modulation is developed at ω_m. A full higher order analysis shows that *only* even harmonic $(2n\omega_m)$ amplitude modulation develops, which does no work on the acoustic mode. [1]

5 Quantitative limitations

Quantitative estimates of those instabilities exciting HTMs depend strongly on the overall interferometer configuration [3]. The very circumstances (high pump mode Q, low

[1] This symmetry is subject strictly to the approximate assumption that $|X|/L << 1$ (that is, an exact analysis where terms $\mathcal{O}(|X|/L)^2$ are neglected). This is an excellent approximation for the gravitational wave interferometers case of interest: $|X|/L \sim 10^{-12}$.

free spectral range or long cavity length, ...) conducive to the instability also severely limit the potentially accessible HTMs.

Practical GW interferometers have pump mode transverse size designed to just fill the test mass mirrors' apertures. HTMs then have larger transverse size, and hence larger finite aperture diffraction losses (higher δ_1). For example the LIGO I cavity round trip diffractive loss for modes TEM_{mn} with $m + n \geq 4$ is already enough to increase the R_0 instability threshold by a factor of ~ 100 over the basic estimate of [1].

A detailed description of the higher transverse mode eigenfrequencies must include interaction with the full interferometer. Although *generated* in the high $\mathcal{F}$inesse Fabry-Perot arm cavities, optical resonance of these HTMs will sensitively depend on the complex coupling of the Fabry-Perot cavity to the rest of the interferometer. Both ω_1 and δ_1 will be affected. The necessity of a full interferometer analysis is implicit in the use of the pump level $\mathcal{E}_0 \simeq 3 \times 10^8$ ergs in [1] corresponding to *full recycled interferometer* optical resonance. We note consistently, for instance, that $\delta_1 >> \delta_0$ by two orders of magnitude when HTMs are excited as in Fig.2, since the scattered SB is generally not resonant in the full interferometer.

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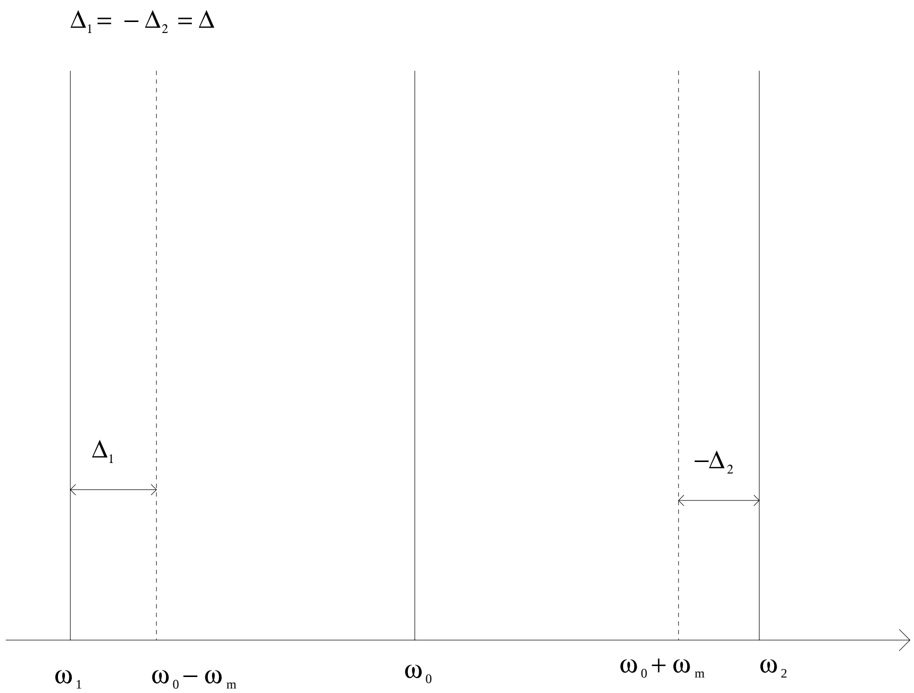

Figure 1: The detuning is symmetric because the modes are equally spaced in the frequency domain

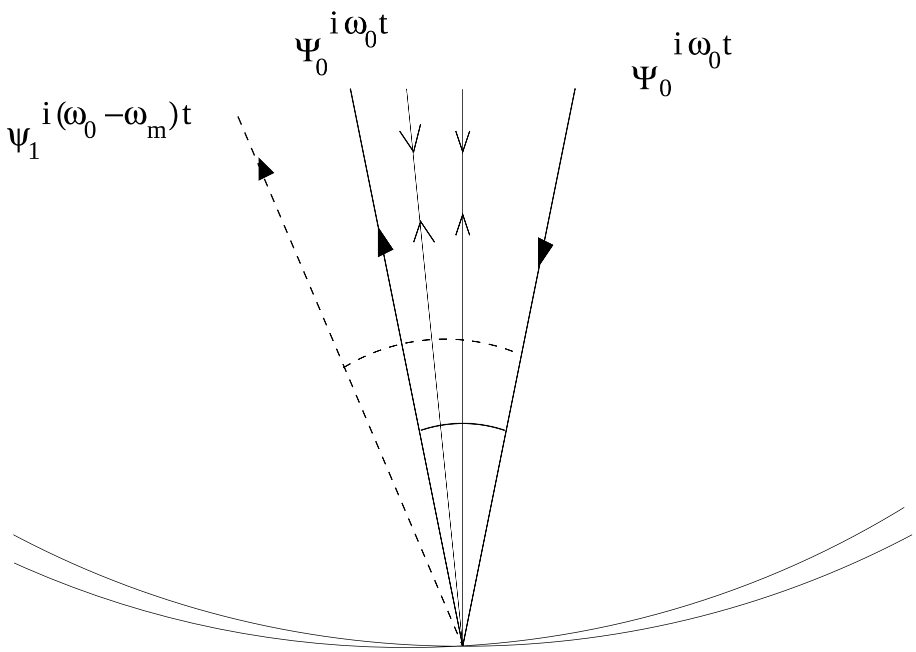

Figure 2: The misalignment is a typical source of excitation of HTMs and the frequency of the scattered field is affected by the frequency of the tilt motion of the mirror