SURF Final Report

EXPECTATIONS ON THE GRAVITATIONAL-WAVE SIGNALS ASSOCIATED WITH COSMIC BREMSSTRAHLUNG EVENTS

by

Bence Kocsis* and Merse Előd Gáspár*

LIGO-T030136-00-D

Advisor: **Szabolcs Márka**

September 2003, LIGO Caltech

* Eötvös Loránd University of Sciences, Hungary

Abstract

The likelihood of close encounters is estimated for stars in globular clusters or galaxies. Close approaches of compact cosmic objects produce bremsstrahlung gravitational-wave radiation of well-known waveforms. We calculated the frequency range of these signals and showed that it would not be regularly detectable with LIGO for sources of currently understood forms of matter.

1 Introduction

The study is inspired by the search for gravitational bremsstrahlung signals (GBS) with presently operating terrestrial gravitational-wave detectors such as LIGO. The GBS waveforms are known analytically for the case of small deflection angles and arbitrary velocities [1], for the arbitrary deflections and small velocities in the Newtonian limit [2], as well as in the post-Newtonian limit [3]. These signals are short period waveforms ("bursts") and are special since they can be trusted to high accuracy as the physics behind them is well understood.

In this paper we shall estimate the average time, $\langle t \rangle$, between consecutive encounters of stars in systems such as galaxies and globular clusters. Various encounters generate GBS with various characteristic frequencies f. The f value is an independent degree of freedom, besides the parameters describing the system (i.e N the number of stars, R the maximum radius of the system, M the typical mass of one star). Without physical restrictions on the possible sizes and velocities, there is a nonzero GBS rate for any f frequency within the same star system. Thus, the encounter rate is first obtained for fixed infinitesimal frequency bands, which is then integrated over the sensitive ranges of the gravitational wave detectors.

There are three different difficulties associated with the detection of GBS. First we will show that the frequency range for the majority of the GBS will not fall into the sensitive range of current terrestrial detectors. The GBS frequency increases with decreasing the impact parameter b. Decreasing b could lead to a collision before the detectable frequency range is reached, thereby producing a signal, different from the GBS waveforms. The second difficulty is that GBS are generated very rarely in one given star cluster. The third problem could be the relative weakness of any GBS compared to the target sensitivity of detection with the present or near-future terrestrial technologies.

We have used a classical nonrelativistic treatment for our analysis. In §2, we derive a gas model with a unique velocity and homogeneous spacial distributions. In §3, the model is improved to account for the velocity distribution. In §4, the relative velocity distribution is taken into account. Finally, the conclusions are drawn in the last section.

2 Model I

This is the zeroth order approximation: uniform density and uniform magnitude of velocities. In this estimate, we derive the encounter probability as a scattering on a still target lattice. We use the viral theorem to obtain the typical velocities, and we derive the enhancement of the impact parameter due to gravitational deflection. The encounter rate is then calculated for fixed infinitesimal frequency bands.The GBS rate is obtained by integrating over the sensitive ranges of the gravitational wave detectors.

Let us first calculate the characteristic velocities in the system. The Newtonian gravitational potential of a spherical uniform cluster is $U = \frac{3}{5}GNM/R$, where N is the number of stars, R is the radius of the cluster, and M is the mass of one star. The speed is estimated using the virial theorem $2E_{kin}/M = U$,[1]

$$v_{RMS} = v_0 = \sqrt{\frac{3}{5}\frac{GNM}{R}} \tag{1}$$

For this estimate, we shall also take this as the velocity dispersion, which might lead to close encounters between stars. As we shall now demonstrate these velocities are typically too low for the detection of the induced GBS in realistic star systems.

The timescale of the GBS generated by this encounter is $T = b^*/v^*$ (e.g. [1]), where v^* is the velocity and b^* the distance of separation at the closest point[2]. Let us define the relative enhancement of the velocity during the

[1] There is a similar result for a spherical star system of polytropic distribution with a root mean square radius R. The only difference is in the $\frac{3}{5}$ factor, which becomes $\frac{1}{2}$ in that case (e.g. [5]).

[2] For small deflection angles, $b^* \approx b$.

encounter as $\kappa = v^*/v$. This corresponds to a short burst with the maximum frequency

$$f = \frac{1}{T} = \frac{v^*}{b^*} = \kappa\sqrt{\frac{3GNM}{5Rb^{*2}}} \tag{2}$$

For terrestrial gravitational wave interferometry the lower frequency limit is currently around 100 Hz. An encounter without a collision occurs for neutron stars with an impact parameter greater than $b^* = 24\,\mathrm{km}$. The minimum v^* velocity for having the GBS in the right band is $v^* = fb^* = 8 \cdot 10^{-3}c$. However, the typical v velocities of these systems are generally much lower, around $v \approx 10^{-4}c$ for globular clusters of radius 10 ly and star population of $N = 10^6$, and $v \approx 10^{-3}c$ for galaxies with $r = 10^4$ ly and $N = 10^{11}$. Only very close encounters with $\kappa > 100$ or $\kappa > 10$ give the sufficient boost in order to get a signal in the detectable range of such detectors. We shall now demonstrate that this condition is very rarely met in regular systems.

In this approximation the trajectories of objects shall be taken nearly straight between near encounters. The increase in the cross section due to gravitational focusing can be accounted for by introducing an enlarged effective impact parameter. Let b^* be the distance to closest approach along the hyperbolic trajectory. Using the conservation of mechanical energy and angular momentum

$$\frac{1}{2}Mv^2 = \frac{1}{2}Mv^{*2} - \frac{GM^2}{b^*} \tag{3}$$

$$bv = b^*v^* \tag{4}$$

Solving for b and v in terms of b^* and v^* we get

$$b = \frac{b^*}{\sqrt{1-2\gamma}} \tag{5}$$

$$v = v^*\sqrt{1-2\gamma} \tag{6}$$

where $\gamma = \frac{GM}{b^* v^{*2}}$ is the ratio of potential energy and double kinetic energy at the closest point[3]. For large speeds the effective impact parameter b reduces to the original b^*, which is evident since in this case the trajectory is a straight line.

In this section we will neglect the deviation from the homogenous distribution of the positions of stars within the system. The "edge effects" arising

[3]For hyperbolic Newtonian trajectories, this has to be less than 1/2.

from the inhomogeneity observed in the region will not be taken into account. The main contribution is expected to come from the inner parts.

For this estimate let us take a star A moving through the cluster, and assume that the other stars are fixed. Now we derive the number of events when Star A approaches another star after travelling a distance r within a minimum separation of $[b^*, b^* + \mathrm{d}b^*]$. The variable r is held fixed at an arbitrary value. Using the relationship (5) connecting b and b^* leads to the corresponding infinitesimal range for b. The number of close approaches with b and r in the given ranges are

$$\mathrm{d}N \frac{2\pi b \mathrm{d}b}{4\pi r^2} \tag{7}$$

where $2\pi b\mathrm{d}b$ is the cross section of such a close approach, the total area is $4\pi r^2$ and $\mathrm{d}N$ is the number of target objects within r and $r+\mathrm{d}r$. Substituting

$$\mathrm{d}N = \frac{N}{\frac{4\pi}{3}R^3} 4\pi r^2 \mathrm{d}r \tag{8}$$

gives the number of events per unit r and b for star A.

$$\partial_r \partial_b n_1(b^*, r) = \frac{3N}{2R^3} b \tag{9}$$

Note that the result is independent of r. Since A is any one of the N stars in the cluster and encounters occur between two stars, the total number of collisions is $N/2$ times the number obtained for A. Substituting $\mathrm{d}r = v\mathrm{d}t$ gives

$$\partial_t \partial_b n(b^*, t) = \frac{3N^2}{4R^3} v_0 b \tag{10}$$

$\partial_t n$ is the number of encounters per unit time, it's reciprocal gives the time between consecutive encounters. The probability of an encounter to occur between t_0 and $t_0 + \mathrm{d}t$ is independent of t_0. Therefore the consecutive encounters follow an exponential distribution with a decay constant

$$\lambda(b)\mathrm{d}b = \partial_t \partial_b n(b^*, t)\mathrm{d}b = \frac{3N^2}{4R^3} v_0 b \mathrm{d}b \tag{11}$$

Using the identity for the GBS characteristic frequency f for a given b^* encounter, we can obtain an expression for $\lambda(f)\mathrm{d}f$. In reality the sensitivity of the detector sets the detectable f interval, leading to a macroscopic

$\lambda(f_{min}, f_{max})$ parameter. The distribution of the times between consecutive encounters is then

$$P(f_{min}, f_{max}, t) = \lambda(f_{min}, f_{max}) e^{-\lambda(f_{min}, f_{max})t} \tag{12}$$

We will first derive the infinitesimal $\lambda(b^*)\mathrm{d}b^*$ rate, and then will change to the GBS frequency variable f. Using equation (5) for b in (11) gives

$$\begin{aligned}
\lambda(b)\mathrm{d}b &= \frac{3}{4}\frac{N^2}{R^3}v_0 \left(\frac{1}{2}\mathrm{d}\frac{b^{*2}}{1-\frac{2GM}{b^* v^{*2}}} \right) & (13)\\
&= \frac{3}{4}\frac{N^2}{R^3}v_0 \left| \frac{b^*\mathrm{d}b^*}{1-\frac{2GM}{b^* v^{*2}}} - \frac{b^{*2}}{2}\frac{2GM}{(b^* v^{*2})^2}\frac{\mathrm{d}(b^* v^{*2})}{\left(1-\frac{2GM}{b^* v^{*2}}\right)^2} \right| & (14)\\
&= \frac{3}{4}\frac{N^2}{R^3}v_0 \left| \frac{v^{*2}}{v_0^2}b^*\mathrm{d}b^* - \frac{GM}{v_0^4}\mathrm{d}(b^* v^{*2}) \right| & (15)\\
&= \frac{3}{4}\frac{N^2}{R^3}v_0 \left[\left(\frac{2GM}{v_0^2} + b^* \right)\mathrm{d}b^* - \frac{GM}{v_0^4}v_0^2\mathrm{d}b^* \right] & (16)\\
&= \frac{3}{4}\frac{N^2}{R^3}\frac{GM}{v_0}\left(1 + \frac{v_0^2}{GM}b^*\right)\mathrm{d}b^* & (17)\\
&= \frac{\sqrt{15}}{4}\frac{N^{1.5}}{R^{2.5}}\sqrt{GM}\left(1 + \frac{3}{5}N\frac{b^*}{R}\right)\mathrm{d}b^* & (18)
\end{aligned}$$

where in (15) and in (16) we have used an equivalent form of (3), the equation of energy conservation

$$b^* v^{*2} = 2GM + b^* v^2 = 2GM\left(1 + \frac{3}{10}N\frac{b^*}{R}\right) \tag{19}$$

and in (18) we have substituted the v_0 velocity from (1).

Recall that $v^* = b^* f$ connects the variables v^* and b^*, only one of them can be chosen freely. Equation (19) yields a cubic equation for the $f(b^*)$ or $f(v^*)$ dependence.

$$f = \frac{v^{*3}}{2GM} - \frac{3}{10}\frac{N}{R}v^* = \sqrt{\frac{2GM}{b^{*3}}\left(1 + \frac{3}{10}\frac{N}{R}b^*\right)} \tag{20}$$

For realistic systems and detectable b^* values, the second term of the parenthesis is much less then 1. Note that the physics behind this approximation is that the total energy of the system is much less than the magnitude of the kinetic and potential energies respectively at the encounter. Also note, that this corresponds to a parabolic rather than a hyperbolic trajectory, in which case b is undefined. Therefore, this approximation is to be handled with care.

To avoid these complications we shall work to the first non-vanishing order. The solution of (20) is expanded in a Taylor-series around the critical trajectory.

$$b^* \approx \sqrt[3]{2GM} f^{-2/3} + (2GM)^{2/3} \frac{N}{10R} f^{-4/3} + \dots \tag{21}$$

Also note, that for the v^* velocity this implies

$$v^* \approx \sqrt[3]{2GM} f^{1/3} + (2GM)^{2/3} \frac{N}{10R} f^{-1/3} + \dots \tag{22}$$

Substituting in (**??**)

$$\lambda(f)\mathrm{d}f \approx \sqrt{\frac{5}{3}} 2^{-2/3} (GM)^{5/6} \frac{N^{1.5}}{R^{2.5}} f^{-2/3} \left(1 + \frac{4}{5} \sqrt[3]{2GM} \frac{N}{R} f^{-2/3}\right) \frac{\mathrm{d}f}{f} \tag{23}$$

to the first order. For dense globular clusters of $R = 10\,\mathrm{ly}$ and $N = 10^6$ we get the rate for consequent signals.

$$\lambda(f)_{\mathrm{glob}}\mathrm{d}f \approx \left(3 \cdot 10^{-11}\,\mathrm{yr}^{-1}\right) \left[1 + 3 \cdot 10^{-6} \left(\frac{f}{100\,\mathrm{Hz}}\right)^{-2/3}\right] \left(\frac{f}{100\,\mathrm{Hz}}\right)^{-2/3} \frac{\mathrm{d}f}{f} \tag{24}$$

For galaxies of $R = 10^4\,\mathrm{ly}$ and $N = 10^{11}$ this becomes

$$\lambda(f)_{\mathrm{gal}}\mathrm{d}f \approx \left(3 \cdot 10^{-11}\,\mathrm{yr}^{-1}\right) \left[1 + 3 \cdot 10^{-4} \left(\frac{f}{100\,\mathrm{Hz}}\right)^{-2/3}\right] \left(\frac{f}{100\,\mathrm{Hz}}\right)^{-2/3} \frac{\mathrm{d}f}{f} \tag{25}$$

Equations (24) and (25) describing the GBS rates are plotted on Figure 1.

Notice how negligible the first correction in the parenthesis is. Therefore the main contribution is due to the zeroth term, which is exact for parabolic trajectories. Hence, this approximation favors the high deflection angle encounters instead of the low deflection angle hyperbolic encounters.

We obtain the macroscopic λ parameter by integrating (23) over the sensitive region of the detector. If the detectable minimum and maximum frequencies are f_{min} and f_{max} respectively then

$$\lambda(f_{min}, f_{max}) \approx \sqrt{15} \cdot 2^{-5/3} (GM)^{5/6} \frac{N^{1.5}}{R^{2.5}} \left(f_{min}^{-2/3} - f_{max}^{-2/3}\right) \tag{26}$$

This is therefore the output of this model for λ. Recall that λ is the decay constant of the exponential distribution (12) describing the time intervals between the consecutive encounters. The expectation value of the time intervals is $1/\lambda$ which for the above cases becomes[4,5]

$$\langle t\rangle_{\text{glob}} = \langle t\rangle_{\text{gal}} = (3\cdot 10^{10}\,\text{yr})\left(\frac{f_{min}}{100\,\text{Hz}}\right)^{2/3}\frac{1}{1-(f_{min}/f_{max})^{2/3}} \tag{27}$$

The expected time between consequent GBS is plotted on Figure 1 for logarithmic frequency bins. The timescale between events is higher at lower frequencies, but is still very large when considering practical time scales. Even space detectors with minimum frequencies of 0.1 mHz (e.g. LISA [4]), will have a chance to encounter one event in every $3\cdot 10^6$ yr per dense cluster or galaxy. Terrestrial detectors like LIGO have event rates of one event per $3\cdot 10^{10}$ yr per cluster.

Equation (27) gives the expected times between GBS in the sensitive bands for a Newtonian system of point masses. However, the theoretical waveforms for these signals can only be trusted within the limits of these approximations. Therefore we shall impose the following constraints:

Accept events with

$$b^* > 24\text{km} \tag{28}$$

$$v^* < 0.1c \tag{29}$$

Substituting in equations (2) and (20) leads to the following constraints

$$f < v^*_{max}/b^*_{min} = 1300\,Hz \tag{30}$$

$$f < \sqrt{\frac{2GM}{b^3_{min}}} = 4400\,\text{Hz} \tag{31}$$

$$f < \frac{(v^*_{max})^3}{2GM} = 101\,Hz \tag{32}$$

[4]Note that these results do depend on the (N, R) values describing the system, and is just a mere coincidence that the special cases of $(10^6, 10^1\,\text{ly})$ and $(10^{11}, 10^4\,\text{ly})$ produce the same numerical result.

[5]The primary potential energy in galaxies come from the dark matter and the distribution of stars is non-spherical. The result for galaxies should therefore be accepted with caution.

The strongest constraint is the third, $f_{max} = 101\,\mathrm{Hz}$, which leaves just a marginal bandwidth for terrestrial detection. Note that if we trusted our results up to $v^* = 0.25c$ instead of $0.1c$, then this bound becomes $f_{max} = 1600\,\mathrm{Hz}$. Hence, the evaluation of relativistic corrections should be considered in the future.

The above calculation gives the rate limit provided that all GBS are detected from a given star system. Depending on the distance and strength of the source, however, only a fraction of these events can be detected. The GBS characteristic strain magnitudes scale as

$$\frac{\delta h}{h} = \frac{4M^2}{rb^*} = (1.1 \cdot 10^{-18}) \left(\frac{M}{M_{SUN}}\right)^2 \left(\frac{10\,\mathrm{kpc}}{r}\right) \left(\frac{24\,\mathrm{km}}{b^*}\right) \tag{33}$$

(see [1]), where r is the distance of the cluster from Earth. This puts a maximum bound on b^* or a maximum bound on r. The detection limit for LIGO [7] is currently about 10^{-21}, and advanced LIGO will have marginal strain magnitudes around 10^{-23}. Substituting these values in (33) yields

$$f > \sqrt{\frac{2GM}{b^3_{max}}} = \begin{cases} (0.11\,\mathrm{Hz}) \left(\frac{r}{10\,\mathrm{kpc}}\right)^{3/2} & \text{for current LIGO sensitivities} \\ (0.11\,\mathrm{mHz}) \left(\frac{r}{10\,\mathrm{kpc}}\right)^{3/2} & \text{for advanced LIGO sensitivities} \end{cases} \tag{34}$$

Since the minimum frequency bound for LIGO is $\propto 100\,\mathrm{Hz}$, the detection rates will not be hampered for nearby globular clusters and galaxies due to sensitivity. The detection cutoff distance according to equation (34) for GBS sources for LIGO is

$$r < \begin{cases} 0.94\,\mathrm{Mpc} & \text{for current LIGO sensitivities} \\ 94\,\mathrm{Mpc} & \text{for advanced LIGO sensitivities} \end{cases} \tag{35}$$

All GBS sources closer than (35), that are in the appropriate frequency bands, satisfy the sensitivity margins of the LIGO detectors.

According to equation (27), the main GBS sources are globular clusters. Let us approximate the number of globular clusters per galaxy with 200. The accumulated number of galaxies is 25 within $1\,\mathrm{Mpc}$ and $5 \cdot 10^4$ within $100\,\mathrm{Mpc}$ (see [6]). The expected resultant timescale between detectable GBS events for all globular clusters is therefore

$$\langle t \rangle = \begin{cases} 6 \cdot 10^5\,\mathrm{yr} & \text{for current LIGO} \\ 3 \cdot 10^3\,\mathrm{yr} & \text{for advanced LIGO} \\ 0.3\,\mathrm{yr} & \text{for LISA} \end{cases} \tag{36}$$

In this calculation, anticipated LISA sensitivities are assumed to reach the advanced LIGO sensitivities, and the lower frequency bound was taken to be 0.1 mHz.

Notice that this estimate is just a theoretical optimal limit for the rate of GBS, realistic expectations are even worse. The signals are only generated by compact objects, such as black holes or neutron stars, which are generally just a small q fraction of the star population[6]. The rate of the signals, according to equation (10), is decreased by q^2, which adds several orders of magnitudes for the result obtained for the timescale of succeeding encounters.

3 Model II

The velocity distribution in reality is some continuous function not just one v_0 value. Depending on how likely the encounter speed, the probability for GBS will be different in all frequency ranges. However the gravitational bremsstrahlung radiation depends on v^*, the relative velocity at the closest point, instead of the initial velocities which are generally much lower. The model of the last section also showed, that the section of the v^* distribution which produced GBS waves in the detectable range was mostly independent of v_0 initial velocities. The velocity buildup due to gravitational attraction was much larger in this regime. We only expect the low v^* speed encounters (i.e. low frequency radiation) to depend strongly on the initial velocity distribution. Therefore, the results should not differ from the last section for the LIGO frequencies.

Our approximation uses a position averaged velocity distribution, and assume a Maxwell-Boltzmann form, valid for ideal gases

$$F(v) = \left(\frac{M}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{\frac{-Mv^2}{2kT}} \tag{37}$$

By equipartition and the virial theorems we can eliminate the kT factors.

$$< E_{kin} >= \frac{1}{2}M < v^2 >= \frac{3}{2}kT = \frac{1}{2} < U > \tag{38}$$

[6] GBS might also be generated by WD–NS or WD–BH encounters, but only at frequencies below 300 Hz for 1000 km size WDs. At higher frequencies the flyby would have to take place with a superluminal velocity. The nonrelativistic limit corresponding to $v^* = 0.1c$ is $f_{max} = 30\,Hz$. To avoid a collision from $b^* > r_{WD}$ eq. (21) gives a limit $f_{max} = 16$ Hz. These limits hold only for the sensitive regions of space detectors. WD – NS and WD – BH encounters therefore do enhance the rate of GBS for space observation.

where U is the average potential energy of the system such as in (1). With the $v_0 = \sqrt{< v^2 >}$ notation

$$F(v) = 4\pi \left(\frac{3}{2\pi}\right)^{3/2} \frac{v^2}{v_0^3} e^{-\frac{3}{2}\frac{v^2}{v_0^2}} \tag{39}$$

Also note that $v_{RMS} = v_0$, the relative RMS velocity fluctuations around v_0 are 0.397.

From now on all probabilities are v dependent. The derivation is similar until eq. (12), with the velocity chosen with the given probability distribution instead of v_0. With $\mu = \frac{3}{4}\frac{N^2}{R^3}$ we have

$$\lambda(b, v)\mathrm{d}b\mathrm{d}v = \mu b F(v) v \mathrm{d}v \mathrm{d}b \tag{40}$$

To get a measure of the encounter likelihood for a given frequency, we should change to the $f = v^*/b^*$ frequency variable.

$$\lambda(b^*, v^*)\mathrm{d}b^*\mathrm{d}v^* = \mu b v F(v(v^*)) \frac{1-\gamma}{1-2\gamma} \mathrm{d}v^*\mathrm{d}b^* \tag{41}$$

$$= \mu b^* v^* F(v(v^*)) \frac{1-\gamma}{1-2\gamma} \mathrm{d}v^*\mathrm{d}b^* \tag{42}$$

where γ is the ratio of $U^*/2K^*$, the potential energy and twice the kinetic energy at the closest point. Recall that $\gamma < \frac{1}{2}$ for hyperbolic trajectories. The fraction $(1-\gamma)/(1-2\gamma)$ is the Jacobi determinant for the variable change, and we have used equation (4). Since $b^* = v^*/f$ for any v^*, we have

$$\lambda(f)\mathrm{d}f = \mu f^{-3}\mathrm{d}f \int \mathrm{d}v^* v^{*3} F(v(v^*)) \frac{1-\gamma}{1-2\gamma} \tag{43}$$

Using the expression (39) for the velocity distribution

$$\lambda(f)\mathrm{d}f = C f^{-3}\mathrm{d}f \int \mathrm{d}v^* v^{*5} \left(1 - \frac{GMf}{(v^*)^3}\right) e^{-\frac{3}{2}\frac{v^{*2}}{v_0^2}(1-\frac{2GMf}{(v^*)^3})} \tag{44}$$

where $C = 4\pi \left(\frac{3}{2\pi}\right)^{3/2} v_0^{-3}\mu$ has been used. Recall that the hyperbolic encounters are defined only for $v^* > v^*_{min} = \sqrt[3]{2GMf}$, and collisions are avoided if $v^* > b^*_{min} f$. Since both of these conditions must be fulfilled, the integrals should be taken from the larger of these values. For $b^*_{min} = 24\,\mathrm{km}$, the first condition is the larger for frequencies below 4400 Hz. We evaluate the result only in this regime.

Changing to the dimensionless integration variable $w = v^*/v^*_{min}$,

$$\lambda(f)\mathrm{d}f = A\frac{\mathrm{d}f}{f}\int_1^\infty \mathrm{d}w w^5\left(1-\frac{1}{2w^3}\right)e^{-\frac{w^2}{2s}(1-\frac{1}{w^3})} \tag{45}$$

where the coefficient comes from

$$A = Cf^{-2}{v^*_{min}}^6 = \sqrt{\frac{3^5}{2^3\pi}\frac{N^2}{R^3}\frac{{v^*_{min}}^6}{v_0^3}}f^{-2} = 26.8\cdot\frac{\sqrt{GMN}}{R^{1.5}} \tag{46}$$

and the variable s is defined as

$$s = \frac{v_0^2}{3{v^*_{min}}^2} = \frac{1}{2^{2/3}5}\sqrt[3]{GM}\frac{N}{R}f^{-2/3} = (3.2\cdot 10^{-7})\left(\frac{f}{100\,\mathrm{Hz}}\right)^{-2/3} \tag{47}$$

the RHS was calculated for globular clusters of $N = 10^6$ and $R = 10\,\mathrm{ly}$. This shows that $s \ll 1$ is a sufficient approximation.

The top bound of the integration is infinity, which is a good approximation for GBS sources closer than $0.5\,\mathrm{Mpc}$. The details about this approximation are discussed in the Appendix.

Changing the integration variable to $u = w - 1$,

$$\lambda(f)\mathrm{d}f = A\frac{\mathrm{d}f}{f}\frac{1}{2}\int_0^\infty \mathrm{d}u(1+u)^2(1+6u+6u^2+2u^3)e^{-\frac{3u}{2s}}e^{-\frac{u^3}{2s(1+u)}} \tag{48}$$

Since $s \ll 1$,

$$\lambda(f)\mathrm{d}f \approx A\frac{\mathrm{d}f}{f}\int_0^\infty \mathrm{d}u\left(\frac{1}{2}+4u\right)e^{-\frac{3u}{2s}} \tag{49}$$

$$= A\frac{\mathrm{d}f}{f}\left(\frac{1}{3}s+\frac{16}{9}s^2\right) \tag{50}$$

After simplifying, we get

$$\lambda(f)\mathrm{d}f = \sqrt{\frac{6}{\pi}}\sqrt{\frac{5}{3}}2^{-2/3}(GM)^{5/6}\frac{N^{1.5}}{R^{2.5}}f^{-2/3}\left(1+\frac{16}{30}\sqrt[3]{2GM}\frac{N}{R}f^{-2/3}\right)\frac{\mathrm{d}f}{f} \tag{51}$$

This is to be compared with the result of the previous model, eq. (23). The difference a factor of $\sqrt{6/\pi}$=1.38, a 38% increase in the GBS rate for the leading order coefficient, and 7.9% for the first correction. The result satisfies the intuition: at large frequencies, v^*, the GBS source velocity is much larger than v_0, the RMS velocities in the star cluster, implying that the velocity uncertainty around v_0 ought not alter the v^* velocities significantly.

4 Model III

So far we have treated the bremsstrahlung encounters in the star cluster as the scattering of stars off fixed background stars. This treatment is also used in [5] for calculating the probability of collisions in spherical star clusters. A better model accounts for the relative movement of the targets. The correction to the GBS rates can be significant if the velocity distribution doesn't vanish rapidly for large initial speeds. In this model we calculate the GBS rate, taking the relative velocity distribution into account.

We now derive the relative velocity distribution for pairs of stars. The relative velocity is $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. Let us denote the relative velocity direction with ζ, i.e. $\zeta = arg(\boldsymbol{v}_1, \boldsymbol{v}_2)$. The individual distribution is assumed to be isotropic, therefore the relative velocity distribution for pairs is also isotropic. The ζ probability density is therefore $P(\zeta) = sin(\zeta)/2$. The v_i velocity magnitudes (with i=1,2) follow the identical distributions $F_0(v_i)$.

By the cosine theorem we have

$$v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos(\zeta)} \tag{52}$$

The relative velocity distribution can be calculated as follows.

$$F(v) = \int\int \mathrm{d}v_1 \mathrm{d}v_2 F_0(v_1) F_0(v_2) \int_0^\pi \frac{\mathrm{d}\zeta\ \sin(\zeta)}{2} \delta\left(v - \sqrt{v_1^2 + v_2^2 + 2v_1 v_2 \cos\zeta}\right) \tag{53}$$

$$= \frac{1}{2} \int\int \mathrm{d}v_1 \mathrm{d}v_2 F_0(v_1) F_0(v_2) \frac{v}{v_1 v_2} \tag{54}$$

$$= \int_{-v/\sqrt{2}}^{v/\sqrt{2}} \mathrm{d}\xi \int_{v/\sqrt{2}}^{\infty} \mathrm{d}\eta\, F_0\left(\frac{\eta - \xi}{\sqrt{2}}\right) F_0\left(\frac{\eta + \xi}{\sqrt{2}}\right) \frac{v}{\eta^2 - \xi^2} \tag{55}$$

where in the last equation we have switched to integration variables (ξ, η), where $\xi = (v_1 - v_2)/\sqrt{2}$ and $\eta = (v_1 + v_2)/\sqrt{2}$. The integration domain is the set of (v_1, v_2) pairs for which $v \in [|v_2 - v_1|, (v_2 + v_1)]$.

The integration can be evaluated symbolically for the Maxwell-Boltzmann distribution, i.e. F_0 defined by (39). The result is also a Maxwell-Boltzmann distribution:

$$F(v) = \sqrt{\frac{27}{4\pi}} \frac{v^2}{v_0^3} e^{-\frac{3v^2}{4v_0^2}} \tag{56}$$

This is just the single velocity distribution with

$$v_{RMS} = \sqrt{2}v_0 \tag{57}$$

The encounter probability for the dynamical star system can be obtained by adding up the contributions of pairs of stars. Assuming that the spatial distribution of stars is homogeneous the encounter probability is identical to the model II, with the velocity distribution changed to the relative velocity distribution. Therefore equation (43) describing the GBS rate is valid, with $F(v)$ substituted from (55). For the Maxwell-Boltzmann distribution, it is sufficient to simply replace v_0 with $v_{RMS} = \sqrt{2}v_0$.

From equations (17) and (50) the results of our models can be summarized as

$$\lambda(f)\mathrm{d}f = \begin{cases} \frac{1}{4}\frac{N^2}{R^3}\frac{{v^*_{min}}^4}{v_0}\left(1+\frac{8}{3}\frac{v_0^2}{{v^*_{min}}^2}\right)\frac{\mathrm{d}f}{f^3} & \text{model I} \\ \sqrt{\frac{3}{8\pi}}\frac{N^2}{R^3}\frac{{v^*_{min}}^4}{v_0}\left(1+\frac{16}{9}\frac{v_0^2}{{v^*_{min}}^2}\right)\frac{\mathrm{d}f}{f^3} & \text{model II} \\ \sqrt{\frac{3}{16\pi}}\frac{N^2}{R^3}\frac{{v^*_{min}}^4}{v_0}\left(1+\frac{32}{9}\frac{v_0^2}{{v^*_{min}}^2}\right)\frac{\mathrm{d}f}{f^3} & \text{model III} \end{cases} \tag{58}$$

where $v^*_{min} = \sqrt[3]{2GMf}$ and v_0 is the virialized speed (1) in the star system.

The leading order terms are proportional to $1/v_0$. The result is slightly counterintuitive if one identifies the star system with an ideal gas, since for ideal gases, the rate of collisions is directly proportional to v_0. In this perspective it seems reasonable to expect the encounter rate a growing function of v_0 for fixed frequency bins. The confusion arises from the fact that the models I, II, and III discussed in this study are using the opposite limit. For star systems the typical velocities are so small that the gravitational interaction dominates the motion of the stars.[7] Increasing the velocities decreases the gravitational focusing, thereby decreasing the encounter likelihood.

Notice the interesting feature in equation (58) that models I and III are practically identical in the leading order terms. The change is a factor of $\sqrt{3/\pi}$, a mere 2% decrease for model III. The inferences obtained for model I are therefore valid for the most sophisticated case as well. The predictions for the overall GBS detection rate for various interferometric detectors are given by equation (36).

[7] The ideal gas model is sufficient only for extremely small GBS frequencies. In this regime the stars' trajectories are only slightly deflected, inferring that gravity, in terms of encounter likelihood, is negligible. This approximation leads to $\lambda(f) \propto v_0^3$, a growing function of v_0 indeed.

5 Conclusions

We have obtained an estimate for the expected rate of the gravitational bremsstrahlung signals (GBS). We gave an estimate using three models. Model I used a uniform velocity model with nonrelativistic stars moving through and scattering off a fixed lattice formed by the other stars in the cluster. In model II we improved this model to account for the velocity distribution statistics. Finally, we obtained model III by relaxing the fixed lattice condition, deriving the relative velocity distribution. The results showed that model I indicates GBS event rates in an excellent approximation.

Our result is in accord with different models such as [5]. They calculate collision times for solar mass stars in spherical star systems. They use a polytropic density distribution and account for the velocity distribution and the increase of the cross section due to gravity. Their results give the same order of magnitudes' timescale for collisions as our estimates for GBS frequency limited encounters.

Our results describe the event rates of detectable GBS for a given frequency interval for one star cluster. The magnitudes of the signals are available [1] to compare with the current and anticipated sensitivities of present and near-future gravitational-wave detectors [7]. We have shown that for sources closer than $\propto 1\,\mathrm{Mpc}$ the GBS events can be detectable with the current LIGO detectors and $100\,\mathrm{Mpc}$ for advanced LIGO. Adding up all the detectable sources, we have obtained the expected overall GBS rate. The results show that even the most sensitive terrestrial gravitational wave detectors such as Advanced LIGO will encounter less than one GBS event in every 3 thousand years. Future space detectors as LISA can have event rates of 1 one event per 4 months. These values are only theoretical upper limits, since the derivation used a pure star population of compact stars (e.g. NS and BH). If the ratio of compact stars was q, than the result is decreased by a factor of q^2.

In conclusion, gravitational bremsstrahlung signals are not likely to be found in regular spherical star systems.[8] High frequency GBS might only be detected by LIGO if the sources of the encounters are neutron stars or black holes with radii less than half their minimal separation, otherwise a collision occurs. If GBS events were to be regularly detected, the above

[8] The estimate assumed spherical symmetry and homogeneity, which was applicable for spherical clusters but breaks down for galaxies. Due to their smaller velocity distributions, event rates are expected to be even lower for galaxies.

estimate would set an upper bound on the typical mass or a lower bound on the number density of the system. As of now we have no evidence for the existence of low mass primordial black holes or dense black hole systems. The detection of GBS with LIGO would prove their existence.

Further studies are needed to investigate the changes in event rates in the post-Newtonian and general relativistic regimes. We have checked the fixed background model for general relativity for two stars having a large mass ratio. By assuming the radius of the individual stars much smaller than the distances in between, we obtained the trajectory for the lighter BH in the background Schwarzschild metric of the massive BH. The results were in accord with the Newtonian solution. Therefore, the post-Newtonian calculation is not expected to change the GBS event rates significantly.

6 Acknowledgements

We are grateful for the Caltech SURF program and the LIGO collaboration, who made our work possible. The authors gratefully acknowledge the support of the United States National Science Foundation for the LIGO Laboratory. This material is based upon work supported by National Science Foundation under Grant [or Cooperative Agreement] No. (NSF grant or cooperative agreement number). This document has been assigned LIGO Laboratory document number LIGO-T030136-00-D.

This paper is one of the three papers submitted for our final report of the LIGOs 2003 SURF program.[9] We gratefully thank our advisor, Szabolcs Marka, for his help and supervision during the work.

7 Appendix

Notice that in equation (45) the top bound of the integration is unconstrained. An exact treatment would introduce the quantity v^*_{max}, defined by two reasons. The first is simply related to the fact that we have used the classical kinematics, which is only valid in the nonrelativistic regime. For relativistic velocities there is a significant Doppler-shift in the frequencies, depending on which direction the flyby occurs. Also the Maxwell-Boltzman distribution was used in the nonrelativistic limit. Therefore $v^*_{max} < c$ is a

[9] The other two papers are [8] and [9].

hard bound, but more realistically $v^*_{max} = 0.25c$ would be a reasonable choice. This implies

$$w_{max} = 2.5 \times (f/100\,\mathrm{Hz})^{-1/3} \tag{59}$$

Since the integrand of equation (45) is negligible everywhere except in the close vicinity of $w = 1$, the constraint (59) does not change the result for $f < 1500\,\mathrm{Hz}$.

The other reason for introducing w^*_{max} in (45) is the limitation of detection caused by signal strength. As it was shown in the previous section, the signal strength puts a low bound on b^*, and thus a high bound on v^* for a given frequency. From equation (33) we get[10]

$$w_{max}(r) = 91.4 \left(\frac{r}{10\,\mathrm{kpc}}\right)^{-1} \left(\frac{f}{100\mathrm{Hz}}\right)^{2/3} \times \begin{cases} 1 & \text{for current LIGO} \\ 100 & \text{for advanced LIGO} \end{cases} \tag{60}$$

Therefore $r < 0.5\,\mathrm{Mpc}$ with LIGO and $r < 50\,\mathrm{Mpc}$ with Advanced LIGO yields $w_{max} \gg 1$. Therefore equation (45) will approximate the true GBS rate, within this distance. For larger r distances, the integration domain ceases with r, decreasing the observable GBS rate.

Substituting in equation (50),

$$\lambda(f)\mathrm{d}f = A\frac{\mathrm{d}f}{f}\int_0^{W(r)=w_{max}(r)-1} \mathrm{d}u \left(\frac{1}{2} + 4u\right) e^{-\frac{3u}{2s}} \tag{61}$$

$$= A\frac{\mathrm{d}f}{f}\frac{s}{3}\left(1 + \frac{16}{3}s\right)\left[1 - \left(1 + \frac{3W(r)}{2s}\right) e^{-\frac{3W(r)}{2s}}\right]\Theta(W(r)) \tag{62}$$

where $\Theta(.)$ is the Heaviside step function and s is defined in equation (47).

Equation (62) describes how the detection rate of GBS depends on the distance r of the source star system from Earth. The r dependance is implicit through the W variable. For $r \ll 1\,\mathrm{Mpc}$, $W \approx \infty$, leading to eq. (62)≈(51). For distances larger than the critical distance (35), the detectable GBS rate is identically zero. The resultant rate from all sources can be obtained by multiplying with the number density of globular clusters and integrating over space. Since the r-cutoff of (62) is very rapid, the r-dependance can be approximated by $\Theta(W)$. This is exactly what has been carried out in the first

[10]Current and advanced LIGO sensitivities are taken $\delta h/h = 10^{-21}$ and 10^{-23} respectively.

model. Therefore the resultant GBS rate of model II is well approximated by the result (36) of model I.

For model III, we show that model II should be modified with the substitution of $v_{RMS} = \sqrt{2}v_0$ for v_0. The GBS rates for model III are identical to model II within 2%. Equation (62) defines the smooth r-dependence for model III. Practically for globular clusters and galaxies, this can be well approximated with the $\Theta(W(r))$

References

[1] Kovács, S.J., and Thorne, K.S. 1978, *Ap.J.* **224**, 62-85.

[2] Turner, M. 1977, *Ap.J.*, **216**, 610.

[3] Wagoner, R.V., and Will, C. M. 1976, *Ap.J.*, **210**, 764.

[4] Larson, S.L., Hiscock W.A., Hellings R.W., 2000, *Phys. Rev. D*, **62**, 062001.

[5] Saslaw, Gravitational physics of stellar and galactical systems, 1985, *Cambridge Univ. Press* **52**.

[6] A. Lazzarini and W. Majid, "Galactic distributions in nearby sky", LIGO-T980090-00-E, http://www.ligo.caltech.edu/docs/T/T980090-00.pdf

[7] Shawhan, P. S., "The search for gravitational waves with LIGO: Status and Plans", LIGO-P000021-00-E, http://www.ligo.caltech.edu/docs/P/P000021-00.pdf

[8] B. Kocsis and M. Gaspar, "The data analysis for short-term gravitational-wave burst signals with a modified maximum likelihood detection method," LIGO technical note LIGO-T030213-00-D, 2003.

[9] M. Gaspar and B. Kocsis, "The development of a digital camera with a high dynamic range" LIGO technical note LIGO-T030232-00-D, 2003.

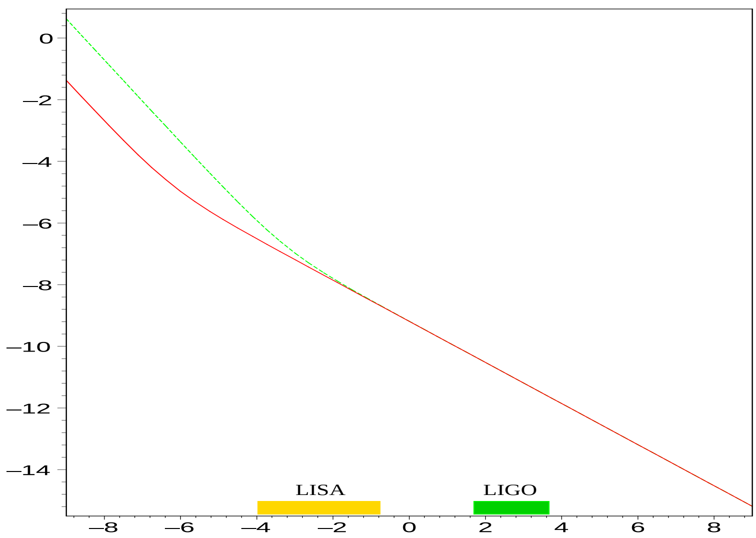

Figure 1: The expected rate of GBS produced in one star cluster is plotted per logarithmic frequency bin, on log-log scale. The solid curve depicts the result for globular clusters, and the dashed curve is for dense galaxies. The rate is expressed in yr^{-1}. Note that these curves show only an upper limit, real GBS rates are further decreased by q^2, the square of the portion of the number of dense cosmic objects (e.g. NS & BH) within the full star population of the cluster. Only a fraction of these events can be detected, depending on the distance of the source. The typical sensitive bands of terrestrial and space gravitational wave detectors are indicated. The maximum distance for a signal to noise ratio of 1 for the current LIGO detectors, is 0.94 Mpc, and 94 Mpc for Advanced LIGO.