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The LIGO Interferometer Signal Response

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Abstract

The response of LIGO's photodetector to a particular signal depends on several factors of the incoming gravitational wave (GW). GW amplitude, polarization, and incidence angle all determine the contraction or expansion of the interferometer arms. This paper is based directly on D. Sigg's Appendix [1] and provides a derivation of how the laser's phase is affected by the presence of a GW.

1 Introduction

In principle, a GW could arrive at earth from any direction and with any polarization. A natural question to ask is: how sensitive is LIGO to a particular type of signal? We will see that in some cases, a perfectly legitimate GW signal may, for example, distort the interferometer arms equally: there would be no chance of observing it. Fortunately, this is a worst case scenario; most directions in space and types of polarization allow for much better detection by LIGO.

LIGO is somewhat of a black box: a GW enters and "somehow" a voltage signal is output by the photodetector. This article aims to partly bridge this gap by mathematically describing how a particular GW affects the phase changes of the laser light in each of the interferometer arms. Ultimately, the information presented here will be implemented in an existing end-to-end (e2e) simulation of LIGO. The e2e software then handles the calculation of the interfering electric fields resulting from phase differences.

2 The Coordinate System

A right-handed coordinate system is constructed by taking the x- and y-axes to coincide with the interferometer arms. These constraints force the origin to be at the beam splitter and the z-axis oriented "vertically". We will make use of spherical coordinates, so note that a point (x, y, z) may also be identified by the usual $(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, where θ is the azimuthal angle measured from the $+z$-axis $(0 \leq \theta < \pi)$, and ϕ is the polar angle measured from the $+x$-axis $(0 \leq \theta < 2\pi)$.

In order to mathematically express a GW in the above coordinate frame, we must perform certain rotation operations. Suppose a GW approaches LIGO with spherical coordinate angles θ and ϕ. To transform a quantity in the wave's frame to LIGO's frame, we first rotate about the y-axis by θ, *then* rotate about the z-axis by ϕ[1]. Or, in matrix notation, if $R(\theta, \phi)$ is the net rotation, it can be decomposed as:

[1] The rotations are carried out in this order so that they are about principal axes.

$$R(\theta, \phi) = R(\phi)R(\theta) \tag{1}$$

where $R(\theta)$ and $R(\phi)$ are the matrices that perform the aforementioned rotations. They are given below:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{2}$$

3 The Induced Laser Phase Change

Let ω be the angular frequency of oscillation of the laser. We are interested in the total phase of the laser during the round-trip from the beam splitter, to the end test mass, and back to the beam splitter. Starting from time t_0, this phase can be found simply by integrating ω over time:

$$\Phi_{RT}(t_0) = \int_{t_0}^{t_0+t(2L)} dt\, \omega \tag{3}$$

where 'RT' denotes 'round-trip,' and $t(2L)$ means the time at which the light has traveled the distance $2L$. In this case we integrate over *time*. LIGO is sensitive to GWs' distortion of *length*, so we wish to change variables from t to, say, x. Some equations from general relativity will be helpful here, because they provide a means of converting from time to distance.

In the vicinity of earth, where there are no particularly strong gravitational fields, the metric $g_{\mu\nu}$ is simply the Minkowski metric $\eta_{\mu\nu}$[2]. If a GW is present, it is represented in the metric by a small, time-dependent adjustment $h_{\mu\nu}$ to the Minkowski metric. Now we can write out an expression for the proper time:

$$d\tau^2 = dx^\mu g_{\mu\nu} dx^\nu = dx^\mu(\eta_{\mu\nu} + h_{\mu\nu})dx^\nu \tag{4}$$

But what is $h_{\mu\nu}$? Consider a GW with angular frequency Ω traveling the $+z$ direction. As with any traveling wave, it can be written as $\cos(\Omega t - kz)$ times some amplitude (k is the magnitude of the wave vector $\vec{k}$). In this case, the amplitude is tensorial (since a tensor is needed to describe the quadrupolar GW's polarization):

$$h_{\mu\nu} = \cos(\Omega t - kz) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Note that all z or t components of this tensor amplitude are 0, since 1) the wave propagates along z and GWs can only be polarized in transverse directions, and 2) polarization of a wave is only for space (i.e. not time). In general, however there will be polarization in all three spatial directions. For a GW of arbitrary incidence angle, we must transform the polarization tensor $\hat{H}_{ij}$ (composed of the nine spatial components of $h_{\mu\nu}$) to LIGO's coordinate system using the matrix $R(\theta, \phi)$ from eq.(1). Using the transformation rule for second-rank tensors, we have:

[2] We use the convention where the coefficient of the time component of $\eta_{\mu\nu}$ is negative.

$$H_{ij} = R(\theta,\phi)\hat{H}_{ij}R(\theta,\phi)^{-1} \tag{6}$$

so that H_{ij} will have the general form:

$$H_{ij} = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix} \tag{7}$$

Similarly, it is necessary to transform the wave vector, $\vec{k}$. Applying the rule to convert spherical coordinates, $\vec{k}$ in the LIGO coordinate system is:

$$(k_x, k_y, k_z) = (k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta). \tag{8}$$

Now, we may begin to make the change of variables for integration. At first we will consider the x-axis interferometer arm, so y and z are fixed at 0 (thus, we require a wave traveling in the $+x$ direction with wavenumber k_x). Recall that for a signal traveling at c, we have the identity $d\tau^2 = 0$. Then, from eq. (4):

$$\begin{aligned} 0 &= d\tau^2 && (9)\\ &= dx^\mu \eta_{\mu\nu} dx^\nu + dx^\mu h_{\mu\nu} dx^\nu && (10)\\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2 + \sum_{i,j=0}^{4} h_{ij}dx^i dx^j && (11)\\ &= -c^2dt^2 + dx^2 + h_{xx}\cos(\Omega t - k_x x)dx^2 && (12) \end{aligned}$$

where numerous cancellations take place because 1) $\eta_{\mu\nu}$ is diagonal, 2) the time components of $h_{\mu\nu}$ are 0, and 3) $dy = dz = 0$. Note that the only non-vanishing component of H_{ij} is h_{xx}[3]. Finally, the conversion from dt to dx is clear:

$$\begin{aligned} 0 &= -c^2dt^2 + dx^2 + h_{xx}\cos(\Omega t - k_x x)dx^2 && (13)\\ c^2dt^2 &= dx^2(1 + h_{xx}\cos(\Omega t - k_x x)) && (14)\\ dt &= \frac{dx}{c}\sqrt{1 + h_{xx}\cos(\Omega t - k_x x)} && (15) \end{aligned}$$

However, the oscillatory part still explictly depends on t. It must be written in terms of x to perform the integration. Over the interval $0 \le x \le L$, t increases linearly with x as:

$$t = t_0 + \frac{x}{c} \tag{16}$$

For the return trip, x ranges from L to 0. The return trip begins at time $t_0 + L/c$, and the time increases linearly as the light gets farther from position $x = L$ (i.e. as $L - x$ increases). Symbolically:

$$\begin{aligned} t &= t_0 + \frac{L}{c} + \frac{1}{c}(L - x) && (17)\\ &= t_0 + \frac{2L}{c} - \frac{x}{c} && (18) \end{aligned}$$

[3]A tool such as Mathematica is quite useful for finding h_{xx} and h_{yy}. It turns out that $h_{xx} = -h_\times\cos\theta\sin 2\phi + h_+(\cos^2\theta\cos^2\phi - \sin^2\phi)$ and $h_{yy} = h_\times\cos\theta\sin 2\phi + h_+(\cos^2\theta\sin^2\phi - \cos^2\phi)$.

We now write out the integral in eq. (3) entirely in terms of x, and use the fact that $\Omega/c = k$:

$$\begin{aligned}
\Phi^x_{RT}(t_0) &= \int_{t_0}^{t_0+t(2L)} dt\,\omega && (19)\\
&= \int_{0\to L} \frac{dx}{c}\sqrt{1+h_{xx}\cos(\Omega t_0+(k-k_x)x)}\;\omega && (20)\\
&\quad+\int_{L\to 0} \frac{dx}{c}\sqrt{1+h_{xx}\cos(\Omega t_0+2Lk-(k+k_x)x)}\;\omega
\end{aligned}$$

4 Simplifications

Up to this point, we have an exact analytical expression for $\Phi^x_{RT}(t_0)$. Now we will use the following first-order Taylor expansion to simplify the integrands: $\sqrt(1+\epsilon) \approx 1+\epsilon/2$.

$$\begin{aligned}
\Phi^x_{RT}(t_0) &\approx \frac{\omega}{c}\int_0^L dx\Bigg[1+\frac{h_{xx}\cos\big(\Omega t_0+(k-k_x)x\big)}{2} && (21)\\
&\quad +1+\frac{h_{xx}\cos\big(\Omega t_0+2Lk-(k+k_x)x\big)}{2}\Bigg]\\
&= \frac{2L\omega}{c}+\frac{h_{xx}\omega}{2c}\Bigg[\frac{1}{k-k_x}\sin\big(\Omega t_0+(k-k_x)x\big) && (22)\\
&\quad -\frac{1}{k+k_x}\sin\big(\Omega t_0+2Lk-(k+k_x)x\big)\Bigg]_{x=0}^{L}\\
&= \frac{2L\omega}{c}+\frac{h_{xx}\omega}{2c}\Bigg[\frac{k+k_x}{k^2-k_x^2}\Big(\sin\big(\Omega t_0+(k-k_x)L\big)-\sin(\Omega t_0)\Big) && (23)\\
&\quad -\frac{k-k_x}{k^2-k_x^2}\Big(\sin\big(\Omega t_0+2Lk-(k+k_x)L\big)-\sin(\Omega t_0+2Lk)\Big)\Bigg]
\end{aligned}$$

Observe that the leading term represents the total phase change of the laser in a round trip in one arm in the absence of a GW. Since we are concerned with *changes* in phase, this term will be neglected. Now, we apply the trigonometric identity $\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$ to two places in the previous equation:

$$\begin{aligned}
\Delta\Phi^x_{RT}(t_0) &\approx \frac{h_{xx}\omega}{2c(k^2-k_x^2)}\Bigg[(k+k_x)2\cos\Big(\frac{2\Omega t_0+(k-k_x)L}{2}\Big)\sin\Big(\frac{(k-k_x)L}{2}\Big) && (24)\\
&\quad -(k-k_x)2\cos\Big(\frac{2\Omega t_0+4Lk-(k+k_x)L}{2}\Big)\sin\Big(\frac{-(k+k_x)L}{2}\Big)\Bigg]
\end{aligned}$$

Consider the arguments of the two sine functions. The one largest in magnitude is the latter, $\frac{(k+k_x)L}{2}$. In the extreme scenario in which $k_x \approx k$, the argument is about kL. However, in the case of a 4 km interferometer with 1 kHz GWs:

$$kL = \frac{2\pi}{\lambda}L = \frac{2\pi f}{c}L \approx \frac{6.28 \times 1000 \times 4000}{3 \times 10^8} \approx 0.08 \tag{25}$$

Thus we may use the small angle approximation $\sin x \approx x$ for both sine terms (note that the approximation's accuracy increases for lower frequency waves):

$$\begin{aligned} \Delta\Phi^x_{RT}(t_0) \quad &\approx \quad \frac{h_{xx}\omega}{2c(k^2-k_x^2)}\left[\frac{L(k^2-k_x^2)}{2}(2)\Big(\cos\big(\Omega t_0 + \frac{1}{2}(k-k_x)L\big)\right. \\ &\left. + \cos\big(\Omega t_0 + 2Lk - \frac{1}{2}(k-k_x)L\big)\Big)\right] \end{aligned} \tag{26}$$

Now, we apply the identity $\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$ to the remaining trigonometric terms:

$$\Delta\Phi^x_{RT}(t_0) \approx \frac{h_{xx}L\omega}{2c}\left[2\cos\left(\frac{2\Omega t_0 + 2Lk - k_xL}{2}\right)\cos\left(\frac{-Lk}{2}\right)\right] \tag{27}$$

Recalling that $k_x/k = \sin\theta\cos\phi$, we may write the phase change of the laser due the a GW in the $x-$axis arm as:

$$\Delta\Phi^x_{RT}(t_0) \approx \frac{h_{xx}L\omega}{c}\Big[\cos\Big(\Omega t_0 + Lk(1 - \frac{1}{2}\sin\theta\cos\phi)\Big)\cos\Big(\frac{Lk}{2}\Big)\Big] \tag{28}$$

Retracing these steps for the quite similar case of the $y-$axis, we find:

$$\Delta\Phi^y_{RT}(t_0) \approx \frac{h_{yy}L\omega}{c}\Big[\cos\Big(\Omega t_0 + Lk(1 - \frac{1}{2}\sin\theta\sin\phi)\Big)\cos\Big(\frac{Lk}{2}\Big)\Big] \tag{29}$$

5 Reference

[1] D. Sigg, "Gravitational Waves" Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, 10/23/1998.