LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
-LIGO-
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Validation of the Source-Detector Simulation Formulation

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Abstract

Details of the implementation of a simulation of a gravitational wave's effect on the LIGO interferometer are given. To help validate program's results, a variety of test cases are presented. For a range of values of the free parameters, the simulation's outputs are compared to physical and mathematical predictions.

1 Introduction

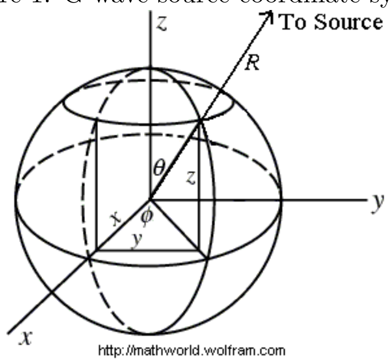
While an end-to-end simulation software package does exist for LIGO, it does not yet allow for gravitational waves to interact with the interferometer. The program described in this document, written in C++, is a first step toward adding this functionality to the simulation. It consists of two main classes: a source class that generates a gravitational wave signal emanating from a binary star system, based on parameters such as star masses, separation, and distance to Earth. The resulting time series is dynamically passed to the detector class, which contains information on a LIGO site's location and orientation. Based on these data, the detector class outputs the phase differences of the laser in each interferometer arm.

The primary coordinate system of interest here has its origin at the center of the Earth. The positive z-axis extends through the rotation axis (toward the north pole), and the x- and y-axes pass through the equator at longitudes of 0° and 90°, respectively. We generally deal with spherical coordinates with radius R, polar angle θ measured from the $+z$-axis, and azimuthal angle ϕ measured in the right-handed sense from the $+x$-axis. For the purposes of this simulation, the point (R, θ, ϕ) gives the location in space of a GW source. See figure 1 for an illustration.

We also require a means of specifying the location and orientation of a LIGO site on Earth. Let the beam detector be located on the Earth's surface at polar angle α and azimuthal angle β, using the same convention as above. Also let the arms of LIGO contain the x- and y-axes of a local right-handed coordinate system, such that the $+z$-axis points away from the earth. Then we define the orientation of the LIGO site using γ, the angle between the prime meridian and the great circle that contains the LIGO x-arm[1]. See figure 2.

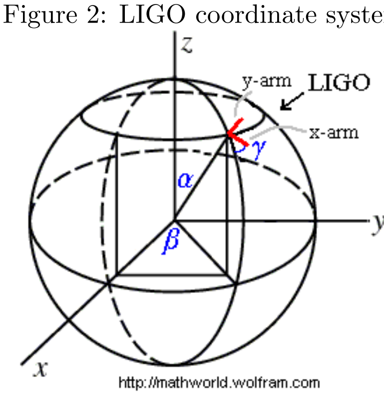
[1]In the actual simulation, γ corresponds to rotation of the source about the propagation axis, looking from the Earth. Mathematically, these two cases are equivalent. However, visualization is more straightforward if we think of rotating LIGO.

Figure 1: G-wave source coordinate system



In a spherical coordinate system, a gravitational wave approaches earth with polar angle θ from the z-axis and azimuthal angle ϕ from the x-axis.

Figure 2: LIGO coordinate system



In the same spherical coordinate system, LIGO's position is defined by α and β, as shown. LIGO's orientation is in terms of the rotation angle γ.

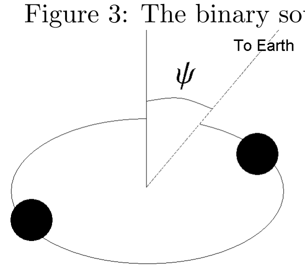
For the simplest case, we simulate GW's from a compact binary system of neutron stars (neglecting any inspiralling). Let the two stars have masses m_1 and m_2 with orbital angular frequency ω. Also let ψ be the angle between the 1) line connecting this system and the earth and 2) a normal of the system's orbital plane (see figure 3). Then [1] states that both polarizations of waves oscillate in phase with angular frequency 2ω and amplitudes:

$$h_+ = \sqrt{2}\frac{\mu}{R}\frac{G^{\frac{5}{3}}}{c^4}\Big(\omega(m_1+m_2)\Big)^{\frac{2}{3}}(1+\cos^2\psi) \quad (1)$$

$$h_\times = 2\sqrt{2}\frac{\mu}{R}\frac{G^{\frac{5}{3}}}{c^4}\Big(\omega(m_1+m_2)\Big)^{\frac{2}{3}}\cos\psi \quad (2)$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the system's reduced mass.

Figure 3: The binary source



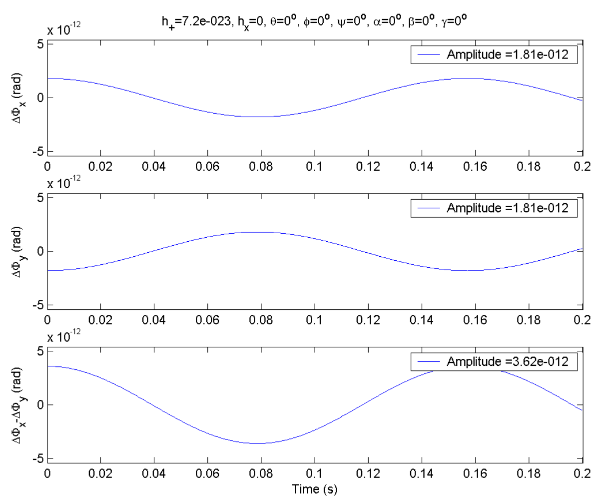
Since the gravitational radiation distribution of a binary system depends on observing angle, the parameter ψ determines what the signal looks like from Earth.

2 Basic Trial Runs

For each of the following cases, a description of the physics is presented, along with an interpretation of the simulation's results. The neutron stars, located 10 Mpc from earth, each have 1.4 times the mass of the sun and are separated by 1000 km. This results in an orbital frequency of 3.2 Hz and a gravitational wave frequency twice that.

Case 0: Simplest Trial As the simplest case, we run the simulation with the source directly above the north pole (with $\psi = 0$), with LIGO oriented with $\gamma = 0$ at the north pole. To allow for clearer analysis, we set the cross polarization component to zero for now. The plus component elongates one arm and shortens the other. Thus, the phase change of the light in each arm is equal in magnitude but opposite in direction to the other. Indeed, the simulation data, when analyzed in MATLAB agreed with this hypothesis and demonstrated the correct frequency (see figure 4).

Figure 4: Results for Case 0

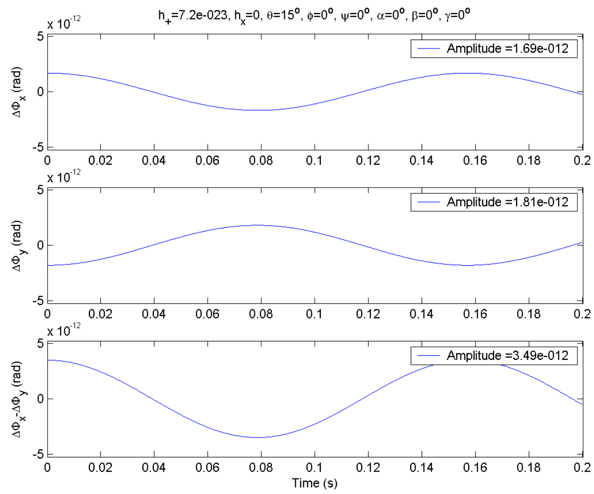


The gravitational wave causes the lengths of the interferometer arms to oscillate 180° out of phase, at a rate twice the orbital frequency of the star system.

Case 1: Vary θ Now all parameters will remain the same, except θ, the polar angle of the source will be moved from 0° to 15°. With $\phi = 0^\circ$, [2] gives $h_{xx} = h_+ \cos^2 \theta$, and $h_{yy} = -h_+$. This makes physical sense, since the x-arm no longer coincides perfectly with the plus polarization component (and thus is affected less). Also, the plus polarization still lines up with the y-arm and so $\Delta\Phi^y$ is the same as in the previous case. In figure 5 we see that the simulation was consistent with our expectations. Mathematically, the amplitude of the laser phase shift in the x-arm was diminshed by a factor of 0.933, which is $\cos^2 15^\circ$ besides rounding errors. This is consisent with the anticipated value found in [2] for the x-arm strain when $h_\times = 0$:

$$h_{xx} = h_+(\cos^2 \theta \cos^2 \phi - \sin^2 \phi) = h_+ \cos^2 \theta \tag{3}$$

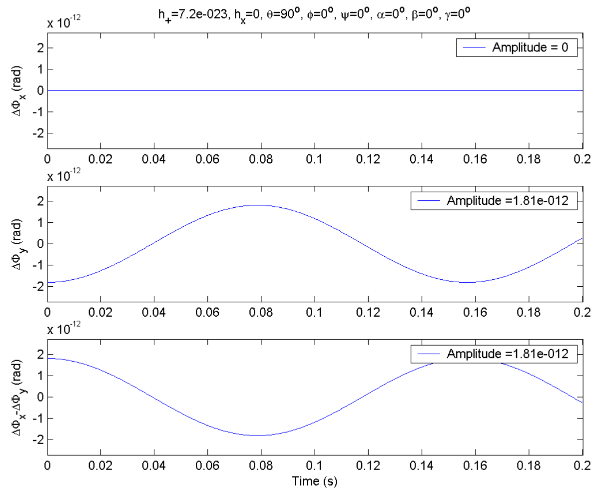
Figure 5: Results for Case 1



Since the strain is no longer parallel to the x-arm, the magnitude of the laser phase shift in that arm is diminished by a factor of the square of the cosine of the polar angle. There is no change to the y-arm.

Case 2: Vary θ more For this trial, the last case is extended by increasing θ to 90°; that is, the gravitational wave is incident on LIGO in a direction parallel to the equator and the great circle containing the prime meridian. The propagation direction is parallel to the x-axis, so there is no phase change of the light in that arm. However, the y-arm is still affected by the maximum amount. See figure 6.

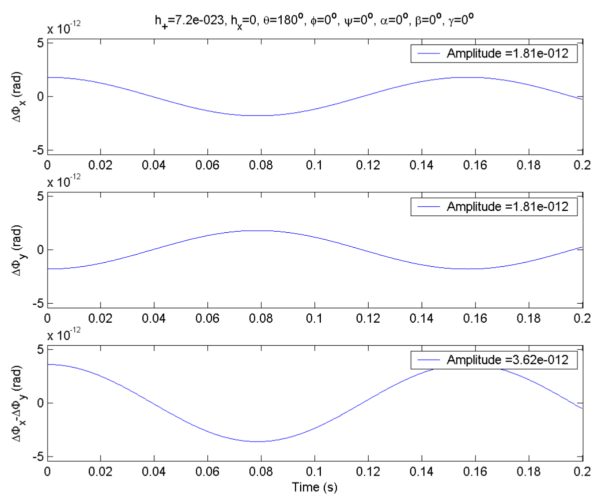
Figure 6: Results for Case 2



The x-arm does not register a change in phase, since the gravitational wave propagates parallel to it.

Case 3: Vary θ to the extreme As a final test of θ, it will be set to 180°, the south pole. Since the new axes are parallel to the original ones, there is no difference between this scenario and case 0's. See figure 7.

Figure 7: Results for Case 3



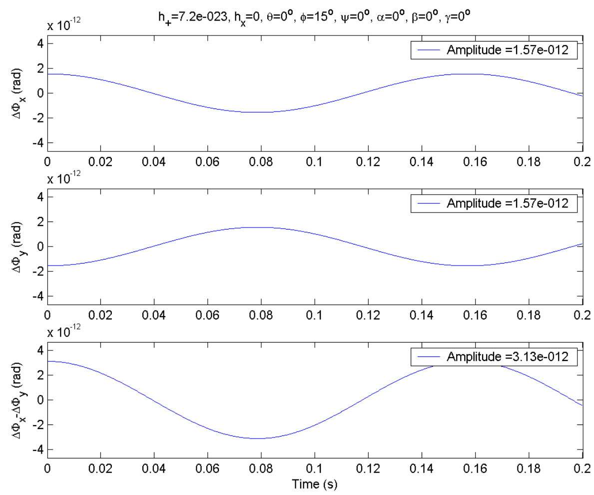
The gravitational wave now approaches from the opposite direction as in the original case, but the effect is identical; the interferometer arm lengths oscillate perfectly out of phase.

Case 4: Vary ϕ The polar angle, θ, is returned to 0°, but ϕ is rotated to 15°. Note that at the north pole, a change in ϕ is equivalent to a rotation of the source about the z-axis. Since the signal was maximized at $\phi = 0$, this small rotation diminishes the signal strength in both arms. In figure 8 we see a reduction in phase change by a factor of about 0.867. This is in agreement with the anticipated strain factor of $\frac{\sqrt{3}}{2}$ for plus-polarized waves with $\theta = 0$, given by [2]:

$$h_{xx} = h_+(\cos^2\theta\cos^2\phi - \sin^2\phi) = h_+\cos 2\phi = \frac{\sqrt{3}}{2}h_+ \quad (4)$$

$$h_{yy} = h_+(\cos^2\theta\sin^2\phi - \cos^2\phi) = -h_+\cos 2\phi = -\frac{\sqrt{3}}{2}h_+ \quad (5)$$

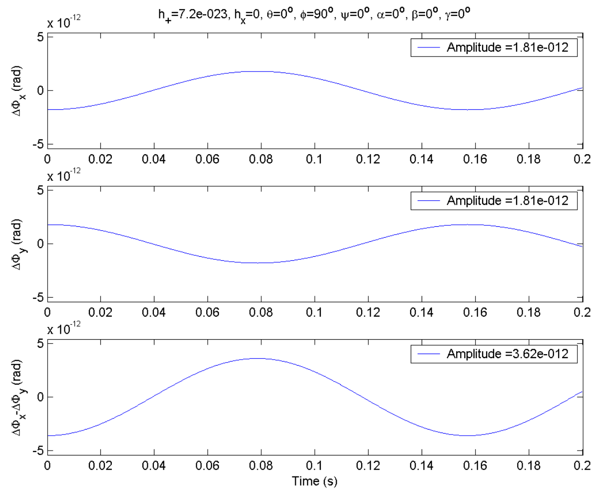
Figure 8: Results for Case 4



With $\theta = 0^\circ$, a change in ϕ is simply a rotation of the source. In this case, the rotation tends to reduce the signal strength by the predicted factor of $\cos 2\phi$.

Case 5: Vary ϕ more The longitude, ϕ, is set at 90°. Now, the stretching component of the gravitational wave aligns with the x-arm, and the contracting component aligns with the y-arm. Thus, the effect of the gravitational wave on the interferometer is precisely reversed: the roles of increasing and decreasing laser phase are swapped. Surely enough, we see this 180° phase flip in the strain vs. time plots of figure 9.

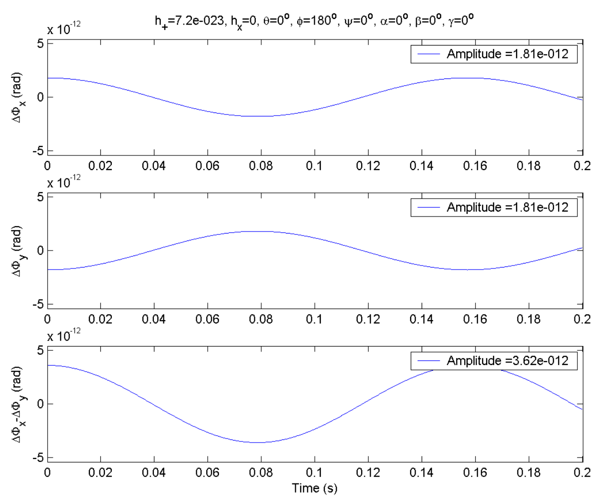
Figure 9: Results for Case 5



Rotating the source by 90° effectively swaps the two interferometer arms so that the laser phase shift in each is moved 180°.

Case 6: Vary ϕ one last time As the last of the basic test cases, ϕ is offset to 180°. Since gravitational waves are symmetric under 180° rotations about the propagation axis, we expect the same results as the initial trial, case 0. For confirmation, see figure 10.

Figure 10: Results for Case 6



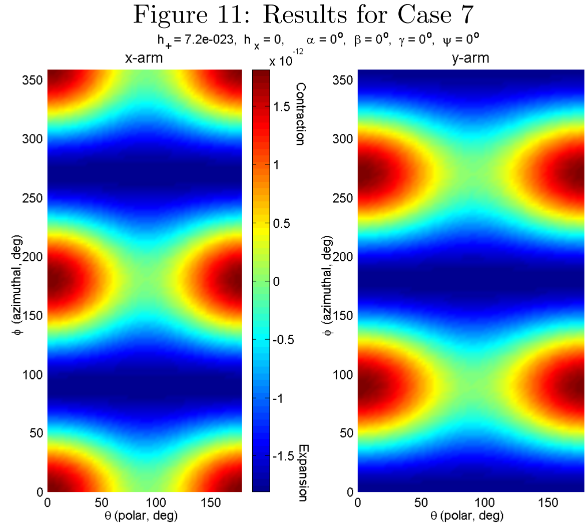
A 180° rotation of the source does not change the signal in either arm, because both arms are still aligned along their original axes.

3 Advanced Trial Runs

The physical properties of the binary system remain fixed at their previous values. However, we now begin to analyze the data in an alternative way. On a rectangular two-dimensional plot, the axes will correspond to $0^\circ \leq \theta < 180^\circ$ and $0^\circ \leq \phi < 360^\circ$ so that the entire range of incidence angles will be considered. There is one plot each from the x- and y-arms (but none for the phase difference). The third dimension, represented by a color scale, corresponds to interferometer laser phase shift[2]. This new technique allows visualization of the effect on all source locations of altering a parameter.

Case 7: Simplest Trial To allow for adjusting to the new plot style, the original case (α=β=γ=ψ=0°) is repeated and shown in figure 11.

Figure 11: Results for Case 7

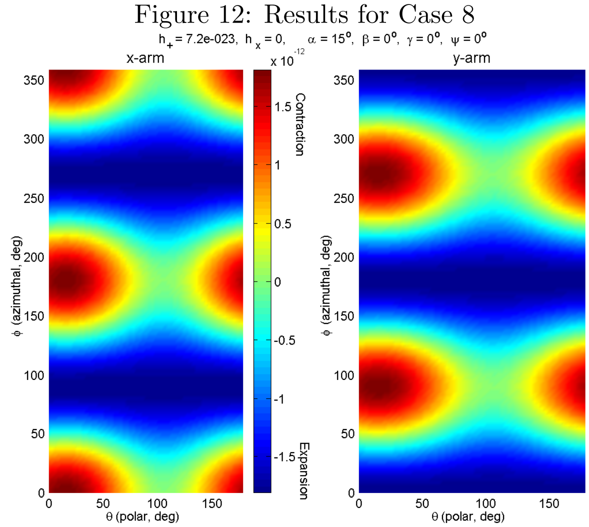


This plot actually contains all the information of the previous trials. For example, the $\theta \equiv 90^\circ$ line registers no signal in the x-arm at $\phi = 0^\circ$ (as in case 2).

Case 8: Shift α We now move α to 15°; physically, this is a shift of a LIGO site from the north pole down by 15° of latitude. To arrange for the source and detector to be in the same relative positions as in the initial case, θ would likewise need to be shifted by 15°. Thus, one would expect the plot for this case to differ from the previous case only by a shift in the positive θ direction. For confirmation, see figure 12.

[2]Of course, the absolute phases of these plots can be chosen arbitrarily. In this investigation they were selected to maximize the phase shift and to make $\theta = 0^\circ, \phi = 0^\circ$ correspond to contraction of the x-arm.

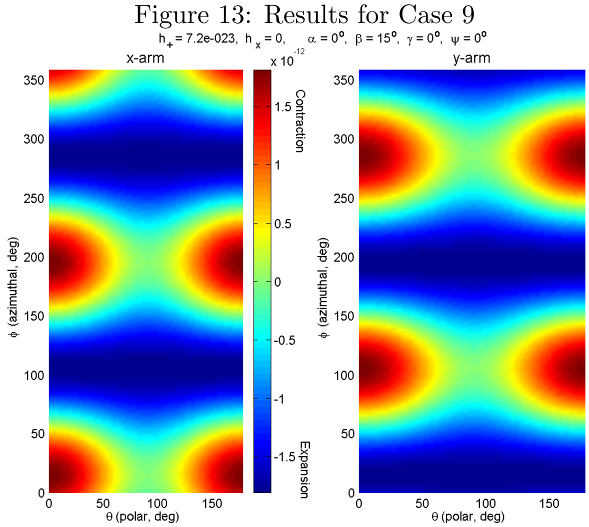
Figure 12: Results for Case 8



Shifting the latitidude of LIGO does not fundamentally change the geometry of the system; we compensate for it by simply shifting the source's angle θ by the same amount. In the above plot, we see this effect.

Case 9: Shift β Similarly, we may shift β (with α held at $0°$). By the same reasoning as above, this alteration shifts the entire plot in the positive ϕ direction (see figure 13).

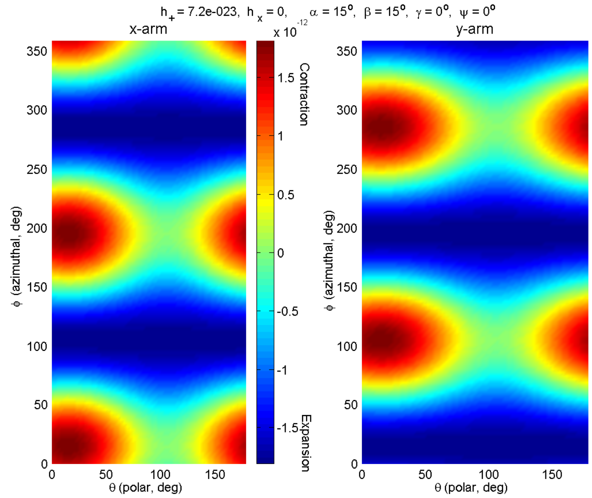
Figure 13: Results for Case 9



The effect of β on ϕ is analogous to the effect of α on θ: any change only shifts plot.

Case 10: Shift α **and** β To demonstrate the independence of α and β, we shift them simultaneously to $15°$ each. Figure 14 shows that the resulting plots are indeed shifted 15° each in both directions [3].

Figure 14: Results for Case 10

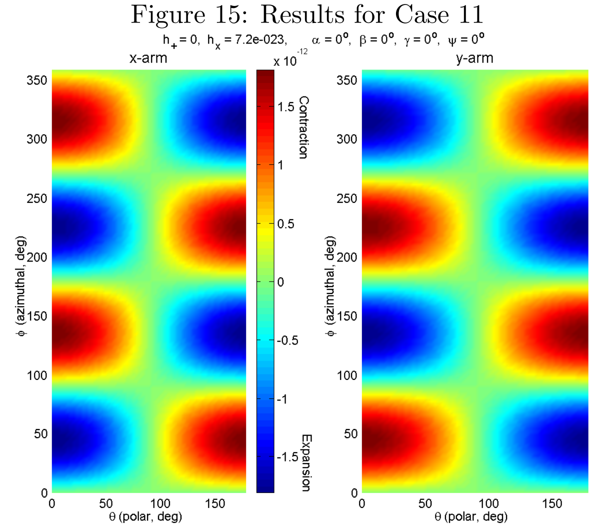


This plot is intended solely for showing that the effect on the plot of either α or β is does not depend on the value of the other.

Case 11: Change the polarization Now we revert to $\alpha = \beta = 0^\circ$ but consider only the *cross* component of the polarization. For $\theta = \phi = 0^\circ$, the spatial effects of the gravitational wave are perpendicular to both interferometer arms, so the phase changes are null. For $\theta = 0^\circ, \phi = 45^\circ$, we rotate the source by 45° relative to the earth. This orientation places the interferometer arms in line with the polarization axes; specifically, the x-arm undergoes stretching and the y-arm undergoes contracting. These effects can be seen in figure 15 below.

[3] This result is no surprise, because the effects α and β are implemented in the simulation by replacing θ with $\theta - \alpha$ and ϕ with $\phi - \beta$, and the rotation matrices for θ and ϕ commute. In other words, the rotations can be applied in either order to achieve the same result

Figure 15: Results for Case 11



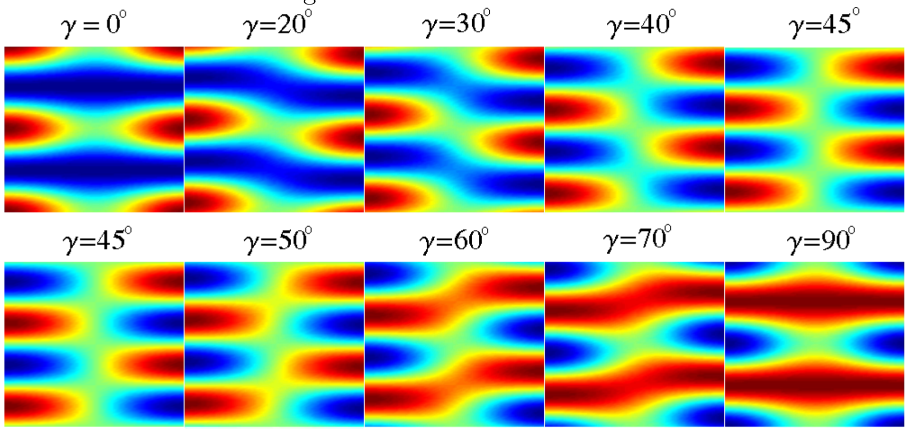
Here, a cross-polarized gravitational wave is incident on LIGO from its z-axis. Note how the response of the interferometer arms as a function of θ and ϕ differs from the case of plus-polarization.

Case 12: Vary γ The only physical difference between the two polarizations of gravitational waves is orientation along the propagation axis. Therefore, switching from plus- to cross-polarization is simply a special case of changing γ (specifically, $\gamma = 45°$). In this case we consider the effect of incrementing gamma by smaller steps. Figure 16 a sequence of plots (of the same style as above, but x-arm only) over a range of γ.

Several properties of these plots show that the simulation's treatment of γ is valid. First, consider gravitational waves arriving from along Earth's axis ($\theta = 0°, 190°$). Since LIGO is positioned at a pole ($\alpha = 0°$), a rotation by γ is equivalent to a rotation of LIGO by β. Indeed, in each case the plot is shifted in the positive ϕ direction by amount γ at $\theta = 0°$, and by amount $-\gamma$ at $\theta = 180°$.

Another validating factor can be seen by considering the strain in the x-arm at the (θ, ϕ) locations $(90°, 0°)$ and $(90°, 180°)$. Physically, these correpond to the two source positions where the x-arm is aligned with the propagation axis. In figure 16 we see that at these points, the laser phase shift is zero for all values of γ.

Figure 16: Results for Case 12



Each of the ten squares above represents a colored plot of laser phase difference in the x-arm vs. θ and ϕ, the x- and y-axes, respectively. From left to right, top to bottom, the plots show the effects of ranging γ from 0° to 90°.

References

[1] Ott, Christian D. "Review on Gravitational Wave Emission from Binary Star Systems." http://physics.arizona.edu/~cott/binary/binary_gw.pdf

[2] Sigg, Daniel. "Gravitational Waves." Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, 10/23/1998.