Estimate of Angular Instability for Mexican-Hat and Gaussian Modes of a Fabry-Perot Interferometer

LIGO Document Number LIGO-T030272-00-Z

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(Dated: 17 November 2003   File Tiltins3.tex)

It is shown that the tilt instability, induced by radiation pressure in an advanced LIGO arm cavity, is 8.5 times stronger for Mexican-Hat mirrors and their mesa beams, than for spherical mirrors and their Gaussian beams, when the beam sizes are chosen to have the same diffraction losses. This result applies to the original version of the advanced-LIGO MH and spherical mirrors, with nearly flat mirrors. A comparison for mirrors with radii of curvature R near $L/2 = 2$ km (and the same beam shapes and sizes on the mirrors as for the nearly flat mirrors) is currently under study in collaboration with members of Thorne's Caltech group.

I. INTRODUCTION

Here we calculate the condition of angular instability considered in [1] for FP interferometer with Mexican-hat (MH) mirrors and compare it with instability condition in gaussian (G) interferometer with spherical mirrors.

For simplicity we consider both mirrors (G or MH) to be identical. We are interesting in symmentical tilt [1] of both mirror by small angle θ as shown on fig. 1b. For this case the new axis of mode is diplaced by small distance $\delta x_{\rm sym}$ being parallel to old axis. Then the distribution of field on each mirror will the same but shifted by value $\delta x_{\rm sym}$. It is easy to see that for the case when only one mirror is tilted (fig.1c) the new axis displaced with some angle relatively old axis and field distributions on each mirror will be shifted by different displacements: $\delta x_{\rm non-sym} \neq \delta x_{\rm sym}$. So to investigate angular instability for MH mode we must analyze symmetrical tilt and not a single tilt analysed in [2].

The difference between symmetric and single tilts is easy to see for gaussian interferometer. For the case when radius R of curvature is smaller than distance L between mirrors tilt we have $\delta x_{\rm non-sym} > \delta x_{\rm sym}$ — it is obvious from fig. 1b,c. In opposite case $R \gg L$ we have $\delta x_{\rm non-sym} < \delta x_{\rm sym}$ — it is illustrated on fig. 1d. It seems that exactly last case corresponds to MH interferometer with nearly flat surfaces of mirrors. It seems this reason why D. Sigg [1] consider the case when R is only slightly larger than $L/2$.

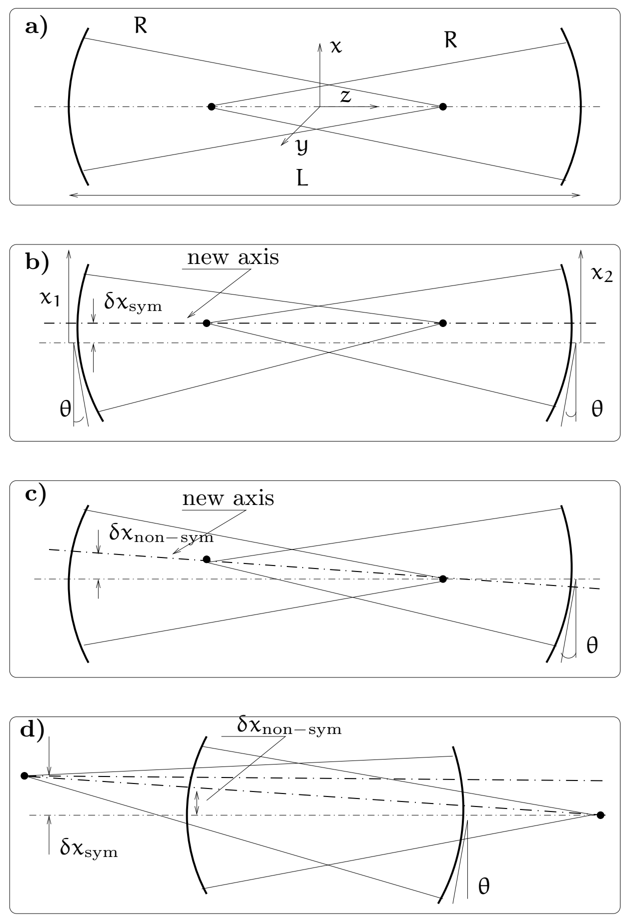

FIG. 1: FP resonator with (a) perfectly positioned mirrors, (b) with symmetricaly tilted mirrors and (c) with single tilted mirror. (d) The difference between symetric and single tilt for the case $R \gg L$.

II. MAIN FORMULAS

Fundamental mode $u_0(\vec r)$ of FP interferometer due to tilt is transformed into perturbed fundamental mode $\tilde u_0(\vec r)$, which can be expanded over set of modes $\{u_n(\vec r)\}$ of unperturbed FP interferometer. We use the lowest (dipolar) order of approximation:

$$\begin{aligned}
\tilde{u}_0(\vec{r}) &\simeq u_0(\vec{r}) + \alpha_1 u_1(\vec{r}), \qquad (2.1)\\
u_0(\vec{r}) &= \frac{u_0(r)}{\sqrt{2\pi}},\\
u_k(\vec{r}) &= \frac{u_k(r)\cos k\phi}{\sqrt{\pi}}, \quad k = 1,\ 2,\ \ldots
\end{aligned}$$

$$\int_0^\infty \left[u_0(r)\right]^2 r\,dr = 1, \qquad \int_0^\infty \left[u_k(r)\right]^2 r\,dr = 1.$$

Here $u_0(\vec{r})$, $u_1(\vec{r})$ are fundamental axisymmetrical and dipolar modes of non-perturbed FP resonator correspondingly, $u_0(r)$, $u_1(r)$ are there parts depending on only radial coordinate r. Coordinate r is assumed to be dimensionless (in units of $b = \sqrt{2\pi L/\lambda} \simeq 2.6$ cm).

Now we can calculate the torque acting on mirror by laser beam of power P

$$\begin{aligned}
T &= \frac{2Pb}{c}\int \left(\tilde{u}_0(\vec{r})\right)^2 r\cos\phi\, r\,dr\,d\phi \simeq\\
&\simeq \frac{2Pb\,2\,\alpha_1}{c}\int u_0(\vec{r})u_1(\vec{r})\, r\cos\phi\, r\,dr\,d\phi \qquad (2.2)
\end{aligned}$$

$$\begin{aligned}
T &\simeq \frac{2\sqrt{2}\,Pb,}{c}\times \alpha_1\, I, \qquad (2.3)\\
I &= \int u_0(r)u_1(r)\, r^2\,dr \qquad (2.4)
\end{aligned}$$

Here c is light speed. These formulas are valid both for MH and G interferometers (of course sets $\{u_n(\vec{r})\}$ are different for MH and G modes). In sections below we calculate values α_1 and I for gaussian and MH interferometer.

III. GAUSSIAN (G) FP INTERFEROMETER

For FP resonator with spherical mirrors we can easy obtain the displacement of optical axis $\delta x_{\rm sym}$:

$$\begin{aligned}
\delta x_{\rm sym} &\simeq \frac{R\theta}{b} = \frac{L\,\theta}{b(1-g)}, \qquad (3.1)\\
g &= 1 - \frac{L}{R}, \qquad (3.2)
\end{aligned}$$

Here g is so called g-parameter, L is distance between mirrors (in cm), R is radius of mirror curvature (in cm). Now we can write down and expand in series the main mode $\tilde{u}_0^G$ of FP resonator with tilted merrors:

$$\begin{aligned}
\tilde{u}_0^G\vec{r}) &= \frac{\sqrt{2}}{r_0}\, e^{-r_{\delta x}^2/2r_0^2}, \qquad (3.3)\\
r_{\delta x}^2 &= \left(r\cos\varphi - \delta x_{\rm sym}\right)^2 + r^2\sin^2\varphi, \qquad (3.4)\\
\tilde{u}_0\vec{r}) &\simeq u_0^G(r)\left(1 - \frac{r\,\delta x_{\rm sym}\,\cos\varphi}{r_0^2}\right) \qquad (3.5)\\
&= u_0^G(r) - \frac{\delta x_{\rm sym}\,\cos\varphi}{r_0}\, u_1^G(r) = \qquad (3.6)\\
&= u_0^G(r) - \underbrace{\frac{\sqrt{\pi}\,\delta x_{\rm sym}}{r_0}}_{\alpha_1^G}\,\underbrace{\frac{u_1^G(r)\,\cos\varphi}{\sqrt{\pi}}}_{u_1^G(\vec{r})} \qquad (3.7)
\end{aligned}$$

See expressions for $u_0(r)$, $u_1(r)$ in Appendix. Using the known formula (see e.g. [3])

$$r_0^2 = \frac{1}{\sqrt{1-g^2}},$$

we can express coefficient α_1^G as following

$$\alpha_1^G = \frac{\sqrt{\pi}\,L\,\theta\,(1+g)^{1/4}}{b\,(1-g)^{3/4}} \simeq 0.0315 \times \frac{\theta}{10^{-8}} \qquad (3.8)$$

We caslculate in Appendix A the integral I:

$$I^G = r_0 = \frac{1}{(1-g)^{1/4}} \simeq 1.807\,. \qquad (3.9)$$

Here for numerical estimates we used the parameters: $b = 2.6$ cm, $L = 4$ km, $g = 0.952$ (the fiducial configuration studied in Sec. IV of [2]).

IV. MEXICAN-HAT (MH) FP INTERFEROMETER

Our numerical calculations for fiducial MH mirrors with the same diffraction loss as our fiducial spherical mirrors (radius of mesa beam $D = 4b \simeq 10.4$ cm; configuration studied in Sec. IV of [2]) gives (see details in Appendix B)

$$\begin{aligned}
I^{MH} &\simeq \int_0^\infty u_0^{MH}(r)u_1^{MH}(r)\, r^2\,dr \simeq 2.65, \qquad (4.1)\\
\alpha_1^{MH} &\simeq 0.182 \times \left(\frac{\theta}{10^{-8}}\right) \qquad (4.2)
\end{aligned}$$

V. COMPARISION OF MH AND G MODE

To compare angular instability in MH and G mode I calculate factor

$$B = \frac{\alpha_1^{MH}\, I^{MH}}{\alpha_1^G\, I^G} \simeq \frac{0.182 \times 2.65}{0.0315 \times 1.807} \simeq 8.47 \qquad (5.1)$$

This factor show how much the torque T (see formula (2.2)) is larger for MH mode than for G mode (with the same power circulating inside), i.e critical power (which is enough to produce angular instability) in interferometer with MH mirror is abour $B \simeq 8.47$ times smaller than critical power in interferometer with sphecical mirrors (which g-factor is $g = 0.952$).

Note that this instability does not look like inevitable one: it can be depressed by introduction of additional yaw rigidity by feed back control system.

APPENDIX A: CALCULATION OF I^G

I write down the expressions for main G axisymmetric and dipolar modes (recall that r is dimensionless radial coordinat in units of b):

$$u_0^G(r) = \frac{\sqrt{2}}{r_0} e^{-r^2/2r_0^2}, \tag{A1}$$

$$u_1^G(r) = \frac{\sqrt{2}\, r}{r_0^2} e^{-r^2/2r_0^2}, \tag{A2}$$

$$\int_0^\infty \left[u_0^G(r)\right]^2 r\,dr = 1, \quad \int_0^\infty \left[u_1^G(r)\right]^2 r\,dr = 1 \tag{A3}$$

Here r_0 is dimensionless radius of beam (the intensity on mirror is proportional to $\sim e^{-r^2/r_0^2}$),

Now I can calculate integral I^G:

$$\begin{aligned} I^G &= \int_0^\infty u_0^G(r) u_1^G(r)\, r^2\,dr = \\ &= \int_0^\infty \frac{2}{r_0^2}\, e^{-r^2/r_0^2}\, r^3\,dr = \\ &= \int_0^\infty r_0 \left(\frac{r^2}{r_0^2}\right) e^{-r^2/r_0^2}\, d\left(\frac{r^2}{r_0^2}\right) = r_0 \end{aligned} \tag{A4}$$

APPENDIX B: EIGEN MODE OF MH FP INTERFEROMETER

a. Perfect positioned MH mirrors. The main axisymmetric mode $u_0^{MH}(\vec r)$ and dipolar mode $u_1^{MN}(\vec r)$ fulfils integral equations:

$$\int G^0(\vec r_1, \vec r_2)\, u_o^{MH}(\vec r_2)\, d\vec r_2 = u_0^{MH}(\vec r_1), \tag{B1}$$

$$\int G^0(\vec r_1, \vec r_2)\, u_1^{MH}(\vec r_1)\, d\vec r_1 = \lambda_1^{MH}\, u_1^{MH}((\vec r_2), \tag{B2}$$

Where we assume that eigen value of main mode $u_0^{MH}(\vec r)$ is equal to 1. The kernel G^0 is the following:

$$G^0(\vec r_1, \vec r_2) = -\frac{i}{2\pi} \exp i \left(\frac{(\vec r_1 - \vec r_2)^2}{2} - h_1(\vec r_1) - h_2(\vec r_2)\right),$$

$$h_1 = kH_1 \qquad h_2 = kH_2, \qquad k = \frac{2\pi}{\lambda}, \tag{B3}$$

where H_1 and H_2 are physical deviations (in cm) of mirror's surface from plane surface.

b. FP interferometer with slightly symmetric tilted MH mirrors as shown on fig. 1b. The tilt is equivalent to small deviations of mirror's position from perfect one:

$$\delta h_1 \simeq 2kb\, r_1\, \cos\varphi_1\, \theta \quad \text{(left mirror)} \tag{B4}$$

$$\delta h_2 \simeq 2kb\, r_2\, \cos\varphi_2\, \theta \quad \text{(right mirror).} \tag{B5}$$

Due to symmetry the complex field distributions on each mirror are the same. We are interesting in main mode $\tilde u_0^{MH}(\vec r)$ which can be expand in series of modes of resonator with perfectly positioned mirrors. We use lowest (dipolar) approximation:

$$\tilde u_0^{MH}(\vec r) \simeq u_0^{MH}(\vec r) + \alpha_1^{MH} u_1^{MH}(\vec r)$$

The eigen value of this mode will be slightly differ from unity: $\tilde\lambda_0 \simeq 1 + \Delta$. Then we have the following integral equation for $\tilde u_0^{MH}(\vec r)$

$$\begin{aligned} &(1+\Delta)\Big(u_0^{MH}(\vec r_1) + \alpha_1^{MH} u_1^{MH}(\vec r_1)\Big) = \\ &= \int G^0(\vec r_1, \vec r_2)\Big(1 - i\,\delta h_1(\vec r_1) - i\delta h_2(\vec r_2)\Big) \times \\ &\quad \times \Big(u_0^{MH}(\vec r) + \alpha_1^{MH} u_1^{MH}(\vec r)\Big)\, d\vec r_2, \end{aligned} \tag{B6}$$

This equation can be simplified:

$$\begin{aligned} &\Delta u_0^{MH}(\vec r_1) + \Big(1 + \Delta - \lambda_1^{MH}\Big)\alpha_1^{MH} u_1^{MH}(\vec r_1) = \\ &= -i \int G^0(\vec r_1, \vec r_2)\Big(\delta h_1(\vec r_1) + \delta h_2(\vec r_2)\Big) \times \\ &\quad \times \Big(u_0^{MH}(\vec r) + \alpha_1^{MH} u_1^{MH}(\vec r)\Big)\, d\vec r_2, \end{aligned} \tag{B7}$$

Multiplying equation (B7) by $u_0^{MH}(\vec r_1)$ and integrating over $d\vec r_1$ one can find that addition Δ has second order of smallnes: $\Delta \sim \theta^2$ and below we assume $\Delta = 0$.

Multiplying equation (B7) by $u_1^{MH}(\vec r_1)$ and integrating over $d\vec r_1$ one can find α_1^{MH}:

$$\begin{aligned} \big(1 - \lambda_1^{MH}\big)\alpha_1^{MH} &= -i\big(1 + \lambda_1^{MH}\big) \times \\ &\quad \times \int u_0^{MH}(\vec r_1)\, u_1^{MH}(\vec r_1)\, \delta h_1(\vec r_1)\, d\vec r_1, \end{aligned} \tag{B8}$$

$$\begin{aligned} \alpha_1^{MH} &= \frac{i2\sqrt{2}\, kb\,\theta\,\big(1 + \lambda_1^{MH}\big)}{\big(1 - \lambda_1^{MH}\big)} \times \\ &\quad \times \underbrace{\int_0^\infty u_0^{MH}(r)\, u_1^{MH}(r)\, r^2\,dr}_{I^{MH}} = \end{aligned} \tag{B9}$$

$$= \frac{i2\sqrt{2}\, L\, I^{MH}\theta(1 + \lambda_1^{MH})}{b(1 - \lambda_1^{MH})} \tag{B10}$$

Our numerical calculations for MH modes with $D = 4$ (radius of MH beam is equal to $Db \simeq 10.4$ cm) gives

$$I^{MH} \simeq 2.65.$$

We can take from [4] (table V) the value of $\lambda_1^{MH} = e^{i\pi\times 0.0404}$ and then calculate α_1^{MH}:

$$\alpha_1^{MH} \simeq 0.182 \times \left(\frac{\theta}{10^{-8}}\right)$$

[1] Daniel Sigg, Angular Instability in High Power FP Cavities, LIGO-T030120-00.

[2] E. d'Ambrosio, R. O'Shaughnessy, S. Strigin, K.Thorne and S. Vyatchanin, "Reducing Thermoelastic Noise in Gravitational-Wave Interferometers by Flattening the Light Beams", currently under LSC review in preparation for submission to *Phys. Rev. D*;
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[4] O'Shaughnessy, S. Strigin and S. Vyatchanin, "The implications of Mexican-hat mirrors: calculations of thermoelastic noise and interferometer sensitivity to perturbation for Mexican-hat mirror proposal for advanced LIGO"; currently under review in preparation for submission to *Phys. Rev. D*;
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