ENCORPORATING THE ANISOTROPY OF THE COATING YOUNG'S MODULUS INTO COATING THERMAL NOISE

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We untilize a more complete mechancial model of the dielectric coating on the LIGO mirrors to include the effect of the anisotropy in the mechanical impedance of the coating. This gives a modified form of the coating thermal noise formula from the familiar Gretarsson/Nakagawa version

Start with a description of an anisotropic multilayer coating from Hoff [1]:

$$\begin{aligned} \epsilon_{rr} &= \sigma_{rr}/Y_{||} - \sigma_{||}\sigma_{\theta\theta}/Y_{||} - \sigma_{\perp}\sigma_{zz}/Y_{\perp} \\ \epsilon_{\theta\theta} &= -\sigma_{||}\sigma_{rr}/Y_{||} + \sigma_{\theta\theta}/Y_{||} - \sigma_{\perp}\sigma_{zz}/Y_{\perp} \\ \epsilon_{zz} &= -\sigma_{\perp}\sigma_{rr}/Y_{\perp} - \sigma_{\perp}\sigma_{\theta\theta}/Y_{\perp} + \sigma_{zz}/Y_{\perp}. \end{aligned} \tag{1}$$

Here, $Y_{||}$ is the Young's modulus for stresses causing strains entirely within the plane parallel to the coating layers. $Y_{\perp}$ is the Young'g modulus for stresses causing strains perpendicular to the coating layers. There are two Poisson's ratios, $\sigma_{||}$ for stresses and strains both with the plane parallel to the coating layers, and $\sigma_{\perp}$ for when either the stress or the stain is perpendicular to the coating layers.

The matrix in Eq. 1 can be inverted, to read

$$\begin{aligned} \sigma_{rr} &= (\lambda_1 + 2\mu_1)\,\epsilon_{rr} + \lambda_1\epsilon_{\theta\theta} + \lambda_2\epsilon_{zz} \\ \sigma_{\theta\theta} &= \lambda_1\epsilon_{rr} + (\lambda_1 + 2\mu_1)\,\epsilon_{\theta\theta} + \lambda_2\epsilon_{zz} \\ \sigma_{zz} &= \lambda_2\epsilon_{rr} + \lambda_2\epsilon_{\theta\theta} + (\lambda_2 + 2\mu_2)\,\epsilon_{zz}, \end{aligned} \tag{2}$$

where

$$\lambda_1 = -Y_{||}\left(\sigma_{\perp}^2 Y_{||} - \sigma_{||}Y_{\perp}\right) / \left(\left(\sigma_{||}+1\right)\left(2\sigma_{\perp}^2 Y_{||} + \left(1-\sigma_{||}\right)Y_{\perp}\right)\right), \tag{3}$$

$$\mu_1 = Y_{||}/\left(2\left(1+\sigma_{||}\right)\right), \tag{4}$$

$$\lambda_2 = \sigma_{\perp}Y_{||}Y_{\perp}/\left(-2\sigma_{\perp}^2 Y_{||} + \left(1-\sigma_{||}\right)Y_{\perp}\right), \tag{5}$$

$$\mu_2 = Y_{\perp}\left(\sigma_{\perp}Y_{||} - \left(1-\sigma_{||}\right)Y_{\perp}\right) / \left(2\left(2\sigma_{\perp}^2 2Y_{||} - \left(1-\sigma_{||}\right)Y_{\perp}\right)\right). \tag{6}$$

Equation 2 is the anisotropic equivalent of Gretarsson's equation A2 in ref. [2]. Following the same procedure as in that reference, the equivalent of Gretarsson's equation A4 is found;

$$\begin{aligned} \epsilon'_{rr} &= \epsilon_{rr} \\ \epsilon'_{\theta\theta} &= \epsilon_{\theta\theta} \\ \epsilon'_{zz} &= (\lambda - \lambda_2)/(\lambda_2 + 2\mu_2)(\epsilon_{rr} + \epsilon_{\theta\theta}) + (\lambda + 2\mu)/(\lambda_2 + 2\mu_2)\,\epsilon_{zz} \\ \epsilon'_{rz} &= \epsilon_{rz} \\ \sigma'_{rr} &= (\lambda_1 + 2\mu_1)\,\epsilon_{rr} + \lambda_1\epsilon_{\theta\theta} + \lambda_2\epsilon'_{zz} \\ \sigma'_{\theta\theta} &= \lambda_1\epsilon_{rr} + (\lambda_1 + 2\mu_2)\,\epsilon_{\theta\theta} + \lambda_2\epsilon'_{zz} \end{aligned} \tag{7}$$

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$$\sigma'_{zz} = \sigma_{zz}$$
$$\sigma'_{rz} = \sigma_{rz}.$$

Here, all the parameters without numerical subscrips refer to values in the substrate.

Following the method of Gretarsson as in [2] with the changes noted above for an anisotropic modulus, the effective loss angle ϕ_{readout} for thermal noise calculations in an interferometer can be found.

$$\begin{aligned}\phi_{\text{readout}} = \phi_{\text{substrate}} + d/\left(\sqrt{\pi} w Y_\perp\right) \\ \left(\left(Y/\left(1-\sigma_\perp^2\right) - 2\sigma_\perp^2 Y Y_{||}/\left(Y_\perp\left(1-\sigma^2\right)\left(1-\sigma_{||}\right)\right)\right)\phi_\perp\right. \\ +Y_{||}\sigma_\perp\left(1-2\sigma\right)/\left(\left(1-\sigma_{||}\right)\left(1-\sigma\right)\right)\left(\phi_{||}-\phi_\perp\right) \\ \left.+Y_{||}Y_\perp\left(1+\sigma\right)\left(1-2\sigma\right)^2/\left(Y\left(1-\sigma_{||}^2\right)\left(1-\sigma\right)\right)\phi_{||}\right).\end{aligned} \tag{8}$$

This is to be compared to equation 20 in reference [2].

The limit of Eq. 8 where all the Poisson ratios are small, $\sigma_{||} = \sigma_\perp = \sigma = 0$, gives

$$\phi_{\text{readout}} = \phi_{\text{substrate}} + d/\left(\sqrt{\pi} w\right)\left(Y/Y_\perp \phi_\perp + Y_{||}/Y \phi_{||}\right). \tag{9}$$

The limit when $Y_{||} = Y_\perp = Y'$ and $\sigma_{||} = \sigma_\perp = \sigma'$, ie when the coating is assumed isotropic except in its loss angles, Eq. 8 reproduces Eq. 21 in reference [2].

The values for $Y_\perp, Y_{||}, \phi_\perp$ and $\phi_{||}$ can be calculated from the values of the isotropic materials that make up the layers of the coating. For a coating of two layers,

$$Y_\perp = \left(d_1 + d_2\right)/\left(d_1/Y_1 + d_2/Y_2\right), \tag{10}$$

$$Y_{||} = \left(Y_1 d_1 + Y_2 d_2\right)/\left(d_1 + d_2\right), \tag{11}$$

$$\phi_\perp = Y_\perp\left(\phi_1 d_1/Y_1 + \phi_2 d_2/Y_2\right)/\left(d_1 + d_2\right), \tag{12}$$

$$\phi_{||} = \left(Y_1 \phi_1 d_1 + Y_2 \phi_2 d_2\right)/\left[Y_{||}\left(d_1 + d_2\right)\right], \tag{13}$$

where d_1 and d_2 are the total thicknesses, Y_1 and Y_2 are the isotropic Young's moduli, and ϕ_1 and ϕ_2 are the isotropic loss angles of the two materials that make up the coating.

References

1. N. J. Hoff in *Engineering Laminates*, ed. A. G. H. Hoff (John Wiley and Sons, New York, 1949).
2. G. M. Harry, A. M. Gretarsson, P. R. Saulson, S. E. Kittleberger, S. D. Penn, W. J. Startin, S. Rowan, M. M. Fejer, D. R. M. Crooks, G. Cagnoli, J. Hough, and N. Nakagawa, Class. Quantum Grav. **19**, 897 (2002).