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Use of overlapping windows in the stochastic background search

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This discussion treats a simple model that serves to identify the correct way to combine segments of analyzed data that may have correlations. This occurs, for example, in the case of analyzing windowed data using overlapping window functions, where each data point is used more than once, with different weights.

1 Statement of the problem

Consider two series of data elements, $\{x_{1\,1}, x_{1\,2}, x_{1\,3}, ..., x_{1\,R}\}$ and $\{x_{2\,1}, x_{2\,2}, x_{2\,3}, ..., x_{2\,R}\}$. The $x_{\alpha i}$ may correspond to AS_Q(t_i) for two different detectors, labelled by $\alpha = 1, 2$. We wish to analyze the statistics of some function of the data when they are analyzed using windowing functions and data partitions which overlap. Let the R data points be partitioned into M non-overlapping segments each containing N points. Further, the data are windowed with 50% overlapping (e.g., Hann) windows, w_i. Figure 1A shows this schematically for one such series x_i. As a concrete example, consider 600s of data with a sample rate of 1024 samples per second. Further assume that this data set is partitioned into $M = 10 \times 60s$ non-overlapping segments. Then, referring to the figure we have the following: $R = 600 \times 1024 = 614400$, $N = 60 \times 1024 = 61440$, and $2M - 1 = 19$ overlapping segments. The odd set, $I \in \{1, 3, 5, \cdots, 19\}$, and even set, $I \in \{2, 4, 6, \cdots, 18\}$, of segments do not overlap amongst themselves. While the odd set of segments covers the full data set, the even set fails to cover the first $N/2$ and last $N/2$ data points.

In particular, we wish to estimate a quantity from each of the overlapping data segments and to then combine them all in order to obtain a more precise estimate of that quantity. Referring once again to Fig. 1A, one sees that an analysis based on using only the odd window functions shown above the data series results in a set of estimates that are independent and non-overlapping. Let Y_I denote the quantities to be estimated and σ_I the corresponding variances. Then from the set of estimates $\{Y_1, Y_3, \cdots, Y_{2M-1}\}$ and associated variances, $\{\sigma_1^2, \sigma_3^2, \cdots, \sigma_{2M-1}^2\}$, one forms the best estimates $Y_{\rm opt}^{\rm odd}$ and $\sigma_{\rm o}^2 \equiv (\sigma_{\rm opt}^{\rm odd})^2$ as follows:

$$Y_{\rm opt}^{\rm odd} \equiv \frac{\sum_{I\in{\rm odd}} \sigma_I^{-2}\, Y_I}{\sum_{I\in{\rm odd}} \sigma_I^{-2}}\,, \tag{1}$$

$$\sigma_{\rm o}^2 \equiv \frac{1}{\sum_{I\in{\rm odd}} \sigma_I^{-2}}\,, \tag{2}$$

Similar expressions hold for the results of analyzing the data using the even window functions shown below the data series in the figure. The best overall estimate $Y_{\rm opt}$ involves further combining the two results, $Y_{\rm opt}^{\rm odd}$ and $Y_{\rm opt}^{\rm even}$. However, to do this requires properly taking into account the correlations between the two estimates. In order to do this, we need a number of intermediate results which will be discussed next.

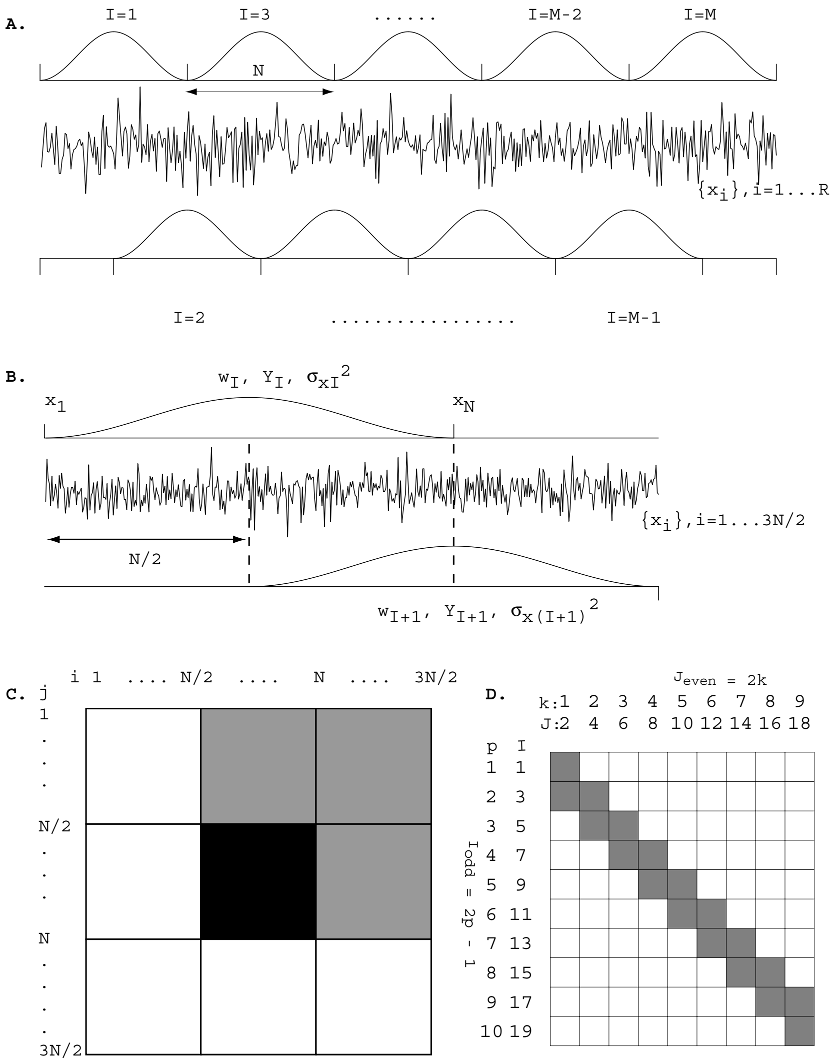

Figure 1: Data analysis schema for overlapping windows. **A**: the full data train of R samples, partitioned into 50% overlapping segments, each of which contains N points. The upper windows (odd-numbered) stride through the R data points and consist of $R/N = M$ *non-overlapping* and thus independent segments. Similarly, the lower windows, offset by 50%, represent a different partition of the data into non-overlapping segments. **B**: Schematic of two adjacent segments, I and $I+1$. Together they span $p = 3N/2$ data points. The $3N/2$ span is composed of three different partitions: interval $(1, N/2)$ containing data only in segment I, interval $(N/2+1, N)$, containing data in common between I and $I+1$, and interval $(N+1, 3N/2)$, with data from segment $I+1$. **C**: Schematic of the indexing for the sums in Eq. 30. As can be visualized in graphic **B** above, there are four components to the double sum, three of which involve uncorrelated data (shown in gray) and one of which (shown in black) contains correlations that must be taken into account. **D**: Schema showing terms contributing to the evaluation of $\langle Y^{\rm even}_{\rm opt} Y^{\rm odd}_{\rm opt} \rangle$.

2 Calculation of various statistics of the $\{x_{\alpha i}\}$

Since the data will be analyzed in segments of length N, it is useful to consider the following different partitions of the data:

$$\begin{aligned} \{x_{\alpha\, i}\} &= \{x_{\alpha\, Ii}\}_{I\in\text{odd}} & (3)\\ &= \{x_{\alpha\, Ii}\}_{I\in\text{even}} \cup \{x_{\alpha\, 1}, \cdots, x_{\alpha\, N/2}\} \cup \{x_{\alpha\, R-N/2}, \cdots, x_{\alpha\, R}\}\,, & (4) \end{aligned}$$

where

$$\begin{aligned} \{x_{\alpha\, Ii}\}_{I\in\text{odd}} &\equiv \{x_{\alpha\, 11}, \cdots, x_{\alpha\, 1N}; x_{\alpha\, 31}, \cdots, x_{\alpha\, 3N}; x_{\alpha\, 2M-1\, 1}, \cdots, x_{\alpha\, 2M-1\, N}\} & (5)\\ \{x_{\alpha\, Ii}\}_{I\in\text{even}} &\equiv \{x_{\alpha\, 21}, \cdots, x_{\alpha\, 2N}; x_{\alpha\, 41}, \cdots, x_{\alpha\, 4N}; x_{\alpha\, 2M-2\, 1}, \cdots, x_{\alpha\, 2M-2\, N}\}\,. & (6) \end{aligned}$$

As mentioned earlier, the segments labelled by even I do not cover the full data set, failing to include the first and last $N/2$ data samples of $\{x_{\alpha\, i}\}$ (hence the need for the unions in Eq. 4).

We will consider the case where the $x_{\alpha\, Ii}$ are given by the sum of noise components $n_{\alpha\, Ii}$ and a common signal component h_{Ii}:

$$x_{\alpha\, Ii} = n_{\alpha\, Ii} + h_{Ii}\,. \tag{7}$$

We will assume that that the noise and signal components are described by zero mean Gaussian random processes

$$\langle n_{\alpha\, Ii}\rangle = 0 = \langle h_{Ii}\rangle\,, \tag{8}$$

and that the different noises are uncorrelated with one another and with the signal:

$$\langle n_{1\, Ii}\, n_{2\, Jj}\rangle = 0 = \langle n_{\alpha\, Ii}\, h_{Jj}\rangle\,. \tag{9}$$

Here $\langle \cdots \rangle$ corresponds to taking the ensemble average of the random variables over many realizations or trials. We will also assume that data from non-overlapping segments are uncorrelated with another, and that while the noise power may fluctuate from one segment to the next, the signal power does not:

$$\begin{aligned} \langle n_{\alpha\, Ii}\, n_{\alpha\, Jj}\rangle &= \delta_{IJ}\, \sigma^2_{n_\alpha\, I}\, \mathcal{R}_{n_\alpha\, I}(|i-j|) + \\ &\quad\ \delta_{I\pm 1\, J}\, \frac{1}{2}\Big[\sigma^2_{n_\alpha\, I}\, \mathcal{R}_{n_\alpha\, I}(|i-j|) + \sigma^2_{n_\alpha\, I\pm 1}\, \mathcal{R}_{n_\alpha\, I\pm 1}(|i-j|)\Big] & (10)\\ \langle h_{Ii}\, h_{Jj}\rangle &= (\delta_{IJ} + \delta_{I\pm 1\, J})\, \sigma_h^2\, \mathcal{R}_h(|i-j|)\,. & (11) \end{aligned}$$

In the above expressions, $\mathcal{R}$ denotes a correlation sequence which is normalized to unity at zero lag. The power spectra of the signal and noise, P_h and $P_{n_\alpha\, I}$, are the inverse Fourier transforms of $\sigma_h^2\, \mathcal{R}_h$ and $\sigma^2_{n_\alpha\, I}\, \mathcal{R}_{n_\alpha\, I}$, respectively. For white Gaussian data, $\mathcal{R}(|i-j|) = \delta_{ij}$, while for colored noise, $\mathcal{R}(|i-j|)$ will have be non-zero over some finite range of $|i-j|$, which we assume is still small relative to the number of samples N in a segment.

Finally, we will assume that the signal power is much smaller than the corresponding noise power:

$$\sigma_h^2 \ll \sigma^2_{n_\alpha\, I}\,, \tag{12}$$

so that in expressions involving both σ_h^2 and $\sigma_{n_\alpha I}^2$, we can ignore terms proportional to σ_h^2.

Now consider the quantity

$$Y_I \equiv \frac{1}{N}\sum_{i=1}^{N} x_{1\,Ii}\, x_{2\,Ii}\, w_i^2 \,. \tag{13}$$

This is an experimental estimate of the covariance of between $x_{1\,Ii}$ and $x_{2\,Ii}$ using a windowed, finite data sample. The window function w_i can be any symmetric function of index (i.e., time) that equals unity in the middle and tapers to zero near the ends. We now determine the statistics of the Y_I given the definition of the processes $x_{\alpha\,I}$.

2.1 The mean of Y_I

$$\begin{aligned}
\langle Y_I \rangle &= \frac{1}{N}\sum_{i=1}^{N}\langle x_{1\,Ii}\, x_{2\,Ii}\rangle\, w_i^2 & (14)\\
&= \frac{1}{N}\sum_{i=1}^{N}\Big(\langle n_{1\,Ii}\, n_{2\,Ii}\rangle + \langle n_{1\,Ii}\, h_{Ii}\rangle + \langle h_{Ii}\, n_{2\,Ii}\rangle + \langle h_{Ii}\, h_{Ii}\rangle\Big)\, w_i^2 & (15)\\
&= \sigma_h^2\,\frac{1}{N}\sum_{i=1}^{N} w_i^2 & (16)\\
&= \sigma_h^2\,\overline{w^2} & (17)
\end{aligned}$$

where

$$\overline{w^2} \equiv \frac{1}{N}\sum_{i=1}^{N} w_i^2 \,. \tag{18}$$

Note that $\langle Y_I \rangle$ is independent of the segment number I.

2.2 The variance of Y_I

$$\begin{aligned}
\sigma_I^2 &= \langle Y_I^2\rangle - \langle Y_I\rangle^2 & (19)\\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\langle x_{1\,Ii}\, x_{2\,Ii}\, x_{1\,Ij}\, x_{2\,Ij}\rangle - \langle x_{1\,Ii}\, x_{2\,Ii}\rangle\langle x_{1\,Ij}\, x_{2\,Ij}\rangle\right)\, w_i^2 w_j^2 & (20)\\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\langle x_{1\,Ii}\, x_{1\,Ij}\rangle\langle x_{2\,Ii}\, x_{2\,Ij}\rangle + \langle x_{1\,Ii}\, x_{2\,Ij}\rangle\langle x_{2\,Ii}\, x_{1\,Ij}\rangle\right)\, w_i^2 w_j^2 & (21)\\
&\approx \sigma_{n_1\,I}^2\sigma_{n_2\,I}^2\,\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{R}_{n_1\,I}(|i-j|)\mathcal{R}_{n_2\,I}(|i-j|)\, w_i^2 w_j^2 & (22)
\end{aligned}$$

where we used the small signal approximation Eq. 12 and the fact that the variances $\sigma^2_{n_\alpha\,I}$ depend only segment number I and not on i, j to obtain the last line. Note that for white noise, where $\mathcal{R}_{n_\alpha\,I}(|i-j|) = \delta_{ij}$, we have

$$\sigma_I^2 \approx \sigma^2_{n_1\,I}\sigma^2_{n_2\,I}\,\frac{1}{N}\overline{w^4}\,, \tag{23}$$

where

$$\overline{w^4} \equiv \frac{1}{N}\sum_{i=1}^{N} w_i^4\,. \tag{24}$$

The generalization to colored noise amounts to defining

$$\overline{w_I^4} \equiv \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{R}_{n_1\,I}(|i-j|)\mathcal{R}_{n_2\,I}(|i-j|)\,w_i^2 w_j^2\,, \tag{25}$$

so that

$$\sigma_I^2 \approx \sigma^2_{n_1\,I}\sigma^2_{n_2\,I}\,\frac{1}{N}\overline{w_I^4}\,. \tag{26}$$

Note the I subscript on w_I^4 in the last two expressions—i.e., window factors for *colored, non-stationary* noise depend on segment number I; there is no I dependence for either white noise or for stationary, colored noise.

2.3 The covariance of $Y_I Y_J$

$$\sigma^2_{IJ} = \langle Y_I Y_J\rangle - \langle Y_I\rangle\langle Y_J\rangle \tag{27}$$

Referring to Figs.1A and 1B, one sees that data segments separated by more than $J = I \pm 1$ involve unrelated, and therefore uncorrelated, data sets. Thus, one immediately obtains $\langle Y_I Y_J\rangle = \langle Y_I\rangle\langle Y_J\rangle$, which implies

$$\sigma^2_{IJ} = 0 \quad \text{for } |I-J| > 1\,. \tag{28}$$

Because of the symmetry of the problem, we need to only consider the covariance between Y_I and Y_{I+1}. For this calculation, data from more than a single segment I are involved. In fact, a total of $3N/2$ points are involved, as can be seen in Fig.1B. Referring to Eqs. 3, 4, it is useful here to consider the data as forming a continuous series of $3N/2$ points. Thus we may rewrite the above equation equivalently as

$$\sigma^2_{I(I+1)} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=N/2+1}^{3N/2}\left(\langle x_{1\,i}\,x_{2\,i}\,x_{1\,j}\,x_{2\,j}\rangle - \langle x_{1\,i}\,x_{2\,i}\rangle\langle x_{1\,j}\,x_{2\,j}\rangle\right)\,w_i^2 w_{j-N/2}^2\,. \tag{29}$$

The summation may be broken up into four terms:

$$\sum_{i=1}^{N}\sum_{j=N/2+1}^{3N/2} = \sum_{i=1}^{N/2}\sum_{j=N/2+1}^{N} + \sum_{i=N/2+1}^{N}\sum_{j=N/2+1}^{N} + \sum_{i=1}^{N/2}\sum_{j=N+1}^{3N/2} + \sum_{i=N/2+1}^{N}\sum_{j=N+1}^{3N/2}\,. \tag{30}$$

The summations on the right-hand side of the above expression correspond to the shaded regions in Fig.1C. The first, third, and fourth summations correspond to the non-overlapping gray regions of the figure, for which $\langle x_{1\,i}\, x_{2\,i}\, x_{1\,j}\, x_{2\,j}\rangle = \langle x_{1\,i}\, x_{2\,i}\rangle\langle x_{1\,j}\, x_{2\,j}\rangle$, and thus do not contribute to $\sigma^2_{I(I+1)}$. The second summation corresponds to the black-shaded region, which has the same data in both segments Y_I and Y_{I+1}. Thus, Eq. 29 for $\sigma_{I(I+1)}$ reduces to

$$\sigma^2_{I(I+1)} = \frac{1}{N^2} \sum_{i=N/2+1}^{N} \sum_{j=N/2+1}^{N} \left(\langle x_{1\,i}\, x_{2\,i}\, x_{1\,j}\, x_{2\,j}\rangle - \langle x_{1\,i}\, x_{2\,i}\rangle\langle x_{1\,j}\, x_{2\,j}\rangle\right)\, w_i^2 w_{j-N/2}^2\,. \tag{31}$$

The calculation of this summation is similar to that for σ_I^2. The only difference is that because the data may not be stationary on time scales comparable to the segment length, care must be taken in the variance we assign to this expression. The most reasonable approach is to use the average of corresponding quantities from segments I and $I+1$. The result is:

$$\sigma^2_{I(I+1)} \approx \frac{1}{2}\Big(\sigma^2_{n_1\,I}\sigma^2_{n_2\,I}\,\frac{1}{N^2}\sum_{i=N/2}^{N}\sum_{j=N/2}^{N}\mathcal{R}_{n_1\,I}(|i-j|)\mathcal{R}_{n_2\,I}(|i-j|)\,w_i^2 w_{j-N/2}^2 + \sigma^2_{n_1\,(I+1)}\sigma^2_{n_2\,(I+1)}\,\frac{1}{N^2}\sum_{i=N/2}^{N}\sum_{j=N/2}^{N}\mathcal{R}_{n_1\,(I+1)}(|i-j|)\mathcal{R}_{n_2\,(I+1)}(|i-j|)\,w_i^2 w_{j-N/2}^2\Big)\,. \tag{32}$$

For white noise, the above expression simplifies to

$$\sigma^2_{I(I+1)} \approx \frac{1}{2}\Big(\sigma^2_{n_1\,I}\sigma^2_{n_2\,I} + \sigma^2_{n_1\,(I+1)}\sigma^2_{n_2\,(I+1)}\Big)\frac{1}{2N}\overline{w^4_{\rm ovl}}\,, \tag{33}$$

where

$$\overline{w^4_{\rm ovl}} \equiv \frac{1}{N/2}\sum_{i=N/2}^{N} w_i^2 w_{j-N/2}^2 \tag{34}$$

is a quantity characterizing the overlap between the last half and first half of the window function. For colored noise, if we define

$$\overline{w^4_{{\rm ovl}\,I}} \equiv \frac{1}{N/2}\sum_{i=N/2}^{N}\sum_{j=N/2}^{N}\mathcal{R}_{n_1\,I}(|i-j|)\mathcal{R}_{n_2\,I}(|i-j|)\,w_i^2 w_{j-N/2}^2\,, \tag{35}$$

then

$$\sigma^2_{I(I+1)} \approx \frac{1}{2}\Big(\sigma^2_{n_1\,I}\sigma^2_{n_2\,I}\,\frac{1}{2N}\overline{w^4_{{\rm ovl}\,I}} + \sigma^2_{n_1\,(I+1)}\sigma^2_{n_2\,(I+1)}\,\frac{1}{2N}\overline{w^4_{{\rm ovl}\,(I+1)}}\Big)\,. \tag{36}$$

Note that the overlapping window factors are convolved with the correlation sequences of the detector noise and depend on the particular segment I.

Furthermore, in terms σ_I^2 and σ_{I+1}^2:

$$\sigma^2_{I(I+1)} = \frac{1}{2}\left(\frac{1}{2}\frac{\overline{w^4_{{\rm ovl}\,I}}}{\overline{w^4_I}}\,\sigma_I^2 + \frac{1}{2}\frac{\overline{w^4_{{\rm ovl}\,(I+1)}}}{\overline{w^4_{(I+1)}}}\,\sigma^2_{I+1}\right)\,. \tag{37}$$

3 Determining the covariance matrix of $Y_{\rm opt}^{\rm odd}\, Y_{\rm opt}^{\rm even}$

Now we can proceed with calculating the covariance matrix of the pair of correlated measurements $Y_{\rm opt}^{\rm odd}$ and $Y_{\rm opt}^{\rm even}$. Using an obvious notation, we write:

$$||\mathbf{C}|| \equiv \begin{bmatrix} \sigma_{\rm o}^2 & \sigma_{\rm oe}^2 \\ \sigma_{\rm eo}^2 & \sigma_{\rm e}^2 \end{bmatrix} \tag{38}$$

From Eq. 2 the diagonal terms are

$$\sigma_{\rm o}^2 = \frac{1}{\sum_{I\in{\rm odd}} \sigma_I^{-2}}\,, \quad \sigma_{\rm e}^2 = \frac{1}{\sum_{I\in{\rm even}} \sigma_I^{-2}}\,. \tag{39}$$

The off diagonal terms are given by

$$\sigma_{\rm oe}^2 = \sigma_{\rm eo}^2 \equiv \langle Y_{\rm opt}^{\rm odd}\, Y_{\rm opt}^{\rm even}\rangle - \langle Y_{\rm opt}^{\rm odd}\rangle\langle Y_{\rm opt}^{\rm even}\rangle\,. \tag{40}$$

Referring to Eq.1, we get

$$\langle Y_{\rm opt}^{\rm odd}\, Y_{\rm opt}^{\rm even}\rangle - \langle Y_{\rm opt}^{\rm odd}\rangle\langle Y_{\rm opt}^{\rm even}\rangle = \frac{\sum_{I\in{\rm odd}}\sum_{J\in{\rm even}} \sigma_I^{-2}\sigma_J^{-2}\left(\langle Y_I\, Y_J\rangle - \langle Y_I\rangle\langle Y_J\rangle\right)}{\sum_{I\in{\rm odd}}\sigma_I^{-2}\sum_{J\in{\rm even}}\sigma_J^{-2}} \tag{41}$$

$$= \sigma_{\rm o}^2\sigma_{\rm e}^2 \sum_{I\in{\rm odd}}\sum_{J\in{\rm even}} \sigma_I^{-2}\sigma_J^{-2}\,\sigma_{IJ}^2\,. \tag{42}$$

To proceed further, note that $\sigma_{IJ}^2 = 0$ for $|I-J|>1$, so that

$$\sigma_{\rm oe}^2 = \sigma_{\rm o}^2\sigma_{\rm e}^2\left(\sum_{I\ {\rm odd}=1}^{2M-3} \sigma_I^{-2}\sigma_{I+1}^{-2}\,\sigma_{I(I+1)}^2 + \sum_{I\ {\rm odd}=3}^{2M-1} \sigma_I^{-2}\sigma_{I-1}^{-2}\,\sigma_{I(I-1)}^2\right) \tag{43}$$

$$= \sigma_{\rm o}^2\sigma_{\rm e}^2\left(\frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2} + \frac{\sigma_{23}^2}{\sigma_2^2\sigma_3^2} + \cdots + \frac{\sigma_{(2M-2)(2M-1)}^2}{\sigma_{2M-2}^2\sigma_{2M-1}^2}\right)\,. \tag{44}$$

Finally, substituting for $\sigma_{I(I+1)}^2$ using Eq. 37, we obtain:

$$\sigma_{\rm oe}^2 = \sigma_{\rm o}^2\sigma_{\rm e}^2\left[\frac{1}{2}\left(\frac{1}{2}\frac{\overline{w_{{\rm ovl}\,1}^4}}{\overline{w_1^4}}\sigma_2^{-2} + \frac{1}{2}\frac{\overline{w_{{\rm ovl}\,2}^4}}{\overline{w_2^4}}\sigma_1^{-2}\right) + \frac{1}{2}\left(\frac{1}{2}\frac{\overline{w_{{\rm ovl}\,2}^4}}{\overline{w_2^4}}\sigma_3^{-2} + \frac{1}{2}\frac{\overline{w_{{\rm ovl}\,3}^4}}{\overline{w_3^4}}\sigma_2^{-2}\right) + \cdots\right] \tag{45}$$

$$= \frac{1}{2}\sigma_{\rm o}^2\sigma_{\rm e}^2\left[\frac{1}{2}\left(\frac{\overline{w_{{\rm ovl}\,2}^4}}{\overline{w_2^4}}\right)\sigma_1^{-2} + \frac{1}{2}\left(\frac{\overline{w_{{\rm ovl}\,1}^4}}{\overline{w_1^4}} + \frac{\overline{w_{{\rm ovl}\,3}^4}}{\overline{w_3^4}}\right)\sigma_2^{-2} + \cdots \right.$$
$$\left. + \frac{1}{2}\left(\frac{\overline{w_{{\rm ovl}\,(2M-3)}^4}}{\overline{w_{2M-3}^4}} + \frac{\overline{w_{{\rm ovl}\,(2M-1)}^4}}{\overline{w_{2M-1}^4}}\right)\sigma_{2M-2}^{-2} + \frac{1}{2}\left(\frac{\overline{w_{{\rm ovl}\,(2M-2)}^4}}{\overline{w_{2M-2}^4}}\right)\sigma_{2M-1}^{-2}\right]\,. \tag{46}$$

For stationary or white noise, the window factors are all independent of segment number, yielding the simplified result

$$\sigma_{\rm oe}^2 = \frac{1}{2}\frac{\overline{w_{\rm ovl}^4}}{\overline{w^4}}\,\sigma_{\rm o}^2\sigma_{\rm e}^2\left[\frac{1}{2}\sigma_1^{-2}+\sigma_2^{-2}+\sigma_3^{-2}+\cdots+\sigma_{2M-2}^{-2}+\frac{1}{2}\sigma_{2M-1}^{-2}\right] \tag{47}$$

$$= \frac{1}{2}\frac{\overline{w_{\rm ovl}^4}}{\overline{w^4}}\,\sigma_{\rm o}^2\sigma_{\rm e}^2\left[\sigma_{\rm e}^{-2}+\sigma_{\rm o}^{-2}-\frac{1}{2}(\sigma_1^{-2}+\sigma_{2M-1}^{-2})\right] \tag{48}$$

$$= \frac{1}{2}\frac{\overline{w_{\rm ovl}^4}}{\overline{w^4}}\left[\sigma_{\rm o}^2+\sigma_{\rm e}^2-\frac{1}{2}\,\sigma_{\rm o}^2\sigma_{\rm e}^2\,\left(\sigma_1^{-2}+\sigma_{2M-1}^{-2}\right)\right]. \tag{49}$$

Since the individual variances σ_I^2 are typically M times larger than either $\sigma_{\rm o}^2$ or $\sigma_{\rm e}^2$, we have

$$\frac{1}{2}\sigma_{\rm o}^2\sigma_{\rm e}^2(\sigma_1^{-2}+\sigma_{2M-1}^{-2}) \sim \frac{1}{2M}\left(\sigma_{\rm o}^2+\sigma_{\rm e}^2\right) \tag{50}$$

so that

$$\sigma_{\rm oe}^2 \sim \frac{\overline{w_{\rm ovl}^4}}{\overline{w^4}}\,\frac{1}{2}\left(\sigma_{\rm o}^2+\sigma_{\rm e}^2\right)\left(1-\frac{1}{2M}\right). \tag{51}$$

The factor $(1-1/2M)$ can be thought of as a finite data stream 'edge-effect' correction. For, e.g., $M = 10$ segments, the term represents a 5% effect.

4 Optimal combination of $Y_{\rm opt}^{\rm odd}$ and $Y_{\rm opt}^{\rm even}$

As shown in Appendix C of "Optimal combination of signals from co-located gravitational wave interferometers for use in searches for a stochastic background," the optimal combination of two measurements Y_1 and Y_2 that have the same theoretical mean and covariance matrix

$$||\mathbf{C}|| \equiv \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{52}$$

is given by the weighted average

$$Y_{\rm opt} \equiv \frac{\sum_{i=1}^2 \lambda_i Y_i}{\sum_{j=1}^2 \lambda_j} \tag{53}$$

where

$$\lambda_i = \sum_{j=1}^2 ||\mathbf{C}||_{ij}^{-1}\,. \tag{54}$$

Explicitly,

$$\lambda_1 = \frac{C_{22}-C_{12}}{\det||\mathbf{C}||}\,,\qquad \lambda_2 = \frac{C_{11}-C_{21}}{\det||\mathbf{C}||}\,, \tag{55}$$

where $\det||\mathbf{C}|| := C_{11}C_{22}-C_{12}C_{21}$. The theoretical variance of $Y_{\rm opt}$ is

$$\sigma_{\rm opt}^2 = \frac{1}{\sum_{k=1}^2 \lambda_k}\sum_{i=1}^2\sum_{j=1}^2 \lambda_i\,C_{ij}\,\lambda_j\,. \tag{56}$$

For the overlapping window analysis, we have $Y_1 = Y_{\rm opt}^{\rm odd}$ and $Y_2 = Y_{\rm opt}^{\rm even}$ with

$$||\mathbf{C}|| \equiv \left[\begin{array}{cc} \sigma_{\rm o}^2 & \sigma_{\rm oe}^2 \\ \sigma_{\rm eo}^2 & \sigma_{\rm e}^2 \end{array} \right] , \tag{57}$$

so

$$\lambda_{\rm o} = \frac{\sigma_{\rm e}^2 - \sigma_{\rm oe}^2}{\sigma_{\rm o}^2\sigma_{\rm e}^2 - (\sigma_{\rm oe}^2)^2} , \quad \lambda_{\rm e} = \frac{\sigma_{\rm o}^2 - \sigma_{\rm oe}^2}{\sigma_{\rm o}^2\sigma_{\rm e}^2 - (\sigma_{\rm oe}^2)^2} . \tag{58}$$

Since we know $\sigma_{\rm o}^2$, $\sigma_{\rm e}^2$, and $\sigma_{\rm oe}^2$ in terms of the σ_I^2 (c.f. Eqs. 39, 46), we can calculate $Y_{\rm opt}$ and $\sigma_{\rm opt}^2$ as desired.