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Parameter Estimation In The Frequency Domain

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Abstract

This technical document investigates how to find the unknown signal parameters, h_0, ι, ψ, and Φ_0, using the method given in Jaranowski, Królak and Schutz (JKS), and extends this method to nonstationary noise. Algebraic expressions for these parameters in terms of the basis amplitudes A_i are found. It is shown how to compute estimates of the A_i's in the frequency domain using Short Fourier Transforms of the data (SFTs) and how the LALDemod function in the LAL library could be used for this purpose.

I. Introduction

A data analysis method for detecting continuous gravitational waves from rotating neutron stars is given by Jaranowski, Królak and Schutz (JKS) [1]. In this technical note we focus on the special case of waves from the quadrupole moment of a triaxial ellipsoid spinning about a principal axis. This note assumes the reader is somewhat familiar with the definition of the quantities given in JKS.

Section II reviews how to find the unknown signal parameters, h_0, ι, ψ, and Φ_0, using the method given in JKS. In Section III we find algebraic expressions for these parameters in terms of the basis amplitudes A_i. Note that JKS shows how to estimate the basis amplitudes from the data by maximizing the likelihood function, for the case of stationary noise. In Section IV we investigates how to generalize the results to the case of nonstationary noise, and how to compute estimates of the parameters in the frequency domain, using Short Fourier Transforms of the data (SFTs). It is shown that the LALDemod function in the LAL library could be used for this purpose.

II. JKS Parameter Estimation

Following JKS, the gravitational-wave strain for a triaxial ellipsoid measured is given by

$$h(t) = F_+ h_+ + F_\times h_\times, \tag{1}$$

$$\begin{aligned} F_+ &= \sin\zeta(a(t)\cos 2\psi + b(t)\sin 2\psi, \quad (2)\\ F_\times &= \sin\zeta(b(t)\cos 2\psi - a(t)\sin 2\psi, \quad (3)\end{aligned}$$

$$\begin{aligned} h_+ =& \ \frac{1}{2}h_0(1+\cos^2\iota)\cos[2\Phi_0 + 2\Phi(t)], \quad (4)\\ h_\times =& \ h_0\cos\iota\sin[2\Phi_0 + 2\Phi(t)]. \quad (5)\end{aligned}$$

The strain can be written as

$$h_2(t) = \sum_{i=1}^{4} A_i h_i, \tag{6}$$

where

$$\begin{aligned} h_1 &= a(t)\cos[2\Phi(t)], \quad h_2 = b(t)\cos[2\Phi(t)], \quad (7)\\ h_3 &= a(t)\sin[2\Phi(t)], \quad h_4 = b(t)\sin[2\Phi(t)]. \quad (8)\end{aligned}$$

Comparing Eq. (1) and Eq. (6) one arrives at JKS Eqs. (32)-(35):

$$\begin{aligned} A_1 &= h_0\sin\zeta\left[\frac{1}{2}(1+\cos^2\iota)\cos 2\psi\cos 2\Phi_0 - \cos\iota\sin 2\psi\sin 2\Phi_0\right], \quad (9)\\ A_2 &= h_0\sin\zeta\left[\frac{1}{2}(1+\cos^2\iota)\sin 2\psi\cos 2\Phi_0 + \cos\iota\cos 2\psi\sin 2\Phi_0\right], \quad (10)\\ A_3 &= h_0\sin\zeta\left[-\frac{1}{2}(1+\cos^2\iota)\cos 2\psi\sin 2\Phi_0 - \cos\iota\sin 2\psi\cos 2\Phi_0\right], (11)\\ A_4 &= h_0\sin\zeta\left[-\frac{1}{2}(1+\cos^2\iota)\sin 2\psi\sin 2\Phi_0 + \cos\iota\cos 2\psi\cos 2\Phi_0\right]. (12)\end{aligned}$$

The log likelihood function can be written in the time domain as

$$\ln \Lambda = \sum_j \left(\frac{x_j h_j}{\sigma_j^2} - \frac{1}{2} \frac{x_j h_j}{\sigma_j^2} \right), \tag{13}$$

where j is a time index. JKS, for the case the σ_j 's are constant, find estimates for the A_i 's by maximizing the likelihood function. The result is

$$A_1 = 2\frac{B(x||h_1) - C(x||h_2)}{D}, \tag{14}$$

$$A_2 = 2\frac{A(x||h_2) - C(x||h_1)}{D}, \tag{15}$$

$$A_3 = 2\frac{B(x||h_3) - C(x||h_4)}{D}, \tag{16}$$

$$A_4 = 2\frac{A(x||h_4) - C(x||h_3)}{D}, \tag{17}$$

This is JKS Eq. (52) with $1i \to i$. In the discrete case the inner product can be written as

$$(x||y) = \frac{1}{N_{\mathrm{T}}} \sum_{j=0}^{N_{\mathrm{T}}-1} x_j y_j. \tag{18}$$

Note that $A = (a||a)$, $B = (b||b)$, $C = (a||b)$, and $D = AB - C^2$, and that JKS make a series of approximations [see their Eqs. (48)-(49)] that should be valid for properly sampled data summed over many cycles.

Thus, Eqs. (14)-(17) gives us the estimates of the basis elements from the data and Eqs. (9)-(12) gives us four equations for the unknown parameters h_0, ι, ψ, and Φ_0. (Note that $\sin\zeta$ is not unknown, but the known angle between the arms of the interferometer.)

III. Algebraically Inverting Eqs. (9)-(12)

In this section we algebraically invert Eqs. (9)-(12) [JKS Eqs. (32)-(35)] to give the unknown parameters $\{p_i\} = \{h_0,\ \iota,\ \psi,\ \Phi_0\}$ in terms of the A_i 's. There are several ways to do this (for example as worked out in Anah Mourant's SURF project at LIGO Hanford Observatory in 2003; mentored by Gregory Mendell). The treatment presented here is the most straight forward, and is due to Yousuke Itoh and Xavier Siemens.

We define two auxiliary variables.

$$A_s^2 \equiv \sum_{i=0}^{4} A_i^2 = \frac{1}{4}h_0^2(1+6\mu^2+\mu^4)\sin^2\zeta \tag{19}$$

$$D_a \equiv A_1A_4 - A_2A_3 = \frac{1}{2}h_0^2\mu(1+\mu^2)\sin^2\zeta, \tag{20}$$

where $\mu = \cos\iota$. It is straight forward to show that

$$A_s^2 + 2D_a = \frac{1}{4}h_0^2\sin^2\zeta(1+\mu)^4, \tag{21}$$

$$A_s^2 - 2D_a = \frac{1}{4}h_0^2\sin^2\zeta(1-\mu)^4. \tag{22}$$

For real h_0 and μ, and since $-1 \leq \mu \leq 1$, only the positive real $4th$-root of Eqs. (21) and (22) applies, and thus h_0 is given by

$$h_0\sin\zeta = \frac{1}{2}\left[(A_s^2-2D_a)^{\frac{1}{4}} + (A_s^2+2D_a)^{\frac{1}{4}}\right]^2. \tag{23}$$

Note that $A_s^2 - 2D_a = (A_1 - A_4)^2 + (A_2 + A_3)^2 \geq 0$ and $A_s^2 + 2D_a = (A_1 + A_4)^2 + (A_2 - A_3)^2 \geq 0$. Note that these inequalities are satisfied independently of the emission mechanisms. Thus, we always obtain a non-imaginary $h_0\sin\zeta$ for any $\{A_i\}$.

Given h_0, we define the third auxiliary variable.

$$r \equiv \frac{A_s^2}{h_0^2\sin^2\zeta} = \frac{4}{[(1-t)^{\frac{1}{4}}+(1+t)^{\frac{1}{4}}]^4} \geq 0. \tag{24}$$

where $t \equiv 2D_a/A_s^2$. Note that $-1 \leq t \leq 1$. $r = r(t)$ is a symmetric function of t and the map from t to r is one to one from $0 \leq t \leq 1$ to $1/4 \leq r \leq 2$. In other words, for any $\{A_i\}$, we have $1/4 \leq r \leq 2$. Again we note that this inequality is satisfied independently of the emission mechanisms.

Given r, we solve Eq. (19) for μ. There are four solutions;

$$\mu_\pm^{\text{unphysical}} = \pm\sqrt{-3-2\sqrt{2+r}} \tag{25}$$

$$\mu_\pm = \pm\sqrt{-3+2\sqrt{2+r}} \tag{26}$$

The first two solutions are obviously unphysical. The other two are physically possible solution only when

$$\frac{1}{4} \leq r \leq 2. \tag{27}$$

In this case, $0 \le \mu_+ \le 1$ and $-1 \le \mu_- \le 0$. This is fulfilled for any $\{A_i\}$, as mentioned above. Thus, we always obtain at least two physical solutions for μ. The degeneracy of the map from t to r corresponds to these two possible solutions of μ and it can be broken by taking the sign of t into account. Namely, if D_a is positive, one should take positive solution (μ_+), and if D_A is negative, one should take negative solution (μ_-).

Now we construct the fourth auxiliary variable β

$$\beta \equiv \frac{1+\mu^2}{2\mu} \tag{28}$$

With β in hand, we may compute ψ and Φ_0. A little subtlety here is that a set of shifts $\{\psi \to \psi + \pi/2, \Phi_0 \to \Phi_0 + \pi/2\}$ taken simultaneously gives the same values of $\{A_i\}$. Needless to say that $\psi \to \psi + \pi$ or $\Phi_0 \to \Phi_0 + \pi$ gives the same values of $\{A_i\}$. Thus we seek ψ and Φ_0 in the ranges of $-\pi/2 \le \psi \le \pi/2$ and $-\pi/2 \le \Phi_0 \le \pi/2$. Using Eqs. (9)-(12) one can show:

$$A_1 \cos 2\psi + A_2 \sin 2\psi = \frac{1}{2}(1+\mu^2) h_0 \sin\zeta \cos 2\Phi_0, \tag{29}$$

$$A_4 \cos 2\psi - A_3 \sin 2\psi = \mu h_0 \sin\zeta \cos 2\Phi_0, \tag{30}$$

$$A_1 \cos 2\Phi_0 - A_3 \sin 2\Phi_0 = \frac{1}{2}(1+\mu^2) h_0 \sin\zeta \cos 2\psi, \tag{31}$$

$$A_4 \cos 2\Phi_0 + A_2 \sin 2\Phi_0 = \mu h_0 \sin\zeta \cos 2\psi, \tag{32}$$

Thus, eliminating $\cos 2\Phi_0$ from Eqs. (29) and (30), and eliminating $\cos 2\psi$ from Eqs. (31) and (32), we obtain the following equations[1] for the final two parameters:

$$\tilde{\psi} = \frac{1}{2}\tan^{-1}\left(\frac{\beta A_4 - A_1}{\beta A_3 + A_2}\right), \tag{33}$$

$$\tilde{\Phi}_0 = \frac{1}{2}\tan^{-1}\left(\frac{A_1 - \beta A_4}{A_3 + \beta A_2}\right). \tag{34}$$

Note that there are always physically acceptable solutions of $\tilde{\psi}$ and $\tilde{\Phi}_0$ for any $\{A_i\}$. We reconstruct A_1 from $\{h_0, \mu, \tilde{\psi}, \tilde{\Phi}_0\}$ which we have computed from $\{A_i\}$. We write A_1^r as the so reconstructed A_1. Then look at the sign of $A_1 \cdot A_1^r$. If $A_1 \cdot A_1^r \ge 0$, then we set $\psi = \tilde{\psi}$ and $\Phi_0 = \tilde{\Phi}_0$. If $A_1 \cdot A_1^r < 0$,

[1] Note that in the real implementation in the C code, one may use "atan", not "atan2".

then we set $\psi = \tilde{\psi}$ and if $\tilde{\Phi}_0 > 0$ then $\Phi_0 = \tilde{\Phi}_0 - \pi/2$ else if $\tilde{\Phi}_0 < 0$ then $\Phi_0 = \tilde{\Phi}_0 + \pi/2$. By construction, $-\pi/2 \leq \psi \leq \pi/2$ and $-\pi/2 \leq \Phi_0 \leq \pi/2$.

It is clear from the arguments given above that for any $\{-\infty < A_i < \infty\}_{i=1}^{4}$, we always obtain physically acceptable $\{p_i\}$, where $0 \leq h_0 \leq \infty$, $-1 \leq \mu \leq 1$, $-\pi/2 \leq \Phi_0 \leq \pi/2$, and $-\pi/2 \leq \psi \leq \pi/2$. Obviously, given $\{p_i\}$ in the ranges specified just above, one can compute $\{-\infty < A_i < \infty\}$. Thus, the map between $\{A_i\}$ and $\{p_i\}$ in the ranges $0 \leq h_0 \leq \infty$, $-1 \leq \mu \leq 1$, $-\pi/2 \leq \Phi_0 \leq \pi/2$, and $-\pi/2 \leq \psi \leq \pi/2$ is onto.

The next question is if the map between $\{p_i\} =$ and $\{A_i\}$ one to one?

By construction, it is clear that the map is one to one apart from a few exceptions discussed below.

The Jacobian J of the transformation is

$$J \equiv \left| \frac{\partial A_i}{\partial p_j} \right| = \frac{1}{2} h_0^3 \sin^4 \zeta (1 - \mu^2) \tag{35}$$

Thus, the transformation can be reversed (one to one) except for $\mu = \pm 1$ or $h_0 = 0$. The situation here is thus similar to the coordinate transformation between the 4 dimensional Cartesian coordinate and 4 dimensional spherical coordinate.

The coordinate singularity $h_0 = 0$ or $\mu = \pm 1$ corresponds to $A_i = 0$ or $A_1 = \pm A_4, A_2 = \pm A_3$ respectively. It is clear that in either case we can not determine $\{p_i\}$ because we do not have enough independent variables. In other words, if $A_i = 0$, then we can take $h_0 = 0$ and any ψ, Φ_0, μ we prefer. If $A_1 = \pm A_4, A_2 = \pm A_3$, then we can determine h_0, $\mu = \pm 1$, and $\psi \pm \Phi_0$ uniquely but we can not determine ψ and Φ_0 independently. But note that we can still reconstruct $\{A_i\}$.

IV. Nonstationary Noise

For nonstationary noise we can no longer treat the σ_j 's as constant in Eq. (13) for the likelihood. However, if we assume that the noise is stationary during short segments of the data we can divide the total data set of N_{T} samples into M segments, each with N samples.

$$\ln \Lambda = \sum_{\alpha=0}^{M-1} \frac{1}{\sigma_\alpha^2} \sum_{n=0}^{N-1} \left(x_{\alpha n} h_{\alpha n} - \frac{1}{2} h_{\alpha n} h_{\alpha n} \right), \tag{36}$$

Here we assume that we can replace $\sigma_{\alpha n}$ with σ_α If we substitute Eq. (6) for h and assume that the detector response amplitude, a and b are roughly constant during each segment so that $a_{\alpha n} \to a_\alpha$ and $b_{\alpha n} \to b_\alpha$ then maximizing the likelihood in Eq. (36) gives

$$
\begin{aligned}
A_1 &= 2\frac{\bar{B}\left(\frac{x}{\sigma}||\frac{h_{21}}{\sigma}\right) - \bar{C}\left(\frac{x}{\sigma}||\frac{h_{22}}{\sigma}\right)}{\bar{D}}, && (37)\\
A_2 &= 2\frac{\bar{A}\left(\frac{x}{\sigma}||\frac{h_{22}}{\sigma}\right) - \bar{C}\left(\frac{x}{\sigma}||\frac{h_{21}}{\sigma}\right)}{\bar{D}}, && (38)\\
A_3 &= 2\frac{\bar{B}\left(\frac{x}{\sigma}||\frac{h_{23}}{\sigma}\right) - \bar{C}\left(\frac{x}{\sigma}||\frac{h_{24}}{\sigma}\right)}{\bar{D}}, && (39)\\
A_4 &= 2\frac{\bar{A}\left(\frac{x}{\sigma}||\frac{h_{24}}{\sigma}\right) - \bar{C}\left(\frac{x}{\sigma}||\frac{h_{23}}{\sigma}\right)}{\bar{D}}, && (40)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{A} &= \frac{1}{M}\sum_{\alpha=0}^{M-1}\frac{a_\alpha a_\alpha}{\sigma_\alpha^2}, && (41)\\
\bar{B} &= \frac{1}{M}\sum_{\alpha=0}^{M-1}\frac{b_\alpha b_\alpha}{\sigma_\alpha^2}, && (42)\\
\bar{C} &= \frac{1}{M}\sum_{\alpha=0}^{M-1}\frac{a_\alpha b_\alpha}{\sigma_\alpha^2}, && (43)\\
\bar{D} &= \bar{A}\bar{B} - \bar{C}^2. && (44)\\
& && (45)
\end{aligned}
$$

We now introduce SFTs in the usual way

$$
x_{\alpha n} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{x}_{\alpha k}e^{2\pi i n k/N}, \quad (46)
$$

where $\tilde{x}$ are the SFTs (just FFTs of the short segments of the data). Now define F_a and F_b as in JKS (but normalized with the σ's) and rewrite these in terms of SFTs and the Dirichlet kernel

$$
F_a = \sum_{j=0}^{N_{\mathrm{T}}-1}\frac{x_j a_j}{\sigma_j^2}e^{i\Phi_j}\Delta t = \sum_{\alpha=0}^{M-1}\frac{a_\alpha}{\sigma_\alpha^2}Q_\alpha\sum_k \tilde{x}_{\alpha k}P_{\alpha k}, \quad (47)
$$

$$F_b = \sum_{j=0}^{N_{\rm T}-1} \frac{x_j b_j}{\sigma_j^2} e^{i\Phi_j} \Delta t = \sum_{\alpha=0}^{M-1} \frac{b_\alpha}{\sigma_\alpha^2} Q_\alpha \sum_k \tilde{x}_{\alpha k} P_{\alpha k}. \tag{48}$$

Note that in practice the sum over k is taken only a narrow band of frequencies centered where $P_{\alpha k}$ is peaked. We are assuming that the noise is roughly stationary during one SFT; if we further assume that the noise is roughly white in the narrow band around the peak in $P_{\alpha k}$ then we can make the replacement

$$2\Delta t \sigma_\alpha^2 \leftrightarrow S_\alpha, \tag{49}$$

where S_α is the one-sided power spectral density of the noise in the narrow band.

$$\bar{F}_a = \sum_{\alpha=0}^{M-1} \left(\frac{a_\alpha}{\sigma_\alpha}\right) Q_\alpha \sum_k \left(\frac{\Delta t \tilde{x}_{\alpha k}/\sqrt{T_{SFT}}}{\sqrt{S_\alpha}}\right) P_{\alpha k}, \tag{50}$$

$$\bar{F}_b = \sum_{\alpha=0}^{M-1} \left(\frac{b_\alpha}{\sigma_\alpha}\right) Q_\alpha \sum_k \left(\frac{\Delta t \tilde{x}_{\alpha k}/\sqrt{T_{SFT}}}{\sqrt{S_\alpha}}\right) P_{\alpha k}, \tag{51}$$

Note that the LALDemod function in the LAL library can easily compute $\bar{F}_a$ and $\bar{F}_b$ as long as the inputs are the normalized detector response amplitudes, a_α/σ_α and b_α/σ_α, and normalized SFTs. The estimates of the basis amplitudes are then:

$$A_1 = \frac{2\sqrt{2N}}{MN\bar{D}} \left[\bar{B}\mathrm{Re}(\bar{F}_a) - \bar{C}\mathrm{Re}(\bar{F}_b)\right], \tag{52}$$

$$A_2 = \frac{2\sqrt{2N}}{MN\bar{D}} \left[\bar{A}\mathrm{Re}(\bar{F}_b) - \bar{C}\mathrm{Re}(\bar{F}_a)\right], \tag{53}$$

$$A_3 = \frac{2\sqrt{2N}}{MN\bar{D}} \left[\bar{B}\mathrm{Im}(\bar{F}_a) - \bar{C}\mathrm{Im}(\bar{F}_b)\right], \tag{54}$$

$$A_4 = \frac{2\sqrt{2N}}{MN\bar{D}} \left[\bar{A}\mathrm{Im}(\bar{F}_b) - \bar{C}\mathrm{Im}(\bar{F}_a)\right], \tag{55}$$

$$\tag{56}$$

These can be used to estimate the unknown parameters, h_0, ι, ψ, and Φ_0, as described in the last section.

References

[1] P. Jaranowski, A. Królak, and B.F. Schutz, Phys. Rev. D **58**, 063001 (1998); gr-qc/9804014.