LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
- LIGO -
CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Technical Note **LIGO-T060162-00-Z** 2006 June 5

Comments on Anisotropic Stochastic Background Searches

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1 Definition of the Stochastic Background Spectrum

The standard plane-wave expansion for a gravitational wave is

$$h_{ab}(\vec{r},t) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{n}}\, h_A(f,\hat{n})\, e_{A\,ab}(\hat{n}) \exp\left(i2\pi f \left[t - \frac{\hat{n}\cdot\vec{r}}{c}\right]\right) \tag{1.1}$$

where $\{\overleftrightarrow{e}_A(\hat{n})\}$ are the usual transverse traceless basis tensors, normalized to obey

$$e_{A\,ab}(\hat{n})\, e^{ab}_{A'}(\hat{n}) = 2\delta_{AA'} \tag{1.2}$$

The spectrum $H(f)$ for an isotropic stochastic background is defined (e.g., in eqn (2.11) of Allen and Romano[1]) by

$$\langle h^*_A(f,\hat{n})\, h_{A'}(f',\hat{n}')\rangle = \delta^2(\hat{n},\hat{n}')\, \delta_{AA'}\, \delta(f-f')\, H(f) \tag{1.3}$$

The natural extension of this definition to a potentially anisotropic, unpolarized background is

$$\langle h^*_A(f,\hat{n})\, h_{A'}(f',\hat{n}')\rangle = \delta^2(\hat{n},\hat{n}')\, \delta_{AA'}\, \delta(f-f')\, H(f,\hat{n}) \tag{1.4}$$

If we further assume that the spatial distribution of the background is non-frequency-dependent[1], we can factor the background strength to get

$$\langle h^*_A(f,\hat{n})\, h_{A'}(f',\hat{n}')\rangle = \delta^2(\hat{n},\hat{n}')\, \delta_{AA'}\, \delta(f-f')\, H(f)\, \mathcal{P}(\hat{n}) \tag{1.5}$$

which is equation (2.8) of Allen and Ottewill [2]. The separation into $H(f)$ and $\mathcal{P}(\hat{n})$ is of course arbitrary, but we will get the closest correspondence to the isotropic formulas if we choose it so that

$$\iint d^2\Omega_{\hat{n}} \mathcal{P}(\hat{n}) = 4\pi\ . \tag{1.6}$$

This is the normalization chosen by Allen and Ottewill [2], in their equation (3.10).

2 Expected Cross-Correlation in a Pair of Detectors

Let detector i at position $\vec{r}_i$ have response tensor $\overleftrightarrow{d_i}$, so that the strain it measures is

$$h_i(t) = d^{ab}_i\, h_{ab}(\vec{r}_i,t) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{n}}\, h_A(f,\hat{n})\, d^{ab}_i\, e_{A\,ab}(\hat{n}) \exp\left(i2\pi f \left[t - \frac{\hat{n}\cdot\vec{r}_i}{c}\right]\right) \tag{2.1}$$

Note that in addition to the explicit time dependence, there is a slow time variation hidden in the quantities $\hat{n}\cdot\vec{r}_i$ and $d^{ab}_i\, e_{A\,ab}(\hat{n}) = F^A_i(\hat{n})$. This is because $\hat{n}$ is a sky direction vector associated with a fixed right ascension and declination while $\vec{r}_i$ and $\overleftrightarrow{d_i}$ are quantities with

[1] This is not a good assumption in general, but we should be able to resolve a background into a sum of contributions which individually satisfy it.

constant components in an Earth-fixed basis, and are thus rotating with respect to the sky-fixed basis. However, if the data are analyzed in chunks which are small compared to the rotation period of one siderial day, we can neglect that time dependence in identifying

$$\widetilde{h}_i(f) \approx \sum_{A=+,\times} \iint d^2\Omega_{\hat{n}}\, h_A(f,\hat{n})\, d_i^{ab}\, e_{A\,ab}(\hat{n}) \exp\left(-i2\pi f \frac{\hat{n}\cdot\vec{r}_i}{c}\right) \tag{2.2}$$

The usual calculation tells us that

$$\begin{aligned} \langle \widetilde{h}_i^*(f)\, \widetilde{h}_j(f') \rangle &= \delta(f-f') \iint d^2\Omega_{\hat{n}} \sum_{A=+,\times} d_i^{ab} e_{A\,ab}(\hat{n})\, e_{A\,ab}(\hat{n})\, e_{A\,cd}(\hat{n})\, d_j^{cd}\, H(f,\hat{n}) e^{i2\pi f \hat{n}\cdot(\vec{r}_i-\vec{r}_j)/c} \\ &= 2\delta(f-f') \iint d^2\Omega_{\hat{n}}\, d_i^{ab}\, P^{\mathrm{TT}\hat{n}ab}{}_{cd}\, d_j^{cd}\, H(f,\hat{n}) e^{i2\pi f \hat{n}\cdot(\vec{r}_i-\vec{r}_j)/c} \end{aligned} \tag{2.3}$$

where $P^{\mathrm{TT}\hat{n}ab}{}_{cd}$ is the projector onto traceless, symmetric tensors transverse to the unit vector $\hat{n}$, which can be expanded in the standard polarization basis as

$$P^{\mathrm{TT}\hat{n}ab}{}_{cd} = \frac{1}{2} \sum_{A=+,\times} e_A^{ab}(\hat{n})\, e_{A\,cd}(\hat{n}) \tag{2.4}$$

If we recall the overlap reduction function appropriate for isotropic stochastic background searches

$$\gamma_{12}(f) = d_{1\,ab}\, d_2^{cd} \frac{5}{4\pi} \iint d^2\Omega_{\hat{n}}\, P^{\mathrm{TT}\hat{n}ab}{}_{cd}\, e^{i2\pi f \hat{n}\cdot(\vec{r}_2-\vec{r}_1)/c} \tag{2.5}$$

and call integrand, which depends on both frequency and sky direction,

$$\frac{d^2\gamma_{12}}{d^2\Omega}(f,\hat{n}) = \frac{5}{4\pi} \left(d_{1\,ab}\, d_2^{cd}\, P^{\mathrm{TT}\hat{n}ab}{}_{cd}\right) \left(e^{i2\pi f \hat{n}\cdot(\vec{r}_2-\vec{r}_1)/c}\right) \tag{2.6}$$

then (2.3) becomes

$$\begin{aligned} \langle \widetilde{h}_i^*(f)\, \widetilde{h}_j(f') \rangle &= \delta(f-f') \frac{8\pi}{5} \iint d^2\Omega_{\hat{n}} \frac{d^2\gamma_{12}}{d^2\Omega}(f,\hat{n})\, H(f,\hat{n}) \\ &= \delta(f-f') \frac{8\pi}{5} H(f) \iint d^2\Omega_{\hat{n}} \frac{d^2\gamma_{12}}{d^2\Omega}(f,\hat{n})\, \mathcal{P}(\hat{n}) \end{aligned} \tag{2.7}$$

If we extend the definition of the overlap reduction function to one specific to a particular background $\mathcal{P}(\hat{n})$ as follows:

$$\gamma_{12}^{\mathcal{P}}(f) = \iint d^2\Omega_{\hat{n}} \frac{d^2\gamma_{12}}{d^2\Omega}(f,\hat{n})\, \mathcal{P}(\hat{n}) \tag{2.8}$$

then we have a generalization of the usual formula:

$$\langle \widetilde{h}_i^*(f)\, \widetilde{h}_j(f') \rangle = \delta(f-f') \frac{8\pi}{5} \gamma_{12}^{\mathcal{P}}(f) H(f) \tag{2.9}$$

This is the equivalent of equation (3.56) in Allen and Romano[1]. [See also Allen & Romano's equation (2.15).]

Note that $\gamma_{12}^{\mathcal{P}}(f)$ depends on the detectors, the spatial distribution of the source, and also on siderial time.

Note also that, subject to the normalization (1.6), a background coming from a single direction $\hat{n}_0$ is described by a distribution

$$\mathcal{P}_{\hat{n}_0}(\hat{n}) = 4\pi\delta^2(\hat{n}, \hat{n}_0) \tag{2.10}$$

which corresponds to the overlap reduction function

$$\begin{aligned}\gamma_{12}^{\hat{n}_0}(f) &= 4\pi\frac{d^2\gamma_{12}}{d^2\Omega}(f, \hat{n}_0) = 5\left(d_{1\,ab}\, d_2^{cd}\, P^{\mathrm{TT}\hat{n}ab}{}_{cd}\right)\left(e^{i2\pi f\hat{n}\cdot(\vec{r}_2-\vec{r}_1)/c}\right) \\ &= \frac{5}{2}\left(\sum_{A=+,\times} F_{A\,1}(\hat{n})\, F_{A\,2}(\hat{n})\right)\left(e^{i2\pi f\hat{n}\cdot(\vec{r}_2-\vec{r}_1)/c}\right)\end{aligned} \tag{2.11}$$

This is 5 times what Ballmer[3] calls $\gamma_{\hat{\Omega}}$.

3 Calculation of Overlap Reduction Function Integrand

For a given sky direction, the overlap reduction function integrand (2.6) factors as shown above into a piece depending only on detector orientation and a factor depending only on the frequency and separation. Note that in a basis co-rotating with the Earth, the components of the detector response tensors $\overleftrightarrow{d_1}$ and $\overleftrightarrow{d_2}$ and the separation vector $\vec{r}_2 - \vec{r}_1$ are fixed, but the direction $\hat{n}$ associated with a particular right ascension and declination changes with siderial time.

The explicit form of the projector $P^{\mathrm{TT}\hat{n}ab}{}_{cd}$, and thus of the overlap reduction function integrand $\frac{d^2\gamma_{12}}{d^2\Omega}(f, \hat{n})$ can be worked out by noting that it must be traceless and symmetric on both pairs of indices ($\{ab\}$ and $\{cd\}$); with $\hat{n}$ as the only preferred direction, there are only three independent tensors which can be created with these properties:

$$T_1{}^{ab}_{cd} = P^{\mathrm{T}ab}{}_{cd} \tag{3.1a}$$

$$T_2{}^{ab}_{cd}(\hat{n}) = P^{\mathrm{T}ab}{}_{ef}\hat{n}^f\hat{n}_g P^{\mathrm{T}eg}{}_{cd} \tag{3.1b}$$

$$T_3{}^{ab}_{cd}(\hat{n}) = P^{\mathrm{T}ab}{}_{ef}\hat{n}^e\hat{n}^f\hat{n}_g\hat{n}_h P^{\mathrm{T}gh}{}_{cd} \tag{3.1c}$$

where

$$P^{\mathrm{T}ab}{}_{cd} = \delta^a_{(c}\delta^b_{d)} - \frac{1}{3}\delta^{ab}\delta_{cd} \tag{3.2}$$

is the projector onto traceless symmetric tensors. We can thus write

$$P^{\mathrm{TT}\hat{n}ab}{}_{cd} = \sum_{n=1}^{3}\beta_n T_n{}^{ab}_{cd}\ ; \tag{3.3}$$

to figure out the coëfficients $\{\beta_n\}$ we just need to contract each of the $\{T_{n}{}^{ab}_{cd}\}$ with $P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd}$ and each other. The former set of contractions is straightforward, because $P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd}$ is a projector onto a two-dimensional subspace which is transverse to $\hat{n}$.

$$T_{1}{}^{cd}_{ab} P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd} = P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}ab} = 2 \tag{3.4a}$$

$$T_{2}{}^{cd}_{ab} P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd} = P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}ac} n_b n^c = 0 \tag{3.4b}$$

$$T_{3}{}^{cd}_{ab} P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd} = P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd} n_a n_b n^c n^d = 0 \tag{3.4c}$$

The latter set of contractions is worked out in the appendix of [4] [equation (21)] and they give us

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} T_{1}{}^{cd}_{ab} \\ T_{2}{}^{cd}_{ab} \\ T_{3}{}^{cd}_{ab} \end{pmatrix} P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd} = \overbrace{\begin{pmatrix} T_{1}{}^{cd}_{ab} \\ T_{2}{}^{cd}_{ab} \\ T_{3}{}^{cd}_{ab} \end{pmatrix} \begin{pmatrix} T_{1}{}^{ab}_{cd} & T_{2}{}^{ab}_{cd} & T_{3}{}^{ab}_{cd} \end{pmatrix}}^{\begin{pmatrix} 5 & 5/3 & 2/3 \\ 5/3 & 17/18 & 4/9 \\ 2/3 & 4/9 & 4/9 \end{pmatrix}} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \tag{3.5}$$

Inverting the matrix gives

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{4} \\ -1 & 4 & -\frac{5}{2} \\ \frac{1}{4} & -\frac{5}{2} & \frac{35}{8} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1/2 \end{pmatrix} \tag{3.6}$$

so

$$P^{\mathrm{TT}\hat{n}ab}_{\phantom{\mathrm{TT}\hat{n}ab}cd} = P^{\mathrm{T}ab}_{\phantom{\mathrm{T}ab}cd} - 2P^{\mathrm{T}ab}_{\phantom{\mathrm{T}ab}ef}\hat{n}^f \hat{n}_g P^{\mathrm{T}eg}_{\phantom{\mathrm{T}eg}cd} + \frac{1}{2} P^{\mathrm{T}ab}_{\phantom{\mathrm{T}ab}ef}\hat{n}^e \hat{n}^f \hat{n}_g \hat{n}_h P^{\mathrm{T}gh}_{\phantom{\mathrm{T}gh}cd} \tag{3.7}$$

which means

$$\frac{d^2\gamma_{12}}{d^2\Omega}(f,\hat{n}) = \frac{5}{4\pi} e^{i2\pi f \hat{n}\cdot(\vec{r}_2-\vec{r}_1)/c} \left[d_1^{\mathrm{T}ab} d^{\mathrm{T}}_{2\,ab} - 2 d_1^{\mathrm{T}ab} \hat{n}_b \hat{n}^c d^{\mathrm{T}}_{2\,ac} + \frac{1}{2} d_1^{\mathrm{T}ab} \hat{n}_a \hat{n}_b \hat{n}^c \hat{n}^d d^{\mathrm{T}}_{2\,cd} \right] \tag{3.8}$$

where

$$d^{\mathrm{T}ab} = P^{\mathrm{T}ab}_{\phantom{\mathrm{T}ab}cd} d^{cd} = d^{ab} - \frac{1}{3}\delta^{ab} d^c_c \tag{3.9}$$

is the traceless part of the response tensor (which, for an interferometer, is just the tensor itself).

This calculation is implemented in the matapps routine `orfintegrand()` in the directory `src/utilities/detgeom/matlab`; if you pass it a $3 \times N$ matrix representing N different sky directions in Earth-fixed Cartesian coördinates, and a column vector ($M \times 1$ matrix) of M frequencies, it will return an $M \times N$ matrix containing the value of $\frac{d^2\gamma_{12}}{d^2\Omega}$ at each frequency and sky direction. The routine `getcartesiandirectionfromsource()` converts the combination of declination and minus hour angle (which is right ascension minus Greenwich Mean Siderial Time) into Earth-fixed Cartesian unit vectors.

References

[1] Allen B and Romano J D 1999 *Phys Rev* **D59** 102001; e-Print: gr-qc/9710117.

[2] Allen B and Ottewill A C 1997 *Phys Rev* **D56** 545; e-Print: gr-qc/9607068.

[3] Ballmer S W "A radiometer for stochastic gravitational waves"; e-Print: gr-qc/0510096

[4] Whelan J T "Stochastic Gravitational Wave Measurements with Bar Detectors: Dependence of Response on Detector Orientation"; submitted to *Class Quant Grav*; e-Print: gr-qc/0509109.