Effect of sampling on frequency domain analysis

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We review the well-known effects of digital sampling on the frequency domain analysis of an analog signal, with emphasis on the effects upon our measurements. This discussion follows the notation of Gaskill[1].
The signal to be sampled is assumed to be a harmonically varying signal, damped in a way that can be characterized by a time constant, τ. That is, we assume the original analog signal is of the form:

$$f(t) = \begin{cases} 0 & t < 0 \\ A\cos(2\pi f_0 t)\cdot e^{-t/\tau} & t > 0 \end{cases} \qquad (1)$$

(Note that the frequency f_0 includes any shift due to the assumed damping). This signal is sampled N = 1400 times at a time interval of $t_S = 0.3$ s, resulting in a total sampling time of $N\cdot t_S = 420$ s. As we will see, it is significant that this is much larger than the damping time, $\tau = 35\text{-}40$ s. The sampled data can be written formally as:

$$f_{SAMP}(t) = f(t)S(t) \qquad (2)$$

where $f(t)$ is from Eq (1) and $S(t)$ is a sampling function, that we write as:

$$S(t) = \sum_{n=0}^{N} \delta(t - nt_S) \qquad (3)$$

That is, the sampling function is a train of N Dirac delta functions, separated by the sampling time, t_S. Using the so-called "rectangle" function, defined as:

$$rect(t) \equiv \begin{cases} 0, t > 1/2 \\ 0.5, t = 1/2 \\ 1, t < 1/2 \end{cases} \qquad (4)$$

we can rewrite the sampling function as:

$$\begin{aligned} S(t) = & \left[\sum_{n=\infty}^{\infty} \delta(t - nt_S)\right] rect\left(\frac{t - Nt_S/2}{Nt_S}\right) \\ = & \frac{1}{t_S} comb\left(\frac{t}{t_S}\right) rect\left(\frac{t - Nt_S/2}{Nt_S}\right) \end{aligned} \qquad (5)$$

(where we've defined the "comb" function, again following Gaskill). This will simplify some of the following discussion [I hope…]. The frequency domain analysis then

[1] J. D. Gaskill, *Linear Systems, Fourier Transforms and Optics,* Wiley, New York, 1978

proceeds by performing a Fourier transform upon the sampled function of Eq. (2). Formally, this is given by:

$$F_{SAMP}(f) = \left[\begin{array}{l} \mathcal{F}\left(A\cos(2\pi f_0 t)\right) * \mathcal{F}\left(e^{-t/\tau}\right) * \\ \mathcal{F}\left(\frac{1}{t_S} comb\left(\frac{t}{t_S}\right)\right) * \mathcal{F}\left(rect\left(\frac{t - Nt_S/2}{Nt_S}\right)\right) \end{array} \right] \tag{6}$$

where "*" denotes the convolution operation, and the script F denotes the Fourier transform operations. We have used the convolution theorem, which holds that the Fourier transform of a product of functions is the convolution of the transform of each function. This transform will give a peak whose location and width are determined, obviously, by f_0 and τ, but also by t_S and N, as we shall discuss. It is useful to break the discussion into two parts, one related to broadening and one related to the location of the peak.

Broadening

Only the exponential decay and the "rectangle" function contribute to broadening (the transform of the cosine and the comb result in Dirac delta functions, as we discuss later). We can then define a broadening function by:

$$F_B(f) = \left[\mathcal{F}\left(e^{-t/\tau}\right) * \mathcal{F}\left(rect\left(\frac{t - Nt_S/2}{Nt_S}\right)\right) \right] \tag{7}$$

If we do the transform explicitly, we have:

$$F_B(f) = \int_0^{Nt_S} e^{-i2\pi ft} e^{-t/\tau} dt = \frac{1 - e^{-\left(i2\pi f + \frac{1}{\tau}\right)Nt_S}}{i2\pi f + \frac{1}{\tau}} \tag{8}$$

The amplitude of $F_B(f)$ is given by:

$$\left|F_B(f)\right| = \left(F_B F_B^*\right)^{\frac{1}{2}} = \tau \sqrt{\frac{1 + e^{-2Nt_S/\tau} - 2e^{-Nt_S/\tau}\cos(2\pi f Nt_S)}{1 + (2\pi f\tau)^2}} \tag{9}$$

and the phase by:

$$\tan\Phi = -\frac{2\pi f\tau + e^{-Nt_S/\tau}\left(\sin(2\pi f Nt_S) - 2\pi f\tau\cos(2\pi f Nt_S)\right)}{1 - e^{-Nt_S/\tau}\left(1 + 2\pi f\tau\sin(2\pi f Nt_S)\right)} \tag{10}$$

These expressions seem forbidding, but the essential point can be seen if we consider two limits: $Nt_S >> \tau$, the condition that the signal is sampled for a time long compared to the decay time (i.e., that it decays significantly during the acquisition of the data), and $Nt_S << \tau$, the condition that the signal is sampled for a time short compared to the decay time (i.e., that it does *not* decay significantly during the acquisition of the data). In the first case, the amplitude and phase take the form:

$$\begin{aligned} \left|F_B(f)\right| &\cong \frac{1}{\sqrt{1+(2\pi f\tau)^2}} \\ \tan\Phi &\cong -2\pi f\tau \end{aligned} \tag{11}$$

and in the latter case:

$$\begin{aligned} \left|F_B(f)\right| &\cong \frac{\sin(\pi f N t_S)}{\pi f} \\ \tan\Phi &\cong -\tan(\pi f N t_S) \cong 0 \end{aligned} \tag{12}$$

That these are the expected forms can be seen most easily by returning to Eq. 8 and taking the limit that $(Nt_S)\rightarrow\infty$ to get Eq. (11) and $\tau\rightarrow\infty$ to get Eq. (12). From Eq. (11), we see that the width of the transformed signal is of order $1/\tau$. That is, if the time during which data is sampled is large compared to the damping time, then the width of the transformed data will be given by the damping time. Conversely, Eq. (12) shows that in the event that the total sampling time is short compared to the damping time, then the width is given by the inverse of the total sampling time, $1/Nt_S$. In general, the width of the transformed data will be reflected by the function of Eq. (9) and will be of order $(1/\tau + 1/Nt_S)$. In our measurements, τ_{PEND} = 35 sec and τ_{PITCH} = 40 sec, while Nt_S is 420 sec, and so we are in the regime described by Eq. (11).

Aliasing

The other two functions in $F_{SAMP}(f)$ (Eq. (6)) determine the location (in frequency space) of the function given by Eq. (8). That is, Eq. (8) determines the shape of the oscillator peak and the remainder:

$$F_{PEAK}(f) = \left[\mathcal{F}\left(A\cos(2\pi f_0 t)\right) * \mathcal{F}\left(\frac{1}{t_S}comb\left(\frac{t}{t_S}\right)\right)\right] \tag{13}$$

determines the location. Performing the transforms, this becomes:

$$F_{PEAK}(f) = A\left[\left[\delta(f-f_0)+\delta(f+f_0)\right] * \left(comb(t_S f)\right)\right] \tag{14}$$

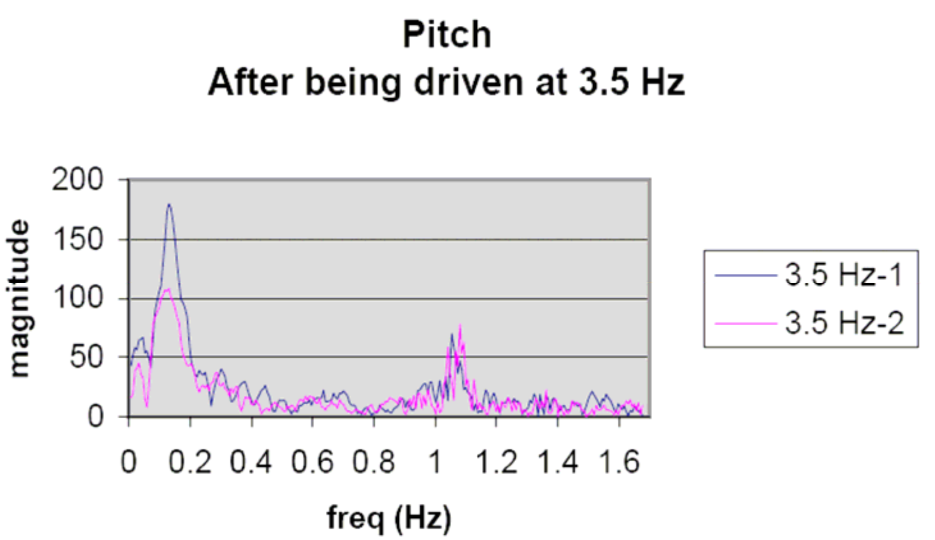
And performing the convolution, we have:

$$F_{PEAK}(f) = A\left[\sum_{n=-\infty}^{n=+\infty}\left[\delta\left(f - f_o - \frac{n}{t_S}\right) + \delta\left(f + f_o - \frac{n}{t_S}\right)\right]\right] \tag{15}$$

Using this, the transform of the sampled data is given by:

$$\begin{aligned} F_{SAMP}(f) &= F_B(f) * F_{PEAK}(f) \\ &= A\left[\sum_{n=-\infty}^{n=+\infty}\left[F_B\left(f - f_o - \frac{n}{t_S}\right) + F_B\left(f + f_o - \frac{n}{t_S}\right)\right]\right] \end{aligned} \tag{16}$$

That is, the transformed data consists of repeated versions of the broadened function from Eq. (8), centered at f_0+n/t_S and $-(f_0-n/t_S)$. There are several conclusions to be drawn. Suppose that the frequency f_0 satisfies $0<f_0<1/(2t_S)$ (this is the Nyquist condition). Then clearly f_0 satisfies $(2n)/(2t_S)<f_0+n/t_S<(2n+1)/(2t_S)$. That is, the first term in Eq. (16) ensures that every other interval ($-1/t_S$ to $-1/2t_S$, 0 to $1/2t_S$, $1/t_S$ to $3/2t_S$, and so on) includes a copy of the peak corresponding to f_0. Further, since $(2n-1)/(2t_S) < n/t_S - f_0 < n/t_S$, the second term in Eq. (16) ensures that there is a copy in the remaining intervals ($-3/2t_S$ to $-1/t_S$, $-1/2t_S$ to 0, $1/2t_S$ to $1/t_S$, and so on). Therefore, only the interval f = 0 to $1/2t_S$ need be considered and there will be precisely one copy of the function $F_B(f)$ in that interval (from the first term in Eq. (16) with n = 0). However, suppose that the frequency f_0 satisfies $1/2t_S < f_0 < 1/t_S$. This is a condition of undersampling, in which the sampling rate is too short to faithfully reproduce the sampled signal. It is again the case that every interval n/t_S to $(n+1/2)/t_S$ contains the same information. But in this case, the peak that appears in the interval 0 to $1/2t_S$ is from the second term in Eq. (16) and occurs at the frequency $f = 1/t_S - f_0$. Referring to Figure:



We see a peak at 1.05 Hz, which is the natural pendular motion of the pendulum and a peak at ~0.15 Hz. The 0.15 Hz peak is an alias of the pitch at 3.18 Hz : (3.33 Hz – 3.18 Hz) = 0.15 Hz.

Multiple Oscillations Simultaneously

Now suppose that we have more than one oscillation in the signal to be sampled. Equation (1) then becomes:

$$f(t)=\begin{cases}0 & t<0\\ \sum_{j=1}^{M} A_j\cos(2\pi f_j t)\cdot e^{-t/\tau_j} & t>0\end{cases}\qquad (17)$$

The sampling function is as given in Eq. (5) and the sampled data is still given formally by Eq. (2). The Fourier transform of the sampled data is then:

$$\begin{aligned}
F_{SAMP}(f) &= \left[\begin{array}{l}\sum_{j=1}^{M}\left[\mathcal{F}\left(A_j\cos(2\pi f_j t)\right)*\mathcal{F}\left(e^{-t/\tau_j}\right)\right]*\\ \mathcal{F}\left(\frac{1}{t_S}comb\left(\frac{t}{t_S}\right)\right)*\mathcal{F}\left(rect\left(\frac{t-Nt_S/2}{Nt_S}\right)\right)\end{array}\right]\\
&=\sum_{j=1}^{M}\left[\begin{array}{l}\mathcal{F}\left(A_j\cos(2\pi f_j t)\right)*\mathcal{F}\left(e^{-t/\tau_j}\right)*\\ \mathcal{F}\left(\frac{1}{t_S}comb\left(\frac{t}{t_S}\right)\right)*\mathcal{F}\left(rect\left(\frac{t-Nt_S/2}{Nt_S}\right)\right)\end{array}\right]\\
&=\sum_{j=1}^{M}\left[\begin{array}{l}\mathcal{F}\left(rect\left(\frac{t-Nt_S/2}{Nt_S}\right)\right)*\mathcal{F}\left(e^{-t/\tau_j}\right)*\\ \mathcal{F}\left(A_j\cos(2\pi f_j t)\right)*\mathcal{F}\left(\frac{1}{t_S}comb\left(\frac{t}{t_S}\right)\right)\end{array}\right]\\
&=\sum_{j=1}^{M}\left[\begin{array}{l}\mathcal{F}\left(A_j\cos(2\pi f_j t)\right)*\mathcal{F}\left(e^{-t/\tau_j}\right)*\\ \mathcal{F}\left(\frac{1}{t_S}comb\left(\frac{t}{t_S}\right)\right)*\mathcal{F}\left(rect\left(\frac{t-Nt_S/2}{Nt_S}\right)\right)\end{array}\right]\\
&=\sum_{j=1}^{M}\left[F_{BROAD,j}(f)*F_{PEAK,j}(f)\right]
\end{aligned}\qquad (18)$$

That is, the transformed sample data will consist of M peaks that show the same properties as discussed above for the single peak. Each peak will be broadened in the same way as discussed before, either by the damping represented by τ_j or by the sampling time, Nt_s. Also, each peak location will be determined by the oscillation frequency and the sampling rate, $1/t_s$, again as discussed previously. Any aliasing will occur for each peak independently, depending upon the frequency of that peak, f_j, and the Nyquist sampling rate, $1/(2\ t_s)$. Note that there are no peaks corresponding to the sum or difference of any of the oscillation frequencies, as might be thought at first. But this is expected for a linear process like a Fourier transform.