Thermorefractive Noise and Electrocaloric Dissipation

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ABSTRACT

In materials with temperature-dependent permittivity, the application of a dynamic electric field induces temperature gradients and thus irreversible heat flow, leading to dissipation of energy. When the applied electric field is the optical field of a laser beam in LIGO, the energy dissipated due to this phenomenon—which is known as the electrocaloric effect—can be related to thermorefractive noise, a particular type of thermal noise in the interferometer, by the fluctuation-dissipation theorem. By computing the amount of electrocaloric dissipation, then, it is possible to determine an analytic expression for thermorefractive noise in the LIGO optics, based on various known quantities and materials parameters. Such calculations may be performed in two distinct cases: the laser beam can be assumed to penetrate the bulk of the optic (as in the beam splitter), or it may be assumed to penetrate only the surface coating and then reflect back (as in the mirrors). This paper examines both situations assuming optics of infinite radius. Future research might extend the present analysis to more strictly realistic geometries, such as the case of optics with finite size.

BACKGROUND

An interferometric gravitational-wave detector such as LIGO must compete with a variety of noise sources in its search for gravitational waves. One major source of noise is the random thermal fluctuations which inevitably occur when materials are not at absolute zero. In fact, thermal noise is one of the fundamental limitations on the sensitivity of interferometric gravitational-wave detectors.

One particular type of thermal noise is thermorefractive noise, which arises from thermal fluctuations of the refractive index in the detector optics. When a laser beam passes through the bulk of an optic in LIGO, such as the beam splitter or an input test mass, it "sees" some index of refraction n. Correspondingly, when the laser penetrates the reflective surface coating of a mirror, it sees two alternating indices of refraction n_1 and n_2, since the reflective coating is comprised of alternating layers of two different materials. In either case, the indices of refraction have some temperature dependence. So any thermal fluctuations in the material lead to fluctuations in the indices of refraction, which amount to fluctuations of optical thickness. This introduces phase noise into the laser beam.

Thermorefractive noise is a reasonably well-understood phenomenon, and expressions for the amount of thermorefractive noise in the LIGO optics have been derived previously[1, 2]. At least one aspect, however, remains unexplored. The fluctuation-dissipation theorem[3] reveals that whenever a mechanism for energy dissipation is present in a system, some source of noise will also be present. In other

words, a dissipative system must include some fluctuating quantity, and vice versa. This provides motivation to search for a relationship between thermorefractive noise and a corresponding mechanism for dissipation.

That dissipation mechanism has been identified as the electrocaloric effect. Like thermorefractive noise, the electrocaloric effect requires a material whose permittivity (and thus index of refraction) depends on temperature. In such a material, the application of a dynamic electric field induces local temperature gradients. This leads to dissipation because heat flows irreversibly down these gradients—hence the term "electrocaloric dissipation." This effect is present in the LIGO optics due to the interaction between the laser optical electric fields and the materials in the LIGO optics. Of course, thermorefractive noise arises as a result of a similar interaction.

From these considerations, it is reasonable to expect that an expression for thermorefractive noise could be calculated in a given optical setup by first calculating the electrocaloric dissipation and then applying the fluctuation-dissipation theorem. This is the strategy employed below; the results are in basic agreement with the previous result derived using purely thermodynamic considerations.

METHODS

The electrocaloric effect is a source of heat and may therefore be included in the heat equation as a source term. From entropy considerations (see Appendix), we find that the source term is

$$\frac{\partial Q}{\partial t} = \frac{\partial \theta_{ad}}{\partial t} \frac{T_0 \varepsilon'}{2 C_V} \frac{\partial}{\partial t} E^2,$$

where θ_{ad} is the local adiabatic temperature change due to the electrocaloric effect, T_0 is the mean temperature, ε' is the temperature derivative of the permittivity, C_V is the volumetric heat capacity, and E is the applied electric field. In LIGO, a sideband electric field is added on to the "carrier" optical field of the laser beam. Both the optical field and the sideband have Gaussian profiles, which means that the fields may be mathematically represented as follows:

$$E_{carrier} = E_{c0} e^{-r^2/w^2} e^{-ikz - i\omega t}$$

and

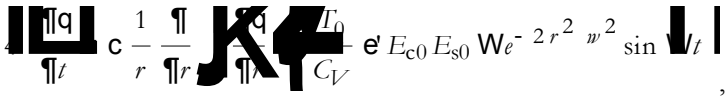
$$E_{sideband} = E_{s0} e^{-r^2/w^2} e^{-ikz - i(\omega + \Omega)t},$$

where $E_{s0} << E_{c0}$ and $\Omega << \omega$. Here k is the wave number, not the thermal conductivity, and w is the laser beam waist. We have used cylindrical coordinates; the z-axis is perpendicular to the surface of the optic.

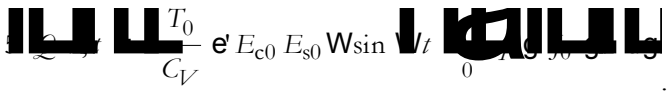
In acoustic beating, two waves of slightly different frequencies are added together. At any given point, the amplitude of the resultant wave varies periodically in time at the beat frequency $\omega_{beat} = |\omega_1 - \omega_2|$, where ω_1 and ω_2 are the frequencies of the contributing waves. A completely analogous electrical effect occurs here. The beat frequency is Ω and the square of the "beat amplitude" is $E^2 = |E|^2 = EE^*$, where E^* is the complex conjugate of $E = E_{carrier} + E_{sideband}$. This beat frequency is the only important frequency in the problem; the oscillation at frequency ω is so rapid that the material in the optic cannot possibly respond to it. On the other hand, the material can keep up with the much smaller beat frequency Ω, so the source term in the heat equation for the optic depends on Ω.

Case 1: The laser penetrates the bulk of an optic, as in the beam splitter or the input test masses.

We wish to solve the relevant heat equation



where χ is the thermal diffusivity and θ is the overall steady-state temperature change in the material of the optic as a function of position and time. In order to accomplish this, we begin by writing the source term as an integral expansion:



Note that the source term is not z-dependent. This is a consequence of assuming that $c/\Omega >> d$, where c is the speed of light and d is the thickness of the optic.

We also expand the solution:

We use the zero-order Bessel function because this is the standard solution to the heat equation in cylindrical coordinates when dependence on φ, the angular polar coordinate, and z both drop out. This is an adaptation of the standard technique of eigenfunction expansion. There is an arbitrary phase shift $\delta(\gamma)$ in equation (6) because the solution is expected to be out of phase with the source term.

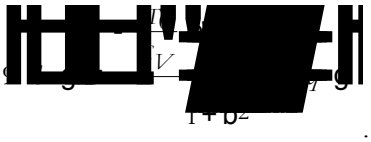
We now insert the expansions (5) and (6) into the heat equation (4). Bringing the derivatives inside the integrals, equating the integrands, and using recurrence relations between different Bessel functions and their derivatives[4] so as to remove J_1 and J_2, gives

after cancellation of the remaining $J_0(\gamma r)$ in each term. Note that $\beta = \chi\gamma^2/\Omega$.

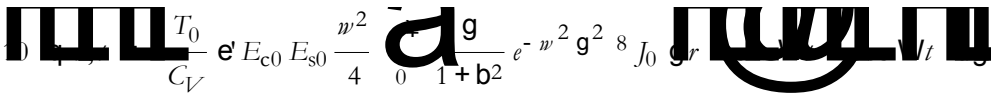
We proceed by expanding the trigonometric functions on the right-hand side using the angle-sum identities. Since only $\sin(\Omega t)$ appears on the LHS, and all the time dependence on the RHS is taken care of by $\cos(\Omega t)$ and $\sin(\Omega t)$, it must be the case that the sum of the $\cos(\Omega t)$ terms on the RHS is zero. After simplifying, we conclude that

(8) $\cot(\delta(\gamma)) = -\beta$

since $b(\gamma) \neq 0$. Equating all the $\sin(\Omega t)$ terms in like manner and using expressions for $\sin(\delta(\gamma))$ and $\cos(\delta(\gamma))$ derived from (8) gives

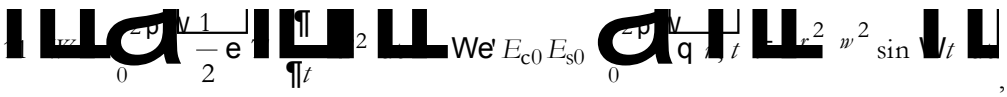


Determining $q(\gamma)$ is straightforward using Hankel transforms and a sophisticated table of integrals[5]. Inserting $q(\gamma)$ into $b(\gamma)$, and in turn inserting $b(\gamma)$ into the solution θ, gives

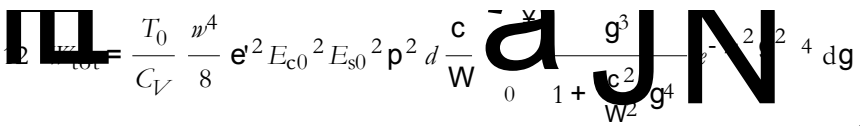


after expanding and simplifying $\sin(\Omega t + \delta(\gamma))$. This gives us a particular solution. The homogenous solutions are the solutions to the homogenous equation corresponding to (4) and therefore decay exponentially in time. Since we assume that a steady state has already been reached (the driving term has been present for all time), the homogenous solutions have already decayed away and the particular solution is the general solution.

We have now determined the temperature distribution throughout the optic and are therefore in a position to calculate the dissipation. The dissipation per unit volume per beat period is given by



where we have performed the Taylor expansion $\varepsilon(T) = \varepsilon + \varepsilon'\theta$. Integrating over the entire volume of the finite-thickness, infinite-radius optic gives a total dissipation of

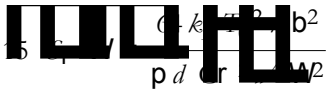


We may then apply the fluctuation-dissipation theorem,

where k_B is Boltzmann's constant and $S_D(\Omega)$ is the spectral density of the electric displacement $D = \varepsilon E$. (The electric displacement D and the electric field E are, in this case, randomly fluctuating quantities rather than the stable quantities introduced at the beginning of this paper.) The quantity $E_{beat} = E_{s0}e^{-r^2/w^2}\cos(\Omega t)$, the component of the applied electric field which oscillates at the beat frequency.

We translate this result to phase noise of the laser:

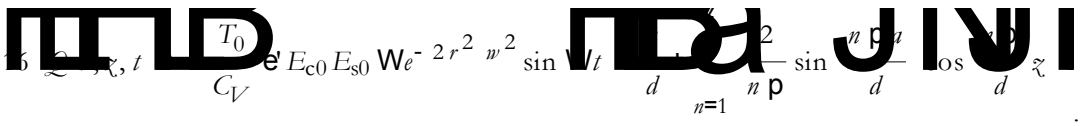
our desired quantification of thermorefractive noise. Note that we have inserted (12) into the expression for $S_D(\Omega)$ and used the parameter $\beta = (1/n)(dn/dT) = (\varepsilon')/(2\varepsilon)$. We have also used $\chi = k/(C\rho)$, where k is the thermal conductivity, C is the heat capacity per unit mass, and ρ is the density. A Jacobi-Anger expansion may be used to justify the conversion between phase noise and electric-field noise in (14). The integral in (14) may be evaluated in terms of hypergeometric functions; if, on the other hand, we operate in the high-frequency/adiabatic regime ($\Omega >> \chi/w^2$), the power spectral density of the phase becomes



This differs by a factor of 2^4 from the previous result obtained by Braginsky and Vyatchanin[1], though the fact that the materials parameters and constants all agree completely with the previous result is noteworthy. The Braginsky and Vyatchanin result holds only in the adiabatic limit (however, see derivation in Appendix C.2 of their paper), while equation (14) is valid in general, in both the adiabatic and isothermal limits. Both analyses fail if $c/\Omega \sim d$.

Case 2: The laser penetrates only the reflective surface coating of a mirror.

The heating due to the electrocaloric effect is now a step function in z. Expanding the z-dependence in a Fourier cosine series gives a source term of



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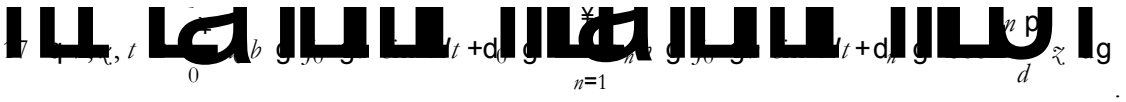
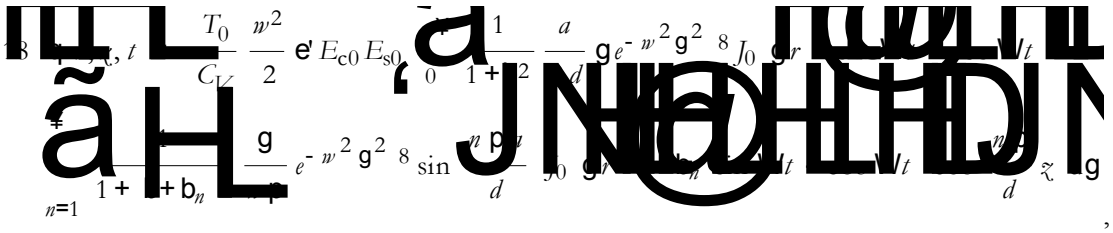
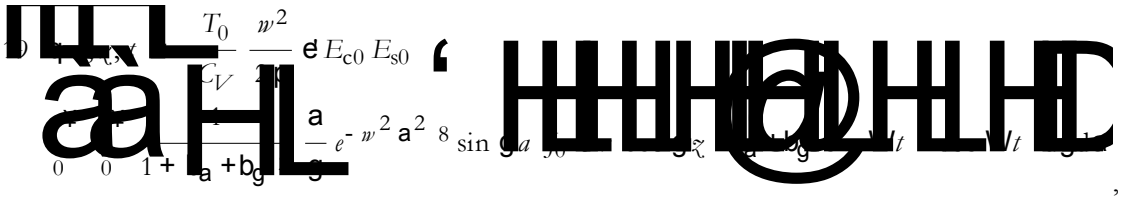
At this point, we may solve the heat equation directly for θ(r,z,t) simultaneously using an integral expansion in r and an infinite-sum expansion in z. The phase shift δ in the argument of the time-dependent sin(Ωt + δ) (see equation (6) for comparison) will now be dependent on both n, the counter in the infinite sum, and on γ, the parameter of integration in the r-expansion:



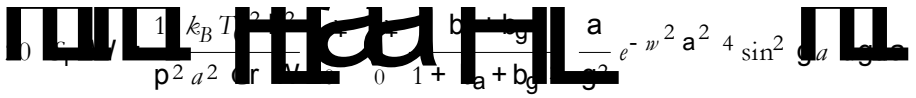
(This expansion is not the most general expansion, as it explicitly satisfies the boundary condition that $\partial\theta(r, 0, t)/\partial z = 0$.) Solving the heat equation by a method completely analogous to that used in Case 1 yields the solution



where a is the thickness of the coating and $\beta_n = (\chi/\Omega)(n\pi/d)^2$; but this is somewhat intractable, and the corresponding expression for dissipation is even messier. We have been treating the mirror as finite in the z-direction (with thickness d). An expression analogous to (18), but cleaner, may be derived if we instead model the mirror as semi-infinite in z ($0 \le z < \infty$). Using a Fourier integral expansion in z rather than a Fourier series expansion, we obtain



where $\beta_\alpha = \chi\alpha^2/\Omega$ and $\beta_\gamma = \chi\gamma^2/\Omega$. The corresponding phase spectral density is



It is conceivable that this double integral could be straightforwardly evaluated if the right substitution and/or approximation were made. However, at present we do not have a suggestion as to what this might be, though it is likely that the adiabatic approximation ($\Omega >> \chi/w^2$) would once again be useful.

Since the above results are mathematically awkward, we might employ a different strategy in solving the heat equation. We first drop the r-dependence by setting r = 0 in

(16) and solve the heat equation for the r-independent solution θ(z,t), again modeling the mirror as having a finite thickness d in the z-direction. Making the infinite-summation expansion

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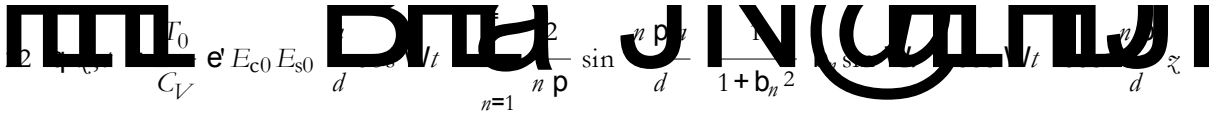
we find (again using a process which is completely analogous to that employed in Case 1) that



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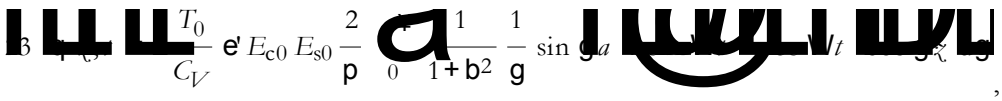
If we instead model the mirror as semi-infinite in z, the analogous expression is



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where again $\beta = \chi\gamma^2/\Omega$.

Using (22) or (23), we can set up an approximate boundary condition for the r-*dependent* problem. We can then solve the homogeneous heat equation subject to this boundary condition, and the result should approximate the exact solution (18) or (19). The details of this calculation are omitted.

This completes the present fluctuation-dissipation-theorem analysis of thermorefractive noise. The calculations performed here, especially that in Case 1, give a nice check of previous results using an entirely different approach. Future research might make use of this strategy as an alternative method of deriving expressions for thermorefractive noise. Specifically, different geometries and finite mirrors might be considered for greater realism, and the calculations suggested in the above paragraph should be carried out. Also, it will be necessary to determine precisely why the result (15) above differs by a factor of 2^4 from the Braginsky & Vyatchanin[1] result. A subtle error involving the assumption that exactly one sideband is present may be to blame. Finally, future research might extend this paper's surface-coating analysis to account for the fact that the surface coatings are composed of alternating layers of two materials with different permittivities/indices of refraction. The result could then be compared to that of Braginsky, Gorodetsky, and Vyatchanin[2], who explicitly take this into consideration by including two indices of refraction, n_1 and n_2, in their derivation of an expression for thermorefractive noise in the surface coatings.

APPENDIX

The free energy of a material with temperature-dependent permittivity ε(T) in the presence of an applied electric field E is

$$F = -\frac{1}{2}\varepsilon(T)E^2 + F_0(T), \quad (A.1)$$

where $F_0(T)$ is the free energy in the absence of an applied electric field. The corresponding entropy is then

$$S = -\frac{\partial F}{\partial T} = \frac{1}{2}\varepsilon' E^2 + S_0, \quad (A.2)$$

where ε' is the temperature derivative of the permittivity and $S_0(T) = -\partial F_0/\partial T$. If we assume that the time scale of the electrocaloric effect and the local temperature changes it induces is small compared to other time scales in our analysis—such as that of heat flow—we may assume that the electrocaloric creation of temperature gradients is an adiabatic process. The entropy then remains constant, and we may write $S(T) = S(T_0)$ for an electrocaloric temperature change from T_0 to T. Defining the entropy zero point to be T_0, this becomes

$$\frac{1}{2}\varepsilon' E^2 + \frac{C_V}{T_0}\theta_{ad} = -\frac{1}{2}\varepsilon' E_0^2, \quad (A.3)$$

where C_V is the volumetric heat capacity and $\theta_{ad} = T - T_0$. Differentiation with respect to time yields equation (1).

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References

[1] V.B. Braginsky and S.P. Vyatchanin, Phys. Lett. A **279**, 154 (2001).
[2] V.B. Braginsky, M.L. Gorodetsky, and S.P. Vyatchanin, Phys. Lett. A **271**, 303 (2000).
[3] H.B. Callen and T.A. Welton, Phys. Rev. **83**, 34 (1951).
[4] see, for example, G. Arfken, *Mathematical Methods for Physicists, Third Edition* (Academic Press, San Diego, 1985).
[5] see, for example, I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products: Corrected and Enlarged Edition* (Academic Press, San Diego, 1980).