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Definitions of the Energy Density Spectrum in Stochastic Gravitational Waves

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1 Total energy density in gravitational waves

Given a description of a gravitational wave as a transverse traceless metric perturbation $h_{ab}(\vec{r}, t)$, one can associate an energy density[1, 2]

$$\rho_{\text{gw}} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}\dot{h}^{ab}\rangle \tag{1.1}$$

with the gravitational field, where the angle brackets indicate an average over several wavelengths, or equivalently cycles, at the frequencies of interest. For a background which is (on average) stationary and homogeneous, the energy density ρ_{gw} will be a single constant number. As with any type of homogeneous mass-energy distribution, one can describe its contribution to the cosmological Ω parameter as the ratio

$$\Omega_{\text{gw}} = \frac{\rho_{\text{gw}}}{\rho_{\text{crit}}} \tag{1.2}$$

where ρ_{crit} is the critical energy density

$$\rho_{\text{crit}} = \frac{3c^2H_0^2}{8\pi G} \tag{1.3}$$

which depends on the current Hubble constant H_0.

2 Alternative definitions of energy density spectrum

Assuming that $h_{ab}(\vec{r}, t)$ can be resolved into modes which are statistically independent of one another, ρ_{gw} (and correspondingly Ω_{gw}) can be separated into contributions from different frequencies. Consider the energy density associated with an infinitesimal frequency range $(f, f + df)$. We will denote this quantity by $\rho_{\text{gw}}(f, f + df)$.[1] If df is truly inifinitesimal (so that the properties of the background do not vary appreciably between f and $f + df$), then $\rho_{\text{gw}}(f, f + df)$ will be proportional to df, and will depend, in general, on the frequency f. How one defines this proportionality is somewhat arbitrary, as evidenced by the several different definitions of energy density spectrum in the literature, e.g., [3, 4, 2]:

(i) For example, one can define

$$\rho_{\text{gw}}(f, f + df) =: \rho^{(1)}_{\text{gw}}(f)\, df\,, \tag{2.1}$$

so that $\rho^{(1)}_{\text{gw}}(f)$ has dimensions of energy density per unit frequency. A consequence of this definition is that the energy density associated with modes in the finite frequency interval (f_1, f_2) is given by an integral

$$\rho_{\text{gw}}(f_1, f_2) = \int_{f_1}^{f_2} \rho^{(1)}_{\text{gw}}(f)\, df\,. \tag{2.2}$$

[1]More generally, $\rho_{\text{gw}}(f_1, f_2)$ will denote the energy associated with modes in the frequency interval (f_1, f_2).

The total energy density in all modes is thus

$$\rho_{\rm gw} \equiv \rho_{\rm gw}(0,\infty) = \int_0^\infty \rho^{(1)}_{\rm gw}(f)\, df\,. \tag{2.3}$$

(ii) Alternatively, one can define

$$\rho_{\rm gw}(f, f+df) =: \rho^{(2)}_{\rm gw}(f)\frac{df}{f}\,. \tag{2.4}$$

Here $\rho^{(2)}_{\rm gw}(f)$ is an energy density per unit *logarithmic frequency interval* $d\ln f := df/f$, evaluated at frequency f. The definition of $\rho^{(2)}_{\rm gw}(f)$ differs from that of $\rho^{(1)}_{\rm gw}(f)$ by a factor of $1/f$. This energy density spectrum satisfies the property

$$\rho_{\rm gw}(f_1, f_2) = \int_{f_1}^{f_2} \frac{\rho^{(2)}_{\rm gw}(f)}{f}\, df = \int_{\ln f_1}^{\ln f_2} \rho^{(2)}_{\rm gw}(f)\, d\ln f \tag{2.5}$$

so that total energy density

$$\rho_{\rm gw} = \int_{-\infty}^{\infty} \rho^{(2)}_{\rm gw}(f)\, d\ln f\,. \tag{2.6}$$

(iii) Finally, if one denotes the infinitesimal quantity $\rho_{\rm gw}(f, f+df)$ by $d\rho_{\rm gw}$, then (2.1) can be rewritten as

$$d\rho_{\rm gw} := \rho^{(1)}_{\rm gw}(f)\, df \tag{2.7}$$

or, equivalently, as

$$\frac{d\rho_{\rm gw}}{df} = \rho^{(1)}_{\rm gw}(f)\,. \tag{2.8}$$

The LHS is simply the derivative of

$$\rho_{\rm gw}(f) := \int \rho^{(1)}_{\rm gw}(f)\, df \tag{2.9}$$

i.e., the *indefinite* integral of $\rho^{(1)}_{\rm gw}(f)$. Since

$$\rho_{\rm gw}(f) = \int_{f_1}^{f} \rho^{(1)}_{\rm gw}(f')\, df' \equiv \rho_{\rm gw}(f_1, f) \tag{2.10}$$

up to an (unimportant) integration constant, we are free to set $f_1 = 0$, so that $\rho_{\rm gw}(f)$ is the *cumulative* energy density in gravitational waves up to frequency f.[2]

[2] We should emphasize that $\rho_{\rm gw}(f)$ is *not* the energy density in gravitational waves at the single frequency f. Energy density at f for a continuous range of frequencies is not defined anymore than the probability at a single value x of a continuous random variable. On the other hand, energy density over some finite range of frequencies (e.g., 0 to f) or energy density per unit frequency (or per unit logarithmic frequency) at a single frequency f *is* well-defined.

3 Definition of $\Omega_{\rm gw}(f)$

Somewhat surprisingly, the gravitational-wave community has agreed upon a single definition of the normalised energy spectrum $\Omega_{\rm gw}(f)$—namely, the contribution to $\Omega_{\rm gw}$ per unit *logarithmic* frequency interval. This means that the normalised energy density in the infinitesimal frequency interval $(f, f+df)$ can be written as

$$\frac{\rho_{\rm gw}(f, f+df)}{\rho_{\rm crit}} =: \Omega_{\rm gw}(f)\,\frac{df}{f}\,. \tag{3.1}$$

For a finite frequency interval (f_1, f_2), one then has

$$\frac{\rho_{\rm gw}(f_1, f_2)}{\rho_{\rm crit}} = \int_{f_1}^{f_2} \frac{\Omega_{\rm gw}(f)}{f}\,df = \int_{\ln f_1}^{\ln f_2} \Omega_{\rm gw}(f)\,d\ln f \tag{3.2}$$

and the total normalised energy density in all modes is given by

$$\Omega_{\rm gw} = \int_{-\infty}^{\infty} \Omega_{\rm gw}(f)\,d\ln f\,. \tag{3.3}$$

Note that $\Omega_{\rm gw}(f)$ and $\Omega_{\rm gw}$ are different quantities.

4 Alternative expressions for $\Omega_{\rm gw}(f)$

The relationship between the various ways of defining a gravitational wave spectrum can be obtained by comparing Equations (2.1), (2.4), (2.8), and (3.1). Solving the expressions for $\Omega_{\rm gw}(f)$, one finds

$$\Omega_{\rm gw}(f) = f\,\frac{\rho^{(1)}_{\rm gw}(f)}{\rho_{\rm crit}} \tag{4.1}$$

which is Eq. (1.3) in [4],

$$\Omega_{\rm gw}(f) = \frac{\rho^{(2)}_{\rm gw}(f)}{\rho_{\rm crit}} \tag{4.2}$$

which is described at the end of Section II.B in [3], and

$$\Omega_{\rm gw}(f) = \frac{f}{\rho_{\rm crit}}\frac{d\rho_{\rm gw}}{df} \equiv \frac{1}{\rho_{\rm crit}}\frac{d\rho_{\rm gw}}{d\ln f}\,. \tag{4.3}$$

which is Eq. (2.1) in [2].

Note the somewhat awkward factor of f that appears in (4.1). This arises because $\Omega_{\rm gw}(f)$ is defined as a spectrum with respect to logaritmic frequency intervals, while $\rho^{(1)}_{\rm gw}(f)$ is a spectrum with respect to ordinary frequency intervals. In (4.2), there is no such factor because $\Omega_{\rm gw}(f)$ and $\rho^{(2)}_{\rm gw}(f)$ are both defined with respect to logarithmic frequency intervals. Finally, (4.3) is a valid expression for $\Omega_{\rm gw}(f)$ if one interprets $d\rho_{\rm gw}/df$ as the derivative of $\rho_{\rm gw}(f)$, the *cumulative energy* density in gravitational waves up to frequency f.

References

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