LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
- LIGO -
CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

| **Technical Note** | **LIGO-T070043-00-Z** | 2007-02-05 |
|---|---|---|

# Geometric Acceptance of the LIGO Interferometers at the FSR Frequencies

T. Fricke, S. Giampanis and A. C. Melissinos
University of Rochester Stochastic Analysis Group

*Distribution of this document:*
Stochastic Analysis Group

This is an internal working
note of the LIGO project

**California Institute of Technology**
**LIGO Project, MS 18-34**
**Pasadena, CA 91125**
Phone (626) 395-2129
Fax (626) 304-9834
E-mail: info@ligo.caltech.edu

**Massachusetts Institute of Technology**
**LIGO Project, Room NW17-161**
**Cambridge, MA 02139**
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

WWW: http://www.ligo.caltech.edu/

# 1 Definitions

We define the *measured* cross-correlation (power) spectrum in the frequency domain by

$$S_m(f) = \langle h_1^*(f) h_2(f) \rangle \tag{1}$$

The $h_{1,2}(f)$ are the amplitude spectral densities of the interferometer output. They are obtained by suitable normalization of the Fourier transform, $\tilde{h}(f)$, of the detected signal time series, $h(t)$,

$$h(f) = \sqrt{2/T}\ \tilde{h}(f)$$

The average in Eq.(1) is over repeated segments of data, each of duration $T$ [1].

The interpretation of $S_m(f)$ follows from the defining properties of a stochastic background (assumed isotropic and unpolarized). Namely

$$\begin{aligned} \langle h_{\alpha\beta}(t) \rangle &= 0 \\ \langle h_{\alpha\beta}(t) h^{\alpha\beta}(t) \rangle &= 32\pi \int_0^\infty H(f) df \end{aligned} \tag{2}$$

where $H(f)$ is a real, non-negative function describing the *spectrum* of the stochastic background

$$H(f) = |h_+(f)|^2 + |h_\times(f)|^2 \tag{3}$$

For interferometers with orthogonal arms

$$S_m(f) = \frac{8\pi}{5} \gamma(f) D(f) H(f) \tag{4}$$

$\gamma(f)$ is the *overlap reduction function.* For co-located and co-aligned interferometers $\gamma(f) = 1$.

$D(f)$ is the *correction* to the acceptance (pattern function) of the interferometer as a function of frequency. It is derived in section 3, but at frequencies $f \lesssim 1$ kHz we can set $D(f) = 1$.

The definition of $H(f)$ through Eq.(2) follows the normalization of Allen and Romano [2]. Maggiore [3] uses instead $S_h(f) = 8\pi H(f)$.

As shown in the following section the local energy density in the gravitational wave is

$$\rho_G = \frac{4\pi^2 c^2}{G} \int_0^\infty f^2 H(f) df \tag{5}$$

Using Eq.(4) and Eq.(5) we can express the energy density per unit frequency interval in terms of the measured cross-correlation spectrum

$$\frac{d\rho_G}{df} = \frac{20\pi c^2}{8G} f^2 S_m(f)/[\gamma(f)D(f)] \tag{6}$$

Note that $d\rho_G/df$ is not a derivative but the integrand in Eq.(5), [4]. Finally we define

$$\Omega(f) \equiv \frac{1}{\rho_c} f \frac{d\rho_G}{df} \tag{7}$$

The convenient notation

$$h_{\rm rms}(f) = \sqrt{S_m(f)} \tag{8}$$

is often encountered.

## 2 Calculation of the local energy density

The local energy density is given by Eq.(35.23) of Misner, Thorne and Wheeler [5]

$$\rho_G = \frac{c^2}{32\pi G} \langle \dot{h}_{\alpha\beta}(t,\vec{x}), \dot{h}^{\alpha\beta}(t,\vec{x}) \rangle \tag{9}$$

The amplitudes $h_{\alpha\beta}(t,\vec{x})$ can be uniquely expressed by a plane wave expansion [2]

$$h_{\alpha\beta}(t,\vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int_{s^2} d\hat{\Omega} h_A(f,\hat{\Omega}) e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} \epsilon^A_{\alpha\beta}(\hat{\Omega}) \tag{10}$$

For a stochastic signal (as defined in the previous section), the following holds in the frequency domain [2]

$$\langle h_A^*(f,\hat{\Omega})h_{A'}(f',\hat{\Omega}')\rangle = \delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')\delta_{AA'}H(f) \tag{11}$$

Using the expansion of Eq.(10) in Eq.(9) and expressing the ensemble average by Eq.(11) we can immediately perform the integrations over $d\hat{\Omega}'$, $df'$ and the summation over $A'$, to obtain

$$\rho_G = \frac{4\pi^2 c^2}{32\pi G} f^2 \sum_A \int_{-\infty}^{\infty} df \int_{s^2} d\hat{\Omega}\epsilon^A_{\alpha\beta}\epsilon^{A\alpha\beta}H(f) \tag{12}$$

The summation over the polarization tensors equals 4 and the integral over $d\hat{\Omega}$ equals $4\pi$. We also limit the integration of $df$ to the range 0 to $\infty$ (because $H(-f) = H(f)$) yielding a further factor of 2, so that

$$\rho_G = \frac{4\pi^2 c^2}{G} \int_0^{\infty} f^2 H(f) df \tag{13}$$

This proves Eq.(5) and the similar steps lead to Eq.(2). Of course both equations are evaluated at the same point in space, $\vec{x}$. Some subtle issues related to infinitely long time series are discussed in [6].

Note that Weinberg [7] gives the local energy density in the gravitational wave [his Eq.(10.37)] as

$$\langle t_{\mu\nu}\rangle \;=\frac{k_\mu k_\nu}{16\pi G}(|h_+|^2+|h_\times|^2) \tag{14}$$

with $c = 1$, $k = \omega = 2\pi f$. If we write

$$\rho_G = \sum_A \int t_{00} d\Omega df \tag{15}$$

the summation over polarizations and the integration over solid angle introduce a factor of $16\pi$ in agreement with Eq.(13).

Finally for convenience in evaluating $\Omega(f)$ in Eq.(7) we give

$$\rho_c = \frac{3c^2 H_0^2}{8\pi G} \tag{16}$$

# 3 The interferometer acceptance as a function of frequency

The gravitational wave signal appears at the antisymmetric (dark) port of the interferometer. It is proportional to the difference in the phase of the carrier light exiting from the two arms. We follow Sigg [8] in calculating the phase shift induced by a gravitational wave of arbitrary incidence and polarization.

The interferometer arms are taken along the $x$ and $y$ axes, and the gravitational wave vector is $\vec{k}$, $\Omega = c|\vec{k}|$. In spherical coordinates

$$\hat{k}_x = \sin\theta\cos\phi \qquad \hat{k}_y = \sin\theta\sin\phi \qquad \hat{k}_z = \cos\theta \tag{17}$$

and $k = |\vec{k}|$. We work in the $TT$ gauge and designate the metric perturbation amplitudes by $h_+$ and $h_\times$ for the two polarizations of the wave. After rotating the gravitational wave tensor into the interferometer coordinate system we find

$$\begin{aligned} h_{xx} &= -\cos\theta\sin 2\phi h_\times + \left(\cos^2\theta\cos^2\phi - \sin^2\phi\right) h_+ \\ h_{yy} &= \cos\theta\sin 2\phi h_\times + \left(\cos^2\theta\sin^2\phi - \cos^2\phi\right) h_+ \end{aligned} \tag{18}$$

The proper time for a light signal is zero

$$d\tau^2 = dx_\mu g_{\mu\nu} dx^\nu = 0 \qquad \text{with} \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The change in the phase of the carrier is obtained by integrating the proper time $d\tau$ over a round trip along the arm

$$\Delta\Phi = \int_0^{2L} \omega d\tau = \int_0^{2L} \frac{\omega}{c} \left[1 + h_{xx}(t)\right]^{\frac{1}{2}} dx \tag{19}$$

or

$$\begin{aligned}\Delta\Phi_x\left(t_0\right) &= \frac{\omega}{c}\int_0^L \left\{1+h_{xx}\cos\left[\Omega t_0+k(1-\hat{k}_x)x\right]\right\}^{\frac{1}{2}}dx + \\ &+ \frac{\omega}{c}\int_0^L \left\{1+h_{xx}\cos\left[\Omega t_0+k\left(2L-(1+\hat{k}_x)x\right)\right]\right\}^{\frac{1}{2}}dx \qquad (20)\end{aligned}$$

Since $h_{xx}$, $h_{yy} \ll 1$ we expand the square roots, convert to exponential notation, discard the time independent term and shift to the time of arrival to find

$$\Delta\Phi_x = \frac{h_{xx}L\omega}{c}e^{-i\Phi_\Omega}\frac{\sin\Phi_\Omega + i\hat{k}_x\cos\Phi_\Omega - i\hat{k}_x e^{ik_x\Phi_\Omega}}{\Phi_\Omega\left(1-\hat{k}_x^2\right)} \qquad (21)$$

and a corresponding expression for $\Delta\Phi_y$. Here

$$\Phi_\Omega = L\Omega/c$$

with $L$ the length of the arm and $\Omega = 2\pi f_G$. The magnitude of Eq. (21) gives the amplitude of the change in carrier phase and the modulus gives the phase shift with respect to the gravitational wave. For normal incidence $k_x = 0$ and we recover the usual expression for the carrier phase.

We must form the phase difference between the two arms for each of the two polarizations separately, and we combine the results in quadrature

$$\begin{aligned} H &= \Delta\Phi_x - \Delta\Phi_y \\ H_+ &= H\left(h_\times = 0, h_+ = 1\right) \\ H_\times &= H\left(h_\times = 1, h_+ = 0\right) \\ \overline{H} &= \sqrt{|H_+|^2 + |H_\times|^2} \qquad (22)\end{aligned}$$

$|H_+|$, $|H_\times|$ and $\overline{H}$ are shown as a function of the angles of incidence $\theta$, $\phi$, of the gravitational wave in Fig. 1 for $f_G = 0$ and in Fig. 2 for $\Phi_\Omega = \pi$ (this corresponds to $f_G = 37.52$ kHz for $L = 4$ km). We refer to $H\left(\theta, \phi, f_G\right)$ as the antenna pattern at frequency $f_G$.

To evaluate the acceptance of the interferometer, i.e., the function $D(f)$ introduced in Eq.(4) we must average over $\theta$, and $\phi$. We do this by integrating

$$V = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{|H(\theta,\phi)|} r^2 dr \qquad (23)$$

and setting

$$D(f) = [V(f)/V(f=0)]^{\frac{1}{3}} \tag{24}$$

Our calculation so far involved a single traversal in the arm. Since the carrier undergoes multiple reflections in the Fabry-Perot cavity we must add the contribution of these repeated traversals. It was shown by Schilling [9] that the result for a single traversal, which we designate $H_1$, is multiplied by the usual transfer function for a resonant optical cavity. Namely

$$H_{FP} = \frac{H_1}{|1 - r_1 r_2 e^{i2\Phi_\Omega}|} \tag{25}$$

where $r_1$, $r_2$ are the reflectivities of the input and end mirrors. Thus the result of Eq.(25) is not modified by the multiple traversals in the arms.

# References

1. W.H. Press, S.A. Teukolsky, W.T. Vetterling and B.P. Flannery, *Numerical Recipes in $C^{++}$*,(Cambridge 2002).

2. B. Allen and J. Romano, Phys. Rev. **D59**, 102001 (1999).

3. M. Maggiore, Physics Reports **331**, 283 (2000).

4. L. Grishchuk and the LSC Stochastic Group, LIGO internal report T060270-00-Z (2006).

5. C. Misner, K. Thorne and J. Wheeler, *Gravitation*, (Freeman 1973).

6. W. Butler and A. Melissinos, *Measurement of a Stochastic Gravitational Background with a Single Laser Interferometer*, arXiv:gr-qc/0501089 v1.(2005).

7. S. Weinberg, *Gravitation and Cosmology*,(Wiley 1972).

8. D. Sigg, *Strain Calibration in LIGO*, T970101-B-D (2003).

9. R. Schilling, Class. Quantum Gravity **14**, 1513 (1997).

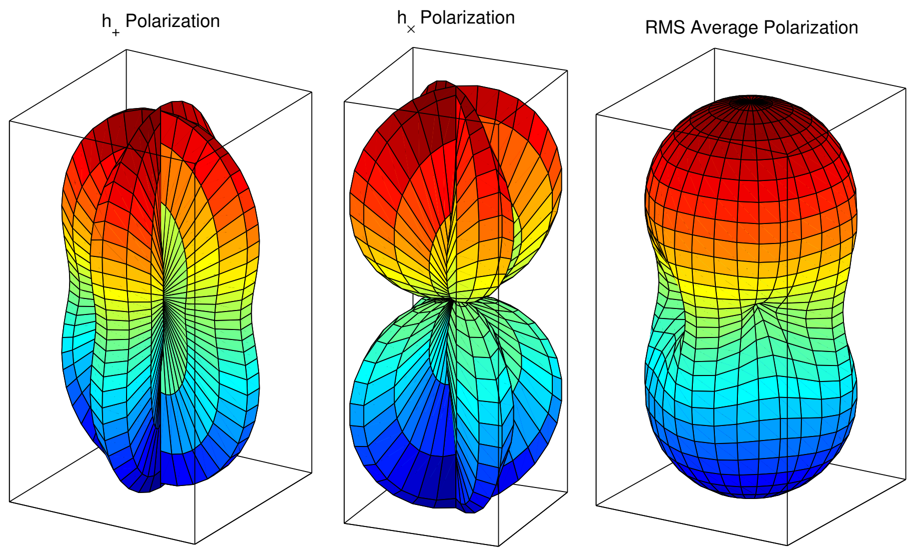


Figure 1: The acceptance pattern at $f < 1$ kHz for the plus and cross polarizations and their rms average

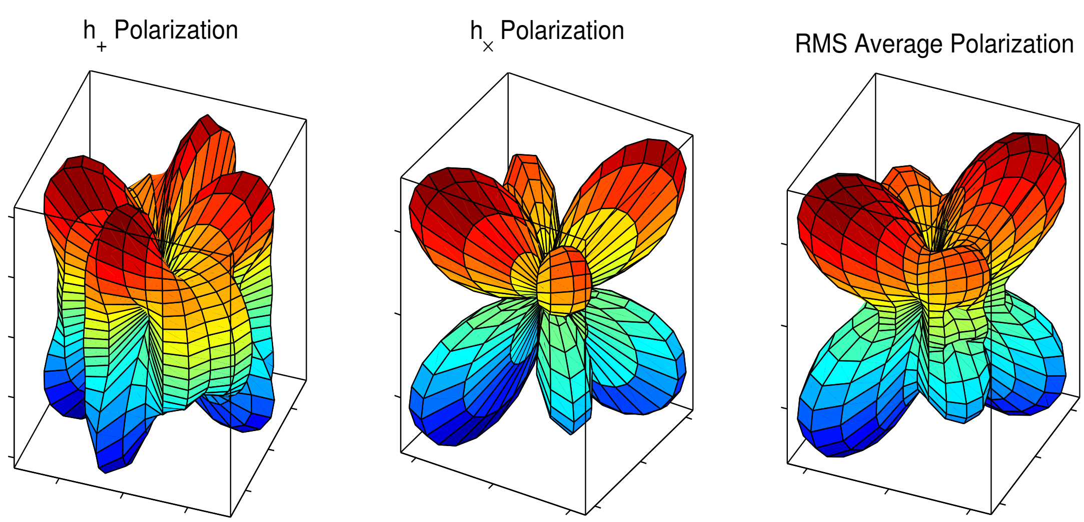


Figure 2: The acceptance pattern at $f = 37.52$ kHz for the plus and cross polarizations and their rms average