Tilt-Horizontal Coupling for a Simple Inverted Pendulum

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1 Summary

In response to some recent discussions, we re-derive the tilt-horizontal coupling for a simple mass-spring, and we show that an inverted pendulum has almost exactly the same response. At low frequency, the tilt-horizontal coupling for a mass-spring system is $\frac{\theta}{x} = \frac{g}{w_0^2} = L_c$. Here, w_0 is the natural frequency of the mass-spring, in rad/sec, and L_c is the characteristic length of a simple pendulum with the same frequency. For a 30 mHz system, $L_c \approx 276$ m. For a simple inverted pendulum, the low frequency coupling asymptotes to $\frac{\theta}{x} = \frac{g}{w_0^2} + L_{IP} = L_c + L_{IP}$. The difference is that the inverted pendulum has an additional response, L_{IP}, equal to the physical length of the inverted pendulum leg, which may be only about 30 cm.

2 Response of the mass and spring to tilt

We begin by modeling an example which is easy to visualize: a horizontal mass-spring system on a tilting floor. This system has a mass, m, on a spring, k, with damping b. The mass is free to slide on the floor. The mass location is x, the floor is allowed to tilt with respect to local gravity by an angle θ. We ignore centrifugal forces, etc.

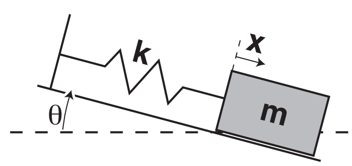

Figure 1: Model of a tilted mass-spring system

The basic equation describing this system is

$$m\,\ddot{x} = -k\,x - b\,\dot{x} + m\,g\,\sin(\theta). \tag{1}$$

We can fourier transform the equation, and assume the tilt angle is small, so equation 1 becomes

$$-m\,\omega^2\,x = -k\,x - i\,b\,\omega\,x + m\,g\,\theta. \tag{2}$$

With algebra, this becomes

$$x(-\omega^2 + i\,\omega\frac{b}{m} + \frac{k}{m}) = g\,\theta. \tag{3}$$

We can write this in a standard form, with $\omega_0^2 = \frac{k}{m}$ and see that

$$x \cdot \omega_0^2\,(-\frac{\omega^2}{\omega_0^2} + i\frac{\omega}{Q\omega_0} + 1) = g\,\theta. \tag{4}$$

With this, it is easy to see that the tilt horizontal coupling of this mass is the common expression

$$\frac{x}{\theta} = \frac{g}{\omega_0^2} \cdot \frac{1}{-\frac{\omega^2}{\omega_0^2} + i\frac{\omega}{Q\omega_0} + 1} \tag{5}$$

At low frequencies, this asymptotes to

$$\frac{x}{\theta} = \frac{g}{\omega_0^2} = L_c \tag{6}$$

where L_c is the characteristic length of a simple pendulum with a frequency of ω_0 rad/sec.

3 Response of a simple inverted pendulum to tilt

We can model an inverted pendulum as a point mass m on top of a massless leg of length L_{IP}. The base of the leg is attached to the floor with a spring of rotational stiffness κ. The floor is tilted by an angle θ, and the inverted pendulum is tilted from the vertical by an angle ϕ. The top mass will have moved away from its vertical location by a distance $x = \phi \cdot L_{IP}$. The moment of inertia of the mass is $M \equiv mL_{IP}^2$.

The basic equation for this system is

$$M\,\ddot{\phi} = -\kappa\,(\phi - \theta) - b(\dot{\phi} - \dot{\theta}) + m\,g\,L_{IP}\sin(\phi). \tag{7}$$

We follow the same procedure as before by assuming small angles and using the fourier transform so that equation 7 becomes

$$-M\,\omega^2\phi = -\kappa\,(\phi - \theta) - ib\omega(\phi - \theta) + m\,g\,L_{IP}\phi. \tag{8}$$

We rewrite this in terms of m and x to get

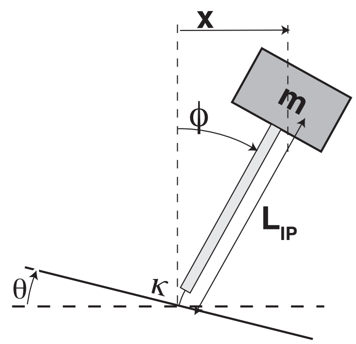

Figure 2: Model of an idealized inverted pendulum

$$-mL_{IP}^2\,\omega^2\frac{x}{L_{IP}} = -\kappa\,(\frac{x}{L_{IP}} - \theta) - ib\omega(\frac{x}{L_{IP}} - \theta) + m\,g\,L_{IP}\frac{x}{L_{IP}}. \tag{9}$$

we collect terms in x, and divide through by mL_{IP}

$$x\Big(- \omega^2 + i\omega\frac{b}{mL_{IP}^2} + (\frac{\kappa}{mL_{IP}^2} - \frac{g}{L_{IP}})\Big) = (i\omega\frac{b}{mL_{IP}} + \frac{\kappa}{mL_{IP}})\theta \tag{10}$$

We put this in the standard form, and see that

$$x \cdot \omega_0^2\,(-\frac{\omega^2}{\omega_0^2} + i\frac{\omega}{Q\omega_0} + 1) = (i\omega\frac{b}{mL_{IP}} + \frac{\kappa}{mL_{IP}})\theta, \tag{11}$$

or

$$\frac{x}{\theta} = \frac{\frac{\kappa}{mL_{IP}}}{\omega_0^2}\frac{i\omega\frac{b}{\kappa} + 1}{(-\frac{\omega^2}{\omega_0^2} + i\frac{\omega}{Q\omega_0} + 1)}. \tag{12}$$

We have defined the natural frequency by

$$\omega_0^2 = \frac{\kappa}{mL_{IP}^2} - \frac{g}{L_{IP}}. \tag{13}$$

The expression for the natural frequency can be rewritten to show:

$$\frac{\kappa}{mL_{IP}} = g + \omega_0^2 \cdot L_{IP}. \tag{14}$$

Thus, at low frequency, the expression for the tilt horizontal coupling of the inverted pendulum becomes

$$\frac{x}{\theta} = \frac{g}{\omega_0^2} + L_{IP} = L_c + L_{IP}, \tag{15}$$

where we have used the previous definition for the characteristic length of $L_c = \frac{g}{\omega_0^2}$.

This is exactly what one would expect. The tilt-horizontal coupling of an inverted pendulum is defined by the characteristic pendulum length, plus a small length which is the physical length of the inverted pendulum leg, since the floor is assumed to pivot about the base of the leg.

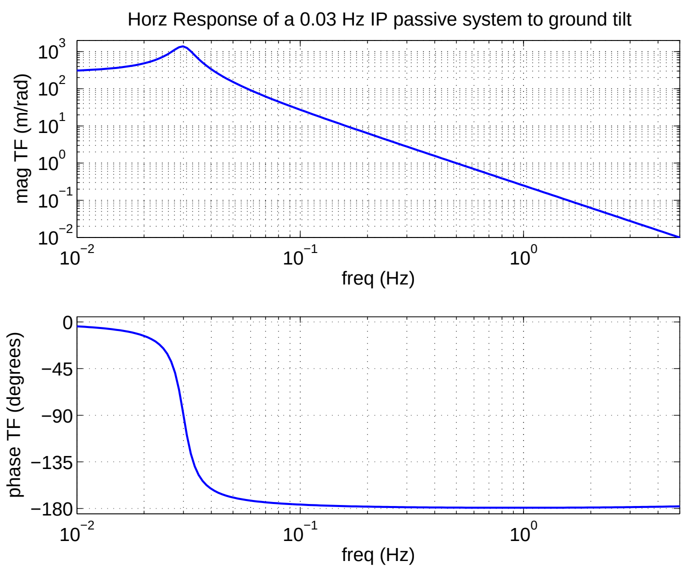

Figure 3: Transfer function of an idealized inverted pendulum

In figure 3 we plot the tilt-horizontal transfer function of an idealized inverted pendulum. The leg length is 30 cm, and the Q of the total system is set to 5. The other parameters are:

mass	2000 kg
leg length	30 cm
natural frequency	30 mHz
IP stiffness ($m\,g\,L_{IP}$)	-5886 N-m
total stiffness	+6.4 N-m

Table 1: Parameters of the simple inverted pendulum shown in figure 3