LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Higher-Frequency Corrections to Stochastic Formulae

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1 Long-Wavelength Limit

The most general tensor gravitational wave in the TT gauge is

$$\overleftrightarrow{h}(t,\vec{r}) = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f,\hat{k})\overleftrightarrow{e}_A(\hat{k})e^{i2\pi f(t-\hat{k}\cdot\vec{r}/c)} \tag{1.1}$$

Where $h_A(f,\hat{k})$ are arbitrary amplitudes and $\{\overleftrightarrow{e}_A(\hat{k})\}$ are the TT polarization basis tensors orthogonal to $\hat{k}$. The spacetime metric it generates is

$$ds^2 = -c^2\,dt^2 + d\vec{r}\cdot\left(\overleftrightarrow{1} + \overleftrightarrow{h}(t,\vec{r})\right)\cdot d\vec{r}\ . \tag{1.2}$$

The long-wavelength-limit (LWL) assumes that a GW detector makes an instantaneous measurement of some projection of the metric perturbation:

$$h^{\text{LWL}}(t) = h_{ab}(t,\vec{r}_{\text{det}})d^{\text{LWL}\,ab} \tag{1.3}$$

In particular, for an IFO with arms along the unit vectors $\hat{u}$ and $\hat{v}$, if $h(t)$ is the fractional differential arm length measured at time t,

$$\overleftrightarrow{d}^{\text{LWL}} = \frac{\hat{u}\otimes\hat{u} - \hat{v}\otimes\hat{v}}{2} \tag{1.4}$$

2 Rigid Adiabatic Approximation

If the interferometer has arms of length L and $(\Delta t)_{\hat{u}}$ is the round-trip travel time down the $\hat{u}$-arm of a photon arriving back at the beam splitter at time t, the strain measured can be written as

$$h(t) := \frac{c(\Delta t)_{\hat{u}} - c(\Delta t)_{\hat{u}}}{2L} = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f,\hat{k})e_{A\,ab}(\hat{k})e^{i2\pi f(t-\hat{k}\cdot\vec{r}/c)}d^{ab}(f,\hat{k}) \tag{2.1}$$

where

$$\overleftrightarrow{d}(f,\hat{k}) = \mathfrak{T}_{\hat{u}}(f,\hat{k})\frac{\hat{u}\otimes\hat{u}}{2} - \mathfrak{T}_{\hat{v}}(f,\hat{k})\frac{\hat{v}\otimes\hat{v}}{2} \tag{2.2}$$

and we have defined in the appendix the notation

$$\mathfrak{T}_{\hat{u}}(f,\hat{k}) = e^{i\frac{\pi f L}{c}(1-\hat{k}\cdot\hat{u})}\operatorname{sinc}\left(\frac{\pi f L}{c}[1+\hat{k}\cdot\hat{u}]\right) + e^{-i\frac{\pi f L}{c}(1+\hat{k}\cdot\hat{u})}\operatorname{sinc}\left(\frac{\pi f L}{c}[1-\hat{k}\cdot\hat{u}]\right)\ . \tag{2.3}$$

Since $\mathfrak{T}_{\hat{u}}(0,\hat{k}) = 1$, we get the expected limit $\overleftrightarrow{d}(0,\hat{k}) = \overleftrightarrow{d}^{\text{LWL}}$.

In the Fourier domain, (2.1) becomes

$$\widetilde{h}(f) = \iint d^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f,\hat{k})e_{A\,ab}(\hat{k})e^{-i2\pi f\hat{k}\cdot\vec{r}/c}d^{ab}(f,\hat{k}) \tag{2.4}$$

3 Stochastic Background Correlations and Overlap Reduction Function

An isotropic background has amplitudes $\{h_A(f,\hat{k})\}$ with correlations

$$\langle h_A^*(f,\hat{k}) h_{A'}(f',\hat{k}')\rangle = \delta^2(\hat{k},\hat{k}')\delta_{AA'}\delta(f-f')\frac{5}{16\pi}S_{\mathrm{gw}}(f) \tag{3.1}$$

which leads to a cross-correlation between the strain in two detectors of

$$\langle \widetilde{h}_1^*(f)\widetilde{h}_2(f')\rangle = \frac{1}{2}\delta(f-f')\,\gamma_{12}(f)\,S_{\mathrm{gw}}(f) \tag{3.2}$$

where the overlap reduction function is

$$\gamma_{12}(f) = \frac{5}{4\pi}\iint d^2\Omega_{\hat{k}}\, d_{1\,ab}^*(f,\hat{k})\, P^{\mathrm{TT}\hat{k}ab}{}_{cd}\, d_2^{cd}(f,\hat{k})\, e^{i2\pi f\hat{k}\cdot(\vec{r}_1-\vec{r}_2)/c} \tag{3.3}$$

in terms of the projector $P^{\mathrm{TT}\hat{k}ab}{}_{cd}$ onto traceless symmetric matrices transverse to $\hat{k}$.

To zeroth order in fL/c, this becomes

$$\gamma_{12}^{\mathrm{LWL}}(f) = \frac{5}{4\pi}\iint d^2\Omega_{\hat{k}}\, d_{1\,ab}^{\mathrm{LWL}}\, P^{\mathrm{TT}\hat{k}ab}{}_{cd}\, d_2^{\mathrm{LWL}cd}\, e^{i2\pi f\hat{k}\cdot(\vec{r}_1-\vec{r}_2)/c} \tag{3.4}$$

which is the expression we usually use.

4 First Order Corrections

To deal with higher-order corrections, it's convenient to define

$$\alpha = 2\pi f\,|\vec{r}_1-\vec{r}_2|\,/c \tag{4.1a}$$
$$\hat{s} = \frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|} \tag{4.1b}$$
$$\beta = \pi fL/c \tag{4.1c}$$

Then

$$\mathfrak{T}_{\hat{u}}(f,\hat{k}) = 1 - i\beta\hat{k}\cdot\hat{u} + \mathcal{O}(\beta^2) \tag{4.2}$$

so

$$\overleftrightarrow{d}(f,\hat{k}) = \overleftrightarrow{d}^{\mathrm{LWL}} - i\beta\frac{(\hat{k}\cdot\hat{u})\hat{u}\otimes\hat{u} - (\hat{k}\cdot\hat{v})\hat{v}\otimes\hat{v}}{2} + \mathcal{O}(\beta^2) \tag{4.3}$$

That means, for a correlation between two IFOs,

$$\gamma_{12}(f) = \gamma_{12}^{\mathrm{LWL}}(f) + \beta_1\frac{\mathcal{G}(\overleftrightarrow{d}_2^{\mathrm{LWL}\,\mathrm{T}},\alpha,\hat{s},\hat{u}_1) - \mathcal{G}(\overleftrightarrow{d}_2^{\mathrm{LWL}\,\mathrm{T}},\alpha,\hat{s},\hat{v}_1)}{2} - \beta_2\frac{\mathcal{G}(\overleftrightarrow{d}_1^{\mathrm{LWL}\,\mathrm{T}},\alpha,\hat{s},\hat{u}_2) - \mathcal{G}(\overleftrightarrow{d}_1^{\mathrm{LWL}\,\mathrm{T}},\alpha,\hat{s},\hat{v}_2)}{2} + \mathcal{O}(\beta^2) \tag{4.4}$$

where

$$\mathcal{G}(\overleftrightarrow{d},\alpha,\hat{s},\hat{u}) = i\frac{5}{4\pi}\iint d^2\Omega_{\hat{k}}\, e^{i\alpha\hat{k}\cdot\hat{s}} d_{ab}\, P^{\mathrm{TT}\hat{k}ab}{}_{cd}\, k_e u^c u^d u^e \ . \tag{4.5}$$

The quantity defined in (4.5) can be calculated by observing that if we define the vector $\vec{\alpha} = \alpha\hat{s}$,

$$\mathcal{G}(\overleftrightarrow{d},\alpha,\hat{s},\hat{u}) = d_{ab}\frac{\partial\Gamma^{ab}_{cd}(\alpha,\hat{s})}{\partial\alpha^e} u^c u^d u^e \tag{4.6}$$

where

$$\Gamma^{ab}_{cd}(\alpha,\hat{s}) = \frac{5}{4\pi}\iint d^2\Omega_{\hat{k}}\, P^{\mathrm{TT}\hat{k}ab}{}_{cd}\, e^{i\alpha\hat{k}\cdot\hat{s}} \tag{4.7}$$

is the usual tensor used in the calculation of the overlap reduction function, which we know to be

$$\Gamma^{ab}_{cd}(\alpha,\hat{s}) = \rho_1(\alpha)P^{\mathrm{T}ab}{}_{cd} + \rho_2(\alpha)P^{\mathrm{T}ab}{}_{fg}s^g s_h P^{\mathrm{T}fh}{}_{cd} + \rho_3(\alpha)P^{\mathrm{T}ab}{}_{fg}s^f s^g s_h s_i P^{\mathrm{T}hi}{}_{cd} \tag{4.8}$$

where

$$\begin{pmatrix}\rho_1(\alpha)\\ \rho_2(\alpha)\\ \rho_3(\alpha)\end{pmatrix} = \begin{pmatrix} 5 & -10 & 5\\ -10 & 40 & -50\\ \frac{5}{2} & -25 & \frac{175}{2}\end{pmatrix}\begin{pmatrix} j_0(\alpha)\\ \frac{j_1(\alpha)}{\alpha}\\ \frac{j_2(\alpha)}{\alpha^2}\end{pmatrix} \tag{4.9}$$

Substituting in for $\hat{s}$ and using

$$\frac{\partial\alpha}{\partial\alpha^e} = \frac{\alpha_e}{\alpha} = s_e \tag{4.10}$$

we have

$$\begin{aligned}
\frac{\partial\Gamma^{ab}_{cd}(\alpha,\hat{s})}{\partial\alpha^e} =& \frac{\partial}{\partial\alpha^e}\left(\rho_1(\alpha)P^{\mathrm{T}ab}{}_{cd} + \frac{\rho_2(\alpha)}{\alpha^2}P^{\mathrm{T}ab}{}_{fg}\alpha^g\alpha_h P^{\mathrm{T}fh}{}_{cd} + \frac{\rho_3(\alpha)}{\alpha^4}P^{\mathrm{T}ab}{}_{fg}\alpha^f\alpha^g\alpha_h\alpha_i P^{\mathrm{T}hi}{}_{cd}\right)\\
=& \rho_1'(\alpha)P^{\mathrm{T}ab}{}_{cd}s_e + \left[\rho_2'(\alpha) - 2\frac{\rho_2(\alpha)}{\alpha}\right]P^{\mathrm{T}ab}{}_{fg}s^g s_h P^{\mathrm{T}fh}{}_{cd}s_e\\
&+ \left[\rho_3'(\alpha) - 4\frac{\rho_3(\alpha)}{\alpha}\right]P^{\mathrm{T}ab}{}_{fg}s^f s^g s_h s_i P^{\mathrm{T}hi}{}_{cd}s_e\\
&+ \frac{\rho_2(\alpha)}{\alpha}\left[P^{\mathrm{T}ab}{}_{fe}s_g P^{\mathrm{T}fg}{}_{cd} + P^{\mathrm{T}ab}{}_{fg}s^g P^{\mathrm{T}fe}{}_{cd}\right]\\
&+ 2\frac{\rho_3(\alpha)}{\alpha}\left[P^{\mathrm{T}ab}{}_{fe}s^f s_g s_h P^{\mathrm{T}gh}{}_{cd} + P^{\mathrm{T}ab}{}_{fg}s^f s^g s_h P^{\mathrm{T}he}{}_{cd}\right]
\end{aligned} \tag{4.11}$$

Working out the tensor contractions (assuming $\overleftrightarrow{d}$ is already traceless) gives

$$d_{ab}P^{\mathrm{T}ab}_{\ \ \ cd}s_e u^c u^d u^e = (\hat{s}\cdot\hat{u})(\hat{u}\cdot\overleftrightarrow{d}\cdot\hat{u}) \tag{4.12a}$$

$$\begin{aligned} d_{ab}P^{\mathrm{T}ab}_{\ \ \ fg}s^g s_h P^{\mathrm{T}fh}_{\ \ \ cd}s_e u^c u^d u^e &= (\hat{s}\cdot\hat{u})d_{fg}s^g s_h\left(u^f u^h - \frac{\delta^{fh}}{3}\right) \\ &= (\hat{s}\cdot\hat{u})^2(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{u}) - \frac{1}{3}(\hat{s}\cdot\hat{u})(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{s}) \end{aligned} \tag{4.12b}$$

$$d_{ab}P^{\mathrm{T}ab}_{\ \ \ fg}s^f s^g s_h s_i P^{\mathrm{T}hi}_{\ \ \ cd}s_e u^c u^d u^e = (\hat{s}\cdot\hat{u})\left((\hat{s}\cdot\hat{u})^2 - \frac{1}{3}\right)(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{s}) \tag{4.12c}$$

$$\begin{aligned} d_{ab}\left[P^{\mathrm{T}ab}_{\ \ \ fe}s_g P^{\mathrm{T}fg}_{\ \ \ cd} + P^{\mathrm{T}ab}_{\ \ \ fg}s^g P^{\mathrm{T}fe}_{\ \ \ cd}\right]u^c u^d u^e &= d_{fe}s_g u^e\left(u^f u^g - \frac{\delta^{fg}}{3}\right) + d_{fg}s_g u_e\left(u^f u^e - \frac{\delta^{fe}}{3}\right) \\ &= (\hat{s}\cdot\hat{u})(\hat{u}\cdot\overleftrightarrow{d}\cdot\hat{u}) + \frac{1}{3}(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{u}) \end{aligned} \tag{4.12d}$$

$$d_{ab}\left[P^{\mathrm{T}ab}_{\ \ \ fe}s^f s_g s_h P^{\mathrm{T}gh}_{\ \ \ cd} + P^{\mathrm{T}ab}_{\ \ \ fg}s^f s^g s_h P^{\mathrm{T}he}_{\ \ \ cd}\right]u^c u^d u^e = \left((\hat{s}\cdot\hat{u})^2 - \frac{1}{3}\right)(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{u}) + \frac{2}{3}(\hat{s}\cdot\hat{u})(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{s}) \tag{4.12e}$$

Combining (4.11), (4.12), and (4.5) gives us

$$\begin{aligned} \mathcal{G}(\overleftrightarrow{d},\alpha,\hat{s},\hat{u}) &= \left[\rho_1'(\alpha) + \frac{\rho_2(\alpha)}{\alpha}\right](\hat{s}\cdot\hat{u})(\hat{u}\cdot\overleftrightarrow{d}\cdot\hat{u}) \\ &+ \left\{(\hat{s}\cdot\hat{u})^2\left[\rho_2'(\alpha) - 2\frac{\rho_2(\alpha)}{\alpha} + 2\frac{\rho_3(\alpha)}{\alpha}\right] + \frac{1}{3}\left[\frac{\rho_2(\alpha)}{\alpha} - 2\frac{\rho_3(\alpha)}{\alpha}\right]\right\}(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{u}) \\ &+ \left\{(\hat{s}\cdot\hat{u})^2\left[\rho_3'(\alpha) - 4\frac{\rho_3(\alpha)}{\alpha}\right] + \frac{1}{3}\left[-\rho_2'(\alpha) + 2\frac{\rho_2(\alpha)}{\alpha} - \rho_3'(\alpha) + 8\frac{\rho_3(\alpha)}{\alpha}\right]\right\}(\hat{s}\cdot\hat{u})(\hat{s}\cdot\overleftrightarrow{d}\cdot\hat{s}) \end{aligned} \tag{4.13}$$

This is relatively easy to evaluate, if we keep in mind the recursion relation

$$\frac{d}{d\alpha}\frac{j_\ell(\alpha)}{\alpha^\ell} = -\alpha\frac{j_{\ell+1}(\alpha)}{\alpha^{\ell+1}} \tag{4.14}$$

and thus

$$\begin{pmatrix}\rho_1'(\alpha)\\ \rho_2'(\alpha)\\ \rho_3'(\alpha)\end{pmatrix} = -\alpha\begin{pmatrix}5 & -10 & 5\\ -10 & 40 & -50\\ \frac{5}{2} & -25 & \frac{175}{2}\end{pmatrix}\begin{pmatrix}\frac{j_1(\alpha)}{\alpha}\\ \frac{j_2(\alpha)}{\alpha^2}\\ \frac{j_3(\alpha)}{\alpha^3}\end{pmatrix} \tag{4.15}$$

Note that since the limiting forms of the spherical Bessel functions tell us that

$$\rho_1(\alpha) = 2 + \mathcal{O}(\alpha^2) \tag{4.16a}$$

$$\rho_2(\alpha) = \mathcal{O}(\alpha^2) \tag{4.16b}$$

$$\rho_3(\alpha) = \mathcal{O}(\alpha^2) \tag{4.16c}$$

all of the coëfficients in (4.13) vanish at $\alpha = 0$. and thus $\mathcal{G}(\overleftrightarrow{d},0,\hat{s},\hat{u}) = 0$, which we could also see by symmetry considerations from (4.5).

	$\gamma^{\text{LWL}}(f)$	$\delta\gamma(f)$	$\delta\gamma(f)/\gamma^{\text{LWL}}(f)$
XARM	0.95333	0.00298	0.00313
YARM	−0.89466	−0.00167	0.00187
NULL	0.03181	−0.00061	−0.01914

Table 1: Impact of first-order corrections on L1-A1 search. The corrections to the overlap reduction function are less than one percent, except for the null orientation. The upper limit results in [4] are not affected to the stated precision by these corrections.

5 Specific Examples

The matlab/octave functions `curlyG.m` and `orfcorrection.m` implement (4.13) and (4.4); they can be found in the CVS at `sgwb/doc/TechNotes/figsources`. We use them to examine the corrections to the overlap reduction function for

1. LLO-ALLEGRO, which has actually been analyzed at 915 Hz.[4]
2. LHO-LLO around 1 kHz, which is being considered for S5 as a counterpart to LIGO-Virgo analyses, and
3. LHO-Virgo and LLO-Virgo around 1 kHz, which are being considered for S5.

5.1 LLO-ALLEGRO

This is fairly easy to consider, since the overlap reduction function (and its first-order correction) is more or less constant across the band of interest. We summarize the corrections in table 5.1. As a check, the scripts used in [4] were re-run with the LWL plus first order overlap reduction functions; the upper limit result was unchanged, while some numbers in tables changed in the third decimal place.

5.2 LHO-LLO

We move on to consider LHO-LLO. Of course, at frequencies previously considered ($\lesssim$ 300 Hz) the effects are negligible. In Figure 1 we show the LWL and first-order overlap reduction functions. However, it's hard to quantify the differences by eye. We can consider the ratio $\delta\gamma(f)/\gamma(f)$ (Fig 2), but this is awkward because $\gamma(f)$ passes through zero. One useful tool for quantifying the size of the corrections is the high-frequency envelope, $\pm\gamma_{\text{env}}/f$, which describes the falloff of the long-wavelength overlap reduction function.[5] For LHO-LLO, this is plotted in Fig. 3. We can thus plot $\frac{\delta\gamma(f)}{\gamma_{\text{env}}/f}$ to get a sense of the size of the corrections. This is done in Fig. 4, which shows that the correction is $> 5\%$ of the LWL amplitude at 1 kHz. We thus conclude that first-order corrections to the overlap reduction function will be necessary if LLO-LHO pairs are included in an analysis around 1 kHz.

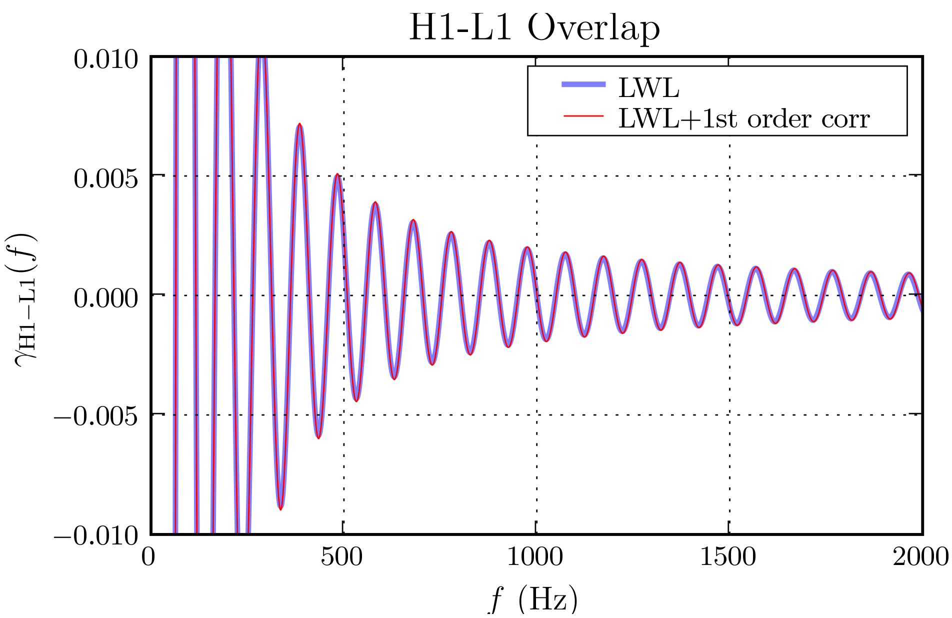

Figure 1: Long-wavelength overlap reduction function for LHO-LLO pair, compared with first-order corrected version. The differences are small, but it's hard to get a quantitative sense with the "eyeball test".

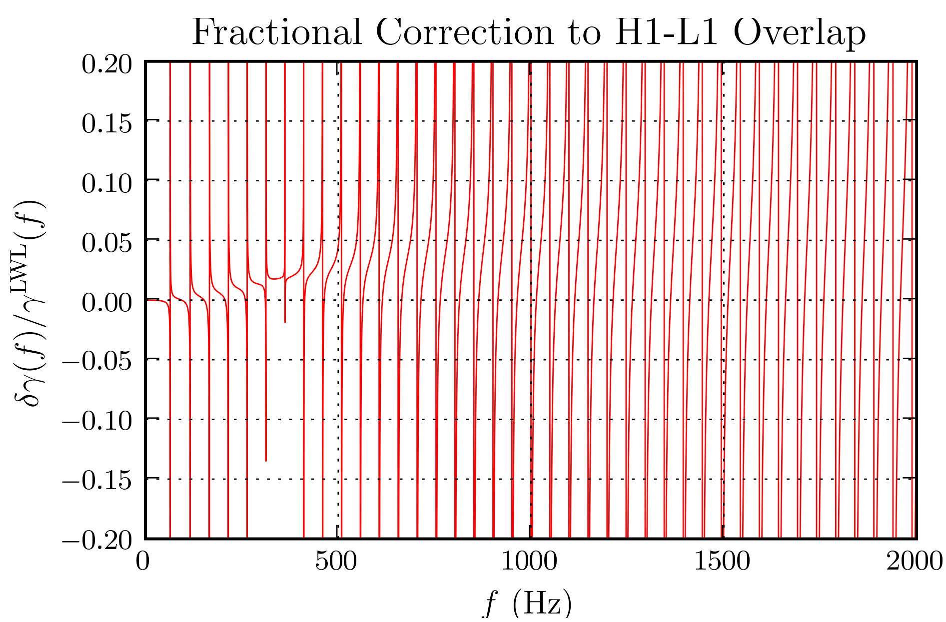

Figure 2: Ratio of first-order LHO-LLO overlap reduction function to long-wavelength value. Because the correction and the long-wavelength form have zeros in different places, the ratio blows up at some frequencies (the ones that contribute least to the search sensitivity) and is therefore not very informative.

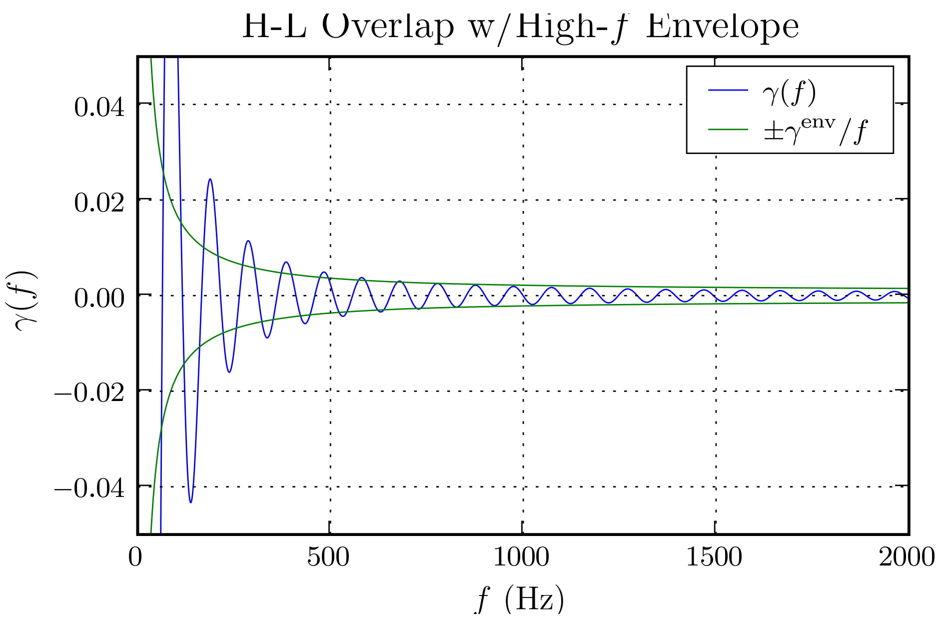

Figure 3: The LHO-LLO overlap reduction function, plotted along with its high-frequency envelope as calculated in [5]. The $1/f$ envelope captures the amplitude of the oscillations at high frequencies.

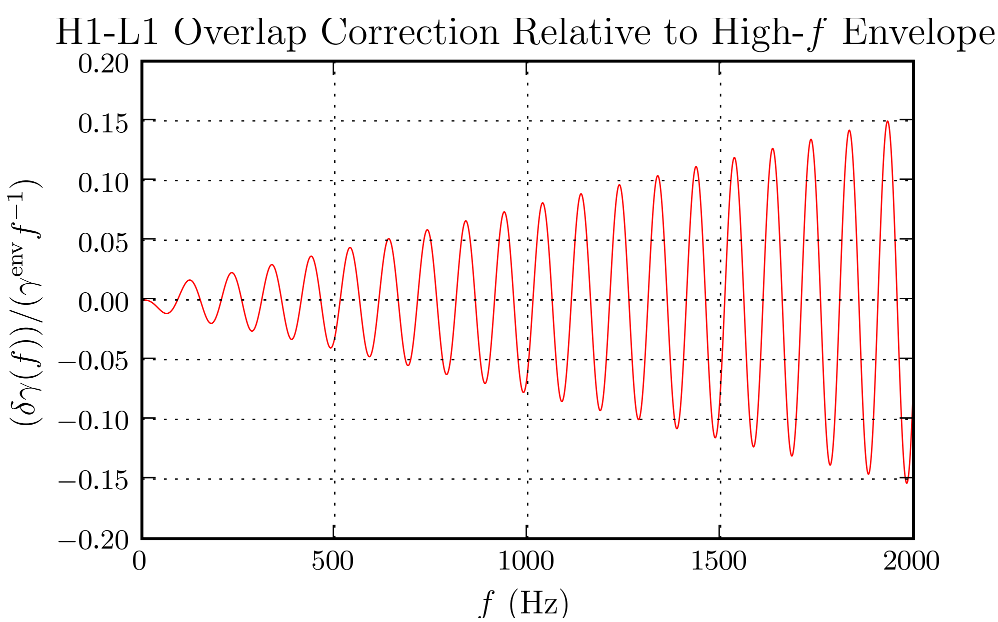

Figure 4: Size of first-order corrections to the LHO-LLO overlap reduction function relative to its overall amplitude. We see that at kilohertz frequencies, $5-10\%$ corrections are necessary.

5.3 LIGO-Virgo

We repeat the same comparison for the LIGO-Virgo detector pairs, plotting $\frac{\delta\gamma(f)}{\gamma_{\text{env}}/f}$ for LHO-Virgo and LLO-Virgo in Fig. 5. In this case, we see that the errors are less than 1%, so the corrections for LIGO-Virgo searches will be negligible.

6 Conclusions

An examination of the (analytically calculated) first-order corrections to the isotropic overlap reduction functions due to finite interferometer arm length, for various detector pairs, shows that

1. Corrections for LLO-ALLEGRO (due to the finite length of the LLO arms) are negligible at 915 Hz
2. Corrections for LHO-LLO near 1 kHz may be 5-10%, so first-order corrections should be incorporated
3. Corrections for LHO-Virgo and LLO-Virgo near 1 kHz are $< 1\%$, so so first-order corrections can be neglected

A Calculation of Rigid Adiabatic Response Tensor

The most general tensor gravitational wave in the TT gauge is

$$\overleftrightarrow{h}(t,\vec{r}) = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \underbrace{\sum_{A=+,\times} h_A(f,\hat{k})\overleftrightarrow{e}_A(\hat{k})}_{\overleftrightarrow{h}(f,\hat{k})} e^{i2\pi f(t-\hat{k}\cdot\vec{r}/c)} \tag{A.1}$$

Where $h_A(f,\hat{k})$ are arbitrary amplitudes and $\{\overleftrightarrow{e}_A(\hat{k})\}$ are the TT polarization basis orthogonal to $\hat{k}$. The spacetime metric it generates is

$$ds^2 = -c^2\,dt^2 + d\vec{r}\cdot\left(\overleftrightarrow{1} + \overleftrightarrow{h}(t,\vec{r})\right)\cdot d\vec{r}\ . \tag{A.2}$$

A.1 Propagation Time Down a Finite-Length Arm

Consider two wordlines with fixed spatial coördinates; in the TT gauge, these will be geodesics. Let their separation vector be $L\hat{n}$ so that a photon travels from the spacetime

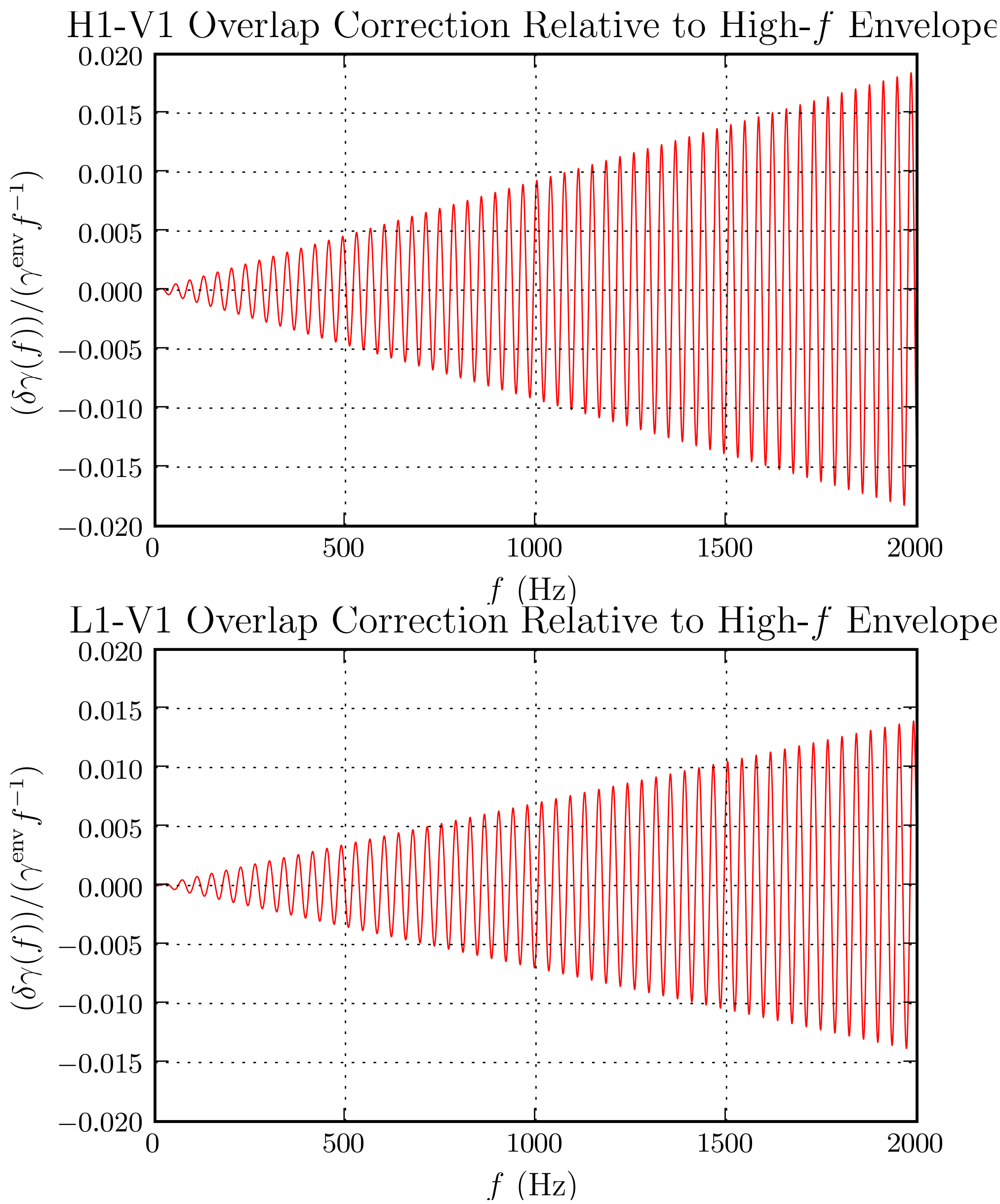

Figure 5: Size of first-order corrections to the LIGO-Virgo overlap reduction functions relative to their overall amplitude. Note that the vertical scale here is different from that used in Fig. 4 and in fact the corrections are $< 1\%$.

point $(t_i, \vec{r}_{\text{mid}} - \frac{L}{2}\hat{n})$ to $(t_f, \vec{r}_{\text{mid}} + \frac{L}{2}\hat{n})$. To lowest order in the metric perturbation, the photon's spatial trajectory can be parametrized as

$$\vec{r}(\lambda) = \vec{r}_{\text{mid}} + \lambda \frac{L}{2}\hat{n} \tag{A.3}$$

where λ goes from -1 to 1.[1] The elapsed time can be obtained from the fact that the photon's trajectory is null:

$$dt = c\sqrt{d\vec{r} \cdot \left(\overleftrightarrow{1} + \overleftrightarrow{h}(t, \vec{r})\right) \cdot d\vec{r}} = \frac{L}{2c}\left(1 + \hat{n} \cdot \overleftrightarrow{h}(t, \vec{r}) \cdot \hat{n}\right)^{1/2} d\lambda \tag{A.4}$$

and integrating this gives (defining $t_{\text{mid}} = t_i + L/2c$)

$$\begin{aligned} t_f - t_i =& \frac{L}{2c}\int_{-1}^{1}\left[1 + \frac{1}{2}\hat{n} \cdot \overleftrightarrow{h}\left(t_{\text{mid}} + \lambda\frac{L}{2c}, \vec{r}_{\text{mid}} + \lambda\frac{L}{2}\hat{n}\right) \cdot \hat{n}\right] + \mathcal{O}(h^2) \\ =& \frac{L}{c}\left(1 + \frac{1}{2}\int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \overleftrightarrow{h}(f, \hat{k}) : (\hat{n} \otimes \hat{n}) e^{i2\pi f(t_{\text{mid}} - \hat{k}\cdot\vec{r}_{\text{mid}}/c)} \frac{1}{2}\int_{-1}^{1} e^{i2\pi f \frac{L}{2c}(1 - \hat{k}\cdot\hat{n})\lambda}\right) \end{aligned} \tag{A.5}$$

The integral over λ is just

$$\frac{1}{2}\int_{-1}^{1} e^{i2\pi f \frac{L}{2c}(1 - \hat{k}\cdot\hat{n})\lambda} = \frac{e^{i2\pi f \frac{L}{2c}(1 - \hat{k}\cdot\hat{n})} - e^{-i2\pi f \frac{L}{2c}(1 - \hat{k}\cdot\hat{n})}}{2i\left[2\pi f \frac{L}{2c}(1 - \hat{k}\cdot\hat{n})\right]} = \operatorname{sinc}\left(\frac{\pi f L}{c}[1 - \hat{k}\cdot\hat{n}]\right) \tag{A.6}$$

So

$$t_f - t_i = \frac{L}{c}\left\{1 + \frac{1}{2}\int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \overleftrightarrow{h}(f, \hat{k}) : (\hat{n} \otimes \hat{n}) e^{i2\pi f(t_{\text{mid}} - \hat{k}\cdot\vec{r}_{\text{mid}}/c)} \operatorname{sinc}\left(\frac{\pi f L}{c}[1 - \hat{k}\cdot\hat{n}]\right)\right\} \tag{A.7}$$

A.2 Michelson Interferometer Response

Consider an interferometer with arms of length L pointing in directions $\hat{u}$ and $\hat{v}$. Let the vertex be at position $\vec{r}$. Let t be the time that two photons meet at the vertex after travelling down their respective arms and back.

First, consider the round-trip travel time down the first arm. This can be broken into two parts:

[1] I think this step is wrong, since there is an $\mathcal{O}(h)$ correction to $d\vec{r}$ (not necessarily along $\hat{n}$) that I'm leaving out, and this would give an additional $\mathcal{O}(h)$ term in dt. However, I seem to get the same answer as Rubbo, Cornish and Poujade.[1]. We now think the explanation for this is that if you use fractional distance down the arm as a parameter for the timelike geodesic, the missing correction term is perpendicular to $\hat{n}$ and therefore gives no first-order contribution when substituted into (A.4).

- The inbound trip, where $\hat{n} = -\hat{u}$ and to lowest order $t_{\text{mid}} = t - L/2c$ and $\vec{r}_{\text{mid}} = \vec{r} + \hat{u}L/2$, so
$$t_{\text{mid}} - \hat{k} \cdot \vec{r}_{\text{mid}}/c = t - \hat{k} \cdot \vec{r}/c - \frac{L}{2c}(1 + \hat{k} \cdot \hat{u}) \tag{A.8}$$
- The outbound trip, where $\hat{n} = \hat{u}$ and to lowest order $t_{\text{mid}} = t - 3L/2c$ and $\vec{r}_{\text{mid}} = \vec{r} + \hat{u}L/2$, so
$$t_{\text{mid}} - \hat{k} \cdot \vec{r}_{\text{mid}}/c = t - \hat{k} \cdot \vec{r}/c - \frac{L}{2c}(3 + \hat{k} \cdot \hat{u}) \tag{A.9}$$

The fractional change in round-trip travel time down the arm in the $\hat{u}$ direction due to the GW is thus

$$\begin{aligned}\frac{c(\Delta t)_{\hat{u}} - 2L}{2L} = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \overleftrightarrow{h}(f, \hat{k}) e^{i2\pi f(t - \hat{k}\cdot\vec{r}/c)} \\ : e^{-i2\pi fL/c} \frac{\hat{u} \otimes \hat{u}}{2} \left[e^{i\frac{\pi fL}{c}(1-\hat{k}\cdot\hat{u})} \operatorname{sinc}\left(\frac{\pi fL}{c}[1 + \hat{k} \cdot \hat{u}]\right) \right. \\ \left. + e^{-i\frac{\pi fL}{c}(1+\hat{k}\cdot\hat{u})} \operatorname{sinc}\left(\frac{\pi fL}{c}[1 - \hat{k} \cdot \hat{u}]\right) \right] /2\end{aligned} \tag{A.10}$$

In the limit $fL \ll 1$ this reduces to the familiar

$$\frac{c(\Delta t)_{\hat{u}} - 2L}{2L} \to \overleftrightarrow{h}(t, \vec{r}) : \frac{\hat{u} \otimes \hat{u}}{2} \tag{A.11}$$

so we write the generalization as

$$\frac{c(\Delta t)_{\hat{u}} - 2L}{2L} = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \overleftrightarrow{h}(f, \hat{k}) e^{i2\pi f(t - \hat{k}\cdot\vec{r}/c)} : \overleftrightarrow{d}_{\hat{u}}(f, \hat{k}) \tag{A.12}$$

where

$$\overleftrightarrow{d}_{\hat{u}}(f, \hat{k}) = \mathcal{T}_{\hat{u}}(f, \hat{k}) \frac{\hat{u} \otimes \hat{u}}{2} \tag{A.13}$$

borrowing from [1] the notation

$$\mathcal{T}_{\hat{u}}(f, \hat{k}) = \frac{e^{-i2\pi fL/c}}{2} \left[e^{i\frac{\pi fL}{c}(1-\hat{k}\cdot\hat{u})} \operatorname{sinc}\left(\frac{\pi fL}{c}[1 + \hat{k} \cdot \hat{u}]\right) + e^{-i\frac{\pi fL}{c}(1+\hat{k}\cdot\hat{u})} \operatorname{sinc}\left(\frac{\pi fL}{c}[1 - \hat{k} \cdot \hat{u}]\right) \right] . \tag{A.14}$$

Note that this corresponds to $D(i2\pi f, -\hat{k} \cdot \hat{u})$ as defined by [2], albeit in rather different notation.

The standard Michelson interferometer, then, measures

$$\frac{c(\Delta t)_{\hat{u}} - c(\Delta t)_{\hat{v}}}{2L} = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \overleftrightarrow{h}(f, \hat{k}) e^{i2\pi f(t - \hat{k}\cdot\vec{r}/c)} : \left\{ \mathcal{T}_{\hat{u}}(f, \hat{k}) \frac{\hat{u} \otimes \hat{u}}{2} - \mathcal{T}_{\hat{v}}(f, \hat{k}) \frac{\hat{v} \otimes \hat{v}}{2} \right\} \tag{A.15}$$

A.3 Fabry-Perot Effect

The LIGO interferometers are not, however, simple Michelson interferometers. The arms act as Fabry-Perot cavities, which store light which gradually leaks out of the interferometer. The result of this is that a measurement at time t reflects the Michelson response convolved with a time-dependent effect.[2, 3] The result is that the quantity measured is

$$\int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \frac{1 - r_a r_b}{1 - r_a r_b e^{-i4\pi f L/c}} \overleftrightarrow{h}(f, \hat{k}) e^{i2\pi f(t - \hat{k}\cdot\vec{r}/c)} : \left\{ \mathcal{T}_{\hat{u}}(f, \hat{k}) \frac{\hat{u}\otimes\hat{u}}{2} - \mathcal{T}_{\hat{v}}(f, \hat{k}) \frac{\hat{v}\otimes\hat{v}}{2} \right\} \tag{A.16}$$

The LIGO calibration doesn't actually use the full frequency-dependent Fabry-Perot response

$$R_{\text{fp}}(f) = \frac{1 - r_a r_b}{1 - r_a r_b e^{-i4\pi f L/c}} \tag{A.17}$$

but instead approximates it with a single cavity pole

$$R_{\text{cp}}(f) = \frac{1}{1 + if/f_{\text{pole}}} \tag{A.18}$$

writing

$$R_{\text{fp}}(f) = \frac{1}{1 + \frac{r_a r_b}{1 - r_a r_b}(1 - e^{-i4\pi f L/c})} \tag{A.19}$$

we can see that

$$f_{\text{pole}} = \frac{1 - r_a r_b}{r_a r_b} \frac{c}{4\pi L} \ . \tag{A.20}$$

Since we've expanded the exponential to first order in fL/c, this expression would seem to be adequate as long as second-order deviations from the long-wavelength limit don't become important. However, since the reflectivity of the LIGO mirrors is high, $r_a r_b$ is close to one, and $\frac{r_a r_b}{1 - r_a r_b}$ is actually rather large. This means the neglected second-order correction to the exponential, once we multiply it by $\frac{r_a r_b}{1 - r_a r_b}$, is more like the size of a first-order quantity. Fortunately, this apparent problem is resolved if we absorb into the Fabry-Perot response the troublesome prefactor $e^{-i2\pi f L/c}$ in (A.14). Then we have

$$R_{\text{fp}}(f) e^{-i2\pi f L/c} = \frac{1 - r_a r_b}{e^{i2\pi f L/c} - r_a r_b e^{-i2\pi f L/c}} = \frac{(r_a r_b)^{-1/2} - (r_a r_b)^{1/2}}{(r_a r_b)^{-1/2} e^{i2\pi f L/c} - (r_a r_b)^{1/2} e^{-i2\pi f L/c}} \tag{A.21}$$

Now if we define

$$\eta = -\frac{1}{2} \ln(r_a r_b) \tag{A.22}$$

and

$$\beta = \pi f L/c \tag{A.23}$$

which are both small quantities for $f \sim 1\,\text{kHz}$,

$$R_{\text{fp}}(f) e^{-i2\pi f L/c} = \frac{e^{\eta} - e^{-\eta}}{e^{\eta + i2\beta} - e^{-\eta - i2\beta}} = \frac{\sinh \eta}{\sinh(\eta + i2\beta)} = \frac{\eta + \mathcal{O}(\epsilon^3)}{\eta + i2\beta + \mathcal{O}(\epsilon^3)} = \frac{1}{1 + i2\beta/\eta + \mathcal{O}(\epsilon^2)} \tag{A.24}$$

So in fact the cavity pole model is accurate to *second* order in small quantities when used to approximate $R_{\rm fp}(f)e^{-i2\pi fL/c}$. That leads us to define

$$\mathfrak{T}_{\hat{u}}(f,\hat{k}) = e^{i\frac{\pi fL}{c}(1-\hat{k}\cdot\hat{u})}\,\mathrm{sinc}\left(\frac{\pi fL}{c}[1+\hat{k}\cdot\hat{u}]\right) + e^{-i\frac{\pi fL}{c}(1+\hat{k}\cdot\hat{u})}\,\mathrm{sinc}\left(\frac{\pi fL}{c}[1-\hat{k}\cdot\hat{u}]\right) \quad \text{(A.25)}$$

and that observe that the LIGO calibration (valid to second order in f/L) actually gives us a "strain" of

$$h(t) = \int_{-\infty}^{\infty} df \iint d^2\Omega_{\hat{k}} \overleftrightarrow{h}(f,\hat{k}) e^{i2\pi f(t-\hat{k}\cdot\vec{r}/c)} : \left\{ \mathfrak{T}_{\hat{u}}(f,\hat{k})\frac{\hat{u}\otimes\hat{u}}{2} - \mathfrak{T}_{\hat{v}}(f,\hat{k})\frac{\hat{v}\otimes\hat{v}}{2} \right\} \qquad \text{(A.26)}$$

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