LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
- LIGO -
CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Technical Note **LIGO-T070222-00-Z** 2007/09/20

The effect of time shift on a cross-correlation statistic

Adrian Melissinos, Tobin Fricke and Stefanos Giampanis

California Institute of Technology
LIGO Project, MS 18-34
Pasadena, CA 91125
Phone (626) 395-2129
Fax (626) 304-9834
E-mail: info@ligo.caltech.edu

Massachusetts Institute of Technology
LIGO Project, Room NW17-161
Cambridge, MA 02139
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

WWW: http://www.ligo.caltech.edu/

The effect of time shift on a cross-correlation statistic

Adrian Melissinos, Tobin Fricke and Stefanos Giampanis
University of Rochester
2007/09/20

In a note of Feb.6,2007 (LIGO-T070025-00-Z) we discussed "Synchronization issues affecting the S4 stochastic search". Here we expand on that note addressing the case where a large static shift is introduced between the time origin of the two time series that are being correlated. In practice the correlation is carried out in the frequency domain and a "statistic" is evaluated by integrating over a finite frequency interval Δf.

Let us first consider the case of no time shift where (in obvious notation) the two channels are represented in the time domain by

$$h_1(t) = s(t) + n_1(t) \qquad h_2(t) = s(t) + n_2(t) \tag{1}$$

and in the frequency domain by

$$x_1(f) = s(f) + n_1(f) \qquad x_2(f) = s(f) + n_2(f) \tag{2}$$

so that

$$\langle x_1^*(f) x_2(f) \rangle = |s(f)|^2 \tag{3}$$

where the average of the noise terms has been set to zero.

If the origin of time in the second channel if shifted (advanced) by τ, namely $t' = (t - \tau)$ then we have instead

$$h_1(t) = s(t) + n_1(t) \qquad h_2(t') = s(t - \tau) + n_2(t') \tag{4}$$

and in the frequency domain by

$$x_1(f) = s(f) + n_1(f) \qquad x_2(f) = s(f) e^{2\pi i f \tau} + n_2(f) \tag{5}$$

and as before

$$\langle x_1^*(f) x_2(f) \rangle = |s(f)|^2 e^{2\pi i f \tau} \tag{6}$$

Namely the cross-correlation which is real for the case $\tau = 0$, has now acquired a frequency dependent phase $\phi = 2\pi f \tau$.

The statistic is the integral over a finite bandwidth $\Delta f = 2\delta$ centered at f_0

$$S = \int_{f_0 - \delta}^{f_0 + \delta} |s(f)|^2 e^{2\pi i f \tau} df. \tag{7}$$

If we assume that $|s(f)|^2$ is reasonably constant in the integration interval we can replace it by $|s(f_0)|^2$ and setting $x = f - f_0$

$$S = |s(f_0)|^2 e^{2\pi i f_0 \tau} \int_{\delta}^{-\delta} e^{2\pi i x \tau} dx = |s(f_0)|^2 e^{2\pi i f_0 \tau} \Delta f \frac{\sin(\pi \Delta \mathrm{f} \tau)}{\pi \Delta f \tau} \tag{8}$$

As long as the *sinc* function remains close to unity the statistic is modified from its value at $\tau = 0$ only by the fixed phase factor $\phi_0 = 2\pi f_0 \tau$. The condition for this to hold is

$$\Delta f \ \tau \leq \ 1 \tag{9}$$

and is independent of the central frequency f_0. However f_0 determines the rate at which the phase ϕ rotates for any given time shift τ.

The above conclusions are confirmed by the results of numerical calculations shown in Figs 1-4. We construct two time series with 2048 entries each; the entries are random numbers with zero mean and unit variance. We can shift the time series by an arbitrary number of sampling intervals or by a fraction of the interval; in that case we use a linear extrapolation between the adjacent entries in the time series. We form the normalized cross correlation in the time domain directly, i.e. without recourse to the convolution theorem, and average (integrate) over the entire time interval. By normalized we mean that for each time bin the cross correlation is divided by the absolute value of the entries. Hence the normalized cross correlation is bounded in the interval ± 1. As expected, the normalized cross correlation drops from its value of $+1$ at $\tau = 0$ to zero when τ exceeds the sampling interval.

Next we obtain the frequency spectra of the two time series and form their normalized cross correlation in the frequency domain. We calculate the statistic by integrating over a finite bandwidth Δf centered at f_0. In the figures we plot in blue the real part of the integral (the statistic) vs. time shift in units of the sampling interval Δt; this is the natural unit. The normalized cross correlation in the time domain is shown in red. Fig.1 is obtained by integrating over the entire bandwidth, namely $\Delta f = 1/(2\Delta t)$ and for this case the frequency and time correlations are identical. Fig.2 is calculated with $\Delta f = 1/8$ of the full bandwidth and centered at 1/4 of the Nyquist frequency, $f_0 = 1/(8\Delta t)$. In this case the real part of the correlation in the frequency domain survives well beyond a time shift exceeding the sampling period.We have superimposed the corresponding envelope predicted by Eq(8). In Fig.3 the integration is over a smaller frequency band $\Delta f = 1/32$ of the full bandwidth centered at the same frequency as in Fig.2. The effect of the sinc function modulation is now clearly visible. Finally Fig.4 shows the results for integration over an even smaller frequency band $\Delta f = 1/128$ of the full bandwidth, still centered at $f_0 = 1/(8\Delta t)$. Increasing the center frequency produces faster oscillations but does not affect the envelope which is controlled by the integration bandwidth Δf.

An intuitive explanation is as follows. Since we are integrating only over a fraction of the Nyquist frequency, say α, we are not utilizing the complete range of the sampling rate. We could have obtained the same results with a lower sampling rate, i.e a sampling period increased by α. But then the time delay before loosing coherence (which equals the sampling period) also increases by the same factor.

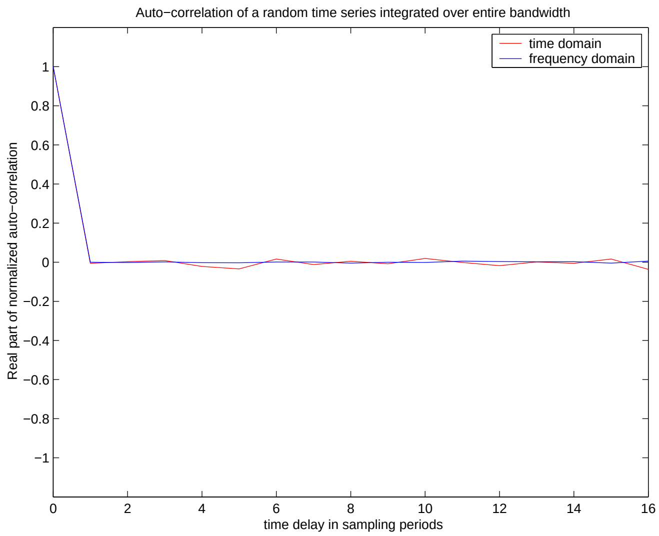

Figure 1: The real part of the normalized cross correlation statistic when integrating over the entire bandwidth

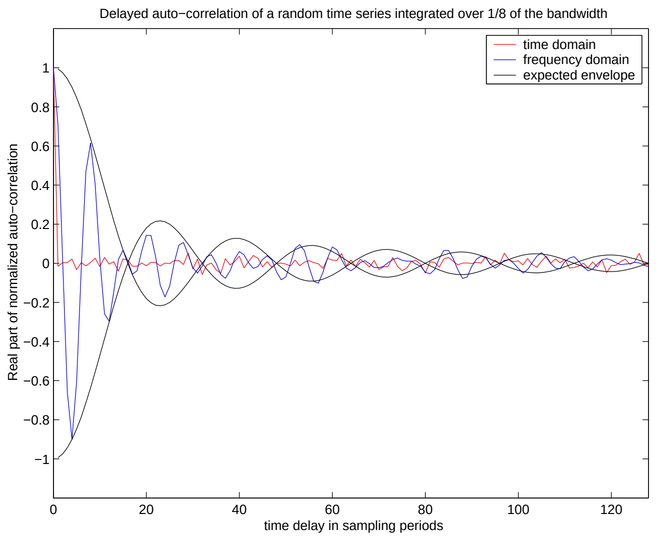

Figure 2: The real part of the normalized cross correlation statistic when integrating over $\Delta f = 1/8$ of the full bandwidth

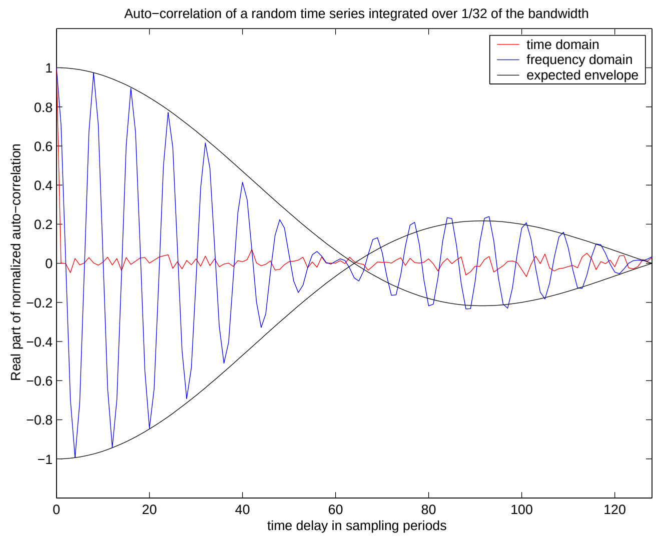

Figure 3: The real part of the normalized cross correlation statistic when integrating over $\Delta f = 1/32$ of the full bandwidth

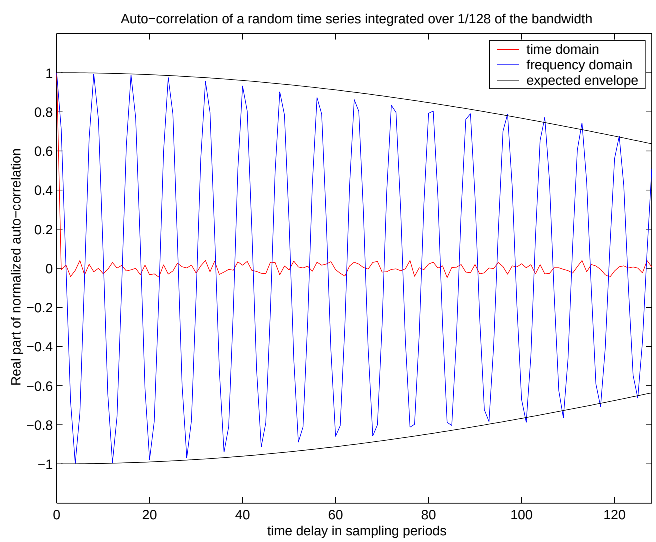

Figure 4: The real part of the normalized cross correlation statistic when integrating over $\Delta f = 1/128$ of the full bandwidth