SHOT NOISE IN THE CALTECH GRAVITATIONAL WAVE DETECTOR - THE MID-1984 CONFIGURATION

Stanley E. Whitcomb

California Institute of Technology
Division of Physics, Mathematics and Astronomy
Pasadena, California 91125 USA

ABSTRACT

The goal of this note is to calculate the shot noise for the Caltech interferometer in the configuration it is actually used now.

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Introduction

The present configuration of the Caltech interferometric gravitational wave detector is shown in Figure 1. Two 40m long optical cavities define the arms of the detector. Separate photodiodes monitor the reflected light from the first mirrors of each cavity. The first arm is used to stabilize the laser frequency while the second arm measures the difference between the laser frequency and the cavity resonance. The comparisons are made using a phase modulation technique. The goal of this paper is to calculate the shot noise for the current interferometer configuration, including the effects of the modulation scheme and certain imperfections in the interferometer (most notably, the nonideal fringe visibility).

The Modulation Scheme

Consider arm 1 first. The input laser light is phase modulated at ~12 MHz, a frequency high compared with the cavity bandwidth. Assume that the incident laser light has α_1 photons/sec. The beam incident on the the cavity can be decomposed into two kinds of light--a fraction β^2 which is properly mode-matched to the cavity and a fraction $1-\beta^2$ which is not. The incorrectly matched part will be entirely reflected onto the photodiode. The properly matched light falling on the photodetector is due to the interference between two amplitudes - the reflected laser light with its phase-modulation and the cavity light which has no phase modulation (See Figure 2a). The intensity on the photodiode is just the sum of the intensities of the properly mode-matched light and the improperly mode-matched light; there is no interference between these two pieces of the light because they are in spatially orthogonal modes (provided the photodiode is large enough to intercept the full beam). Thus the expected intensity on the photodiode is

$$<\eta_1(t)> = \alpha_1[1 + \beta^2 A^2 J_0^2(\Phi_m) - 2\beta^2 A J_0(\Phi_m)\cos(\Phi_m \sin\omega_m t)]$$

where A is the amplitude the cavity light would have in resonance if there were no modulation, J_0 is a Bessel function of zero order, and Φ_m and ω_m are the amplitude and frequency of the phase modulation. A is determined by the properties of the cavity mirrors,

$$A = \frac{|t_1|^2|r_2|}{1-|r_1 r_2|}$$

where t_1, r_1, and r_2 are the amplitude transmissivity and reflectivity of the cavity mirrors. The amplitude of the cavity light is reduced by a factor of $J_0(\Phi_m)$, the fraction of the incident field which is not shifted to one of the sidebands by the phase modulation. The actual number of photons falling on the detector $\eta_1(t)$ will vary about the expected number. The rms variation of η_1 in a short interval dt will have the value given by Poisson statistics, namely, $[\eta_1(t)dt]^{1/2}$.

The effect of a gravity wave is to cause a phase difference φ_1 between the laser light and the cavity light (Figure 2b). In general, the phase difference in arm 2 caused by the gravity wave will have the opposite sign from φ_1, and it is the difference $\varphi_1 - \varphi_2$ which we measure. This phase difference can be related to the gravity wave amplitude $h(\omega)$ by a transfer function which depends on the frequency and angle of incidence of the wave (preprint). The most favorable case is that of a wave incident normal to the plane of the detector. With the additional assumption that the detector is small compared to the gravitational wavelength, the phase differences are given by $\varphi_1 = -\varphi_2$ and

$$\Delta\varphi(\omega) = \frac{4\pi L}{\lambda} \frac{e^{i\omega\tau_t}(1 - |r_1 r_2|)}{1 - 2|r_1 r_2| e^{i\omega\tau_t} \cos \omega\tau_t + |r_1 r_2|^2 e^{i2\omega\tau_t}} h(\omega)$$

$$\approx \frac{4\pi L}{\lambda} \left[\frac{\tau_t}{\tau_s} - 2i\omega\tau_t\right]^{-1} h(\omega)$$

where L is the length of one arm, $\tau_t = L/c$, and τ_s is the cavity storage time,

$$\tau_s = \frac{\tau_t}{1 - |r_1 r_2|}$$

The noise level is frequently expressed as a displacement (m Hz$^{-½}$) using the relation

$$\delta l = hL$$

The photon flux incident on photodetector 1 is the presence of a wave is

$$<\eta_1(t)> = \alpha_1\{1 + \beta^2 A^2 J_0^2 - 2\beta^2 A J_0 \cos\left[\varphi_1(t) + \Phi_m \sin \omega_m t\right]\}$$

where all Bessel functions are henceforth understood to be evaluated at Φ_m. For any realistic case, $\varphi_{1,2} \ll 1$, and we can expand to first order in $\varphi_{1,2}$

$$<\eta_1(t)> \approx \alpha_1[1 + \beta^2 A^2 J_0^2 - 2\beta^2 A J_0 \cos(\Phi_m \sin \omega_m t) + 2\varphi_1(t)\beta^2 A J_0 \sin(\Phi_m \sin \omega_m t)]$$

The signal from the photodiode is demodulated using a mixer. The process is equivalent to multiplying by $\sin \omega_m t$ and averaging with a time constant long compared to ω_m^{-1} but short compared to the time scales interesting for gravitational wave detection.

Signal

Assume a sinusoidal signal

$$\varphi_1 = -\varphi_2 = \Delta\varphi_0 \sin \omega t$$

$\Delta\varphi_0$ can be measured by comparing signals S_1 and S_2 derived from the two photodetectors,

$$S_1(\omega) = \frac{1}{T}\int_0^T \eta_1(t) \sin \omega_m t \sin \omega t \, dt$$

The transfer function shown assumes that $h(t)$ and $\Delta\varphi(t)$ are related to $h(\omega)$ and $\Delta\varphi(\omega)$ by

$$f(t) = \int f(\omega) e^{-i\omega t} d\omega$$

Engineers may prefer a plus sign in the exponential, in which case they should change ω to $-\omega$ in the transfer function.

$$= \frac{\alpha_1 \beta^2 A J_0 J_1 \Delta\varphi_0}{2}$$

It follows trivially that

$$\Delta\varphi_0 = \frac{S_1(\omega)}{\alpha_1 \beta^2 A J_0 J_1} - \frac{S_2(\omega)}{\alpha_2 \beta^2 A J_0 J_1}$$

Noise

To calculate the noise, assume $\varphi_1 = \varphi_2 = 0$. In this case the noise is formally defined to be the sum of the squares of $\Delta\varphi$ measured with both $\sin \omega t$ and $\cos \omega t$ phases, i.e.

$$N_\varphi^2\,(\omega) = \left[\frac{1}{T} \int_0^T \left\{ \frac{\eta_1}{\alpha_1} - \frac{\eta_2}{\alpha_2} \right\} \frac{\sin \omega_m t \, \sin \omega t}{\beta^2 A J_0 J_1} dt \right]^2$$

$$+ \left[\frac{1}{T} \int_0^T \left\{ \frac{\eta_1}{\alpha_1} - \frac{\eta_2}{\alpha_2} \right\} \frac{\sin \omega_m t \, \cos \omega t}{\beta^2 A J_0 J_1} dt \right]^2$$

where < > denotes average value. (This is what we see on the spectrum analyzer.) To evaluate N_φ^2 we can use a generalization of an elementary formula for the propagation of errors. If

$$y = \sum a_i x_i$$

then

$$\sigma_y^2 = \sum \left[\frac{\partial y}{\partial x_i} \right]^2 \sigma_{x_i}{}^2$$
$$= \sum a_i^2 \sigma_{x_i}{}^2$$

Converting this to an integral, remembering that the uncertainty in $\eta(t)dt$ is $[\eta(t)dt]^{1/2}$, yields

$$N_\varphi^2\,(\omega) = \frac{1}{T} \int_0^T \left\{ \frac{\eta_1}{\alpha_1^2} + \frac{\eta_2}{\alpha_2^2} \right\} \frac{\sin^2 \omega_m t}{\beta^4 A^2 J_0^2 J_1^2} dt$$

$$= \left[\frac{\beta^{-2} + A^2 J_0^2 - 2 A J_0^2 + 2 A J_0 J_2}{2 \beta^2 A^2 J_0^2 J_1^2} \right] \left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right]$$

Notice that $N_\varphi^2(\omega)$ has units of radians^{2}Hz^{-1}.

This noise level can be converted to either displacement sensitivity

$$N_x^2(\omega) = \left[\frac{\lambda}{4\pi} \right]^2 \left[\frac{\tau_t^2}{\tau_s^2} + (2\omega\tau_t)^2 \right] \left[\frac{\beta^{-2} + A^2 J_0^2 - 2 A J_0^2 + 2 A J_0 J_2}{2 \beta^2 A^2 J_0^2 J_1^2} \right] \left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right]$$

or gravity wave sensitivity

$$N_h(\omega) = \frac{N_x(\omega)}{L}$$

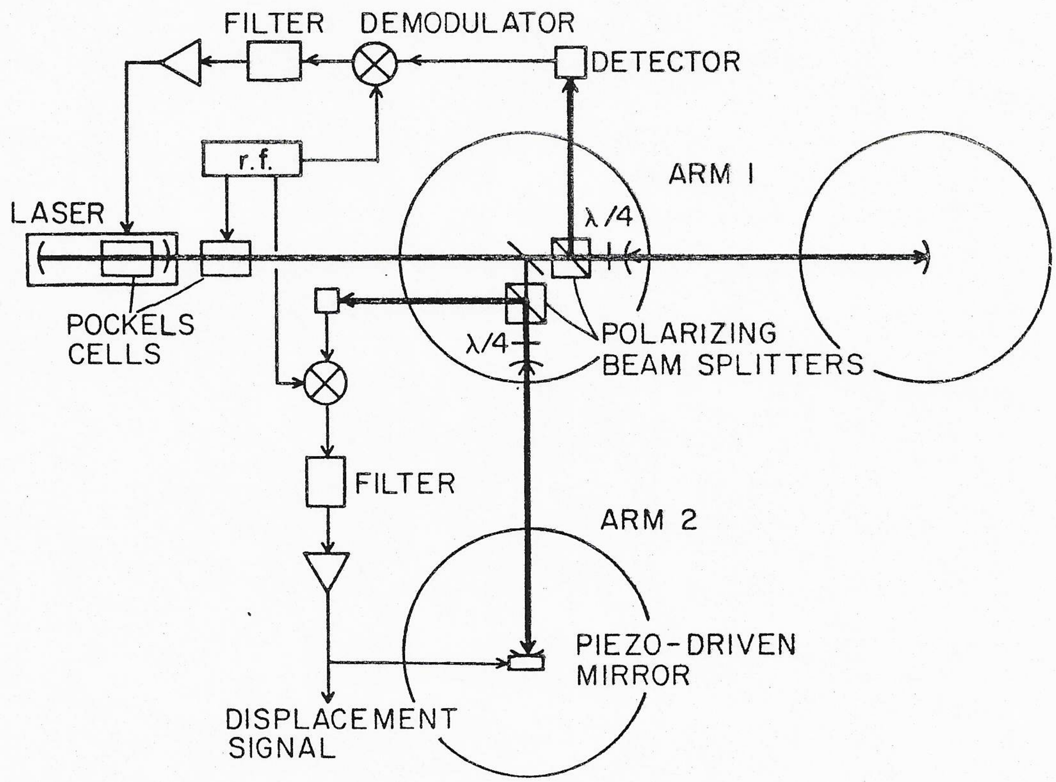

FILTER
DEMODULATOR
DETECTOR
r.f.
ARM 1
LASER
λ/4
POCKELS
CELLS
POLARIZING
BEAM SPLITTERS
λ/4
FILTER
ARM 2
PIEZO-DRIVEN
MIRROR
DISPLACEMENT
SIGNAL

FIGURE 1

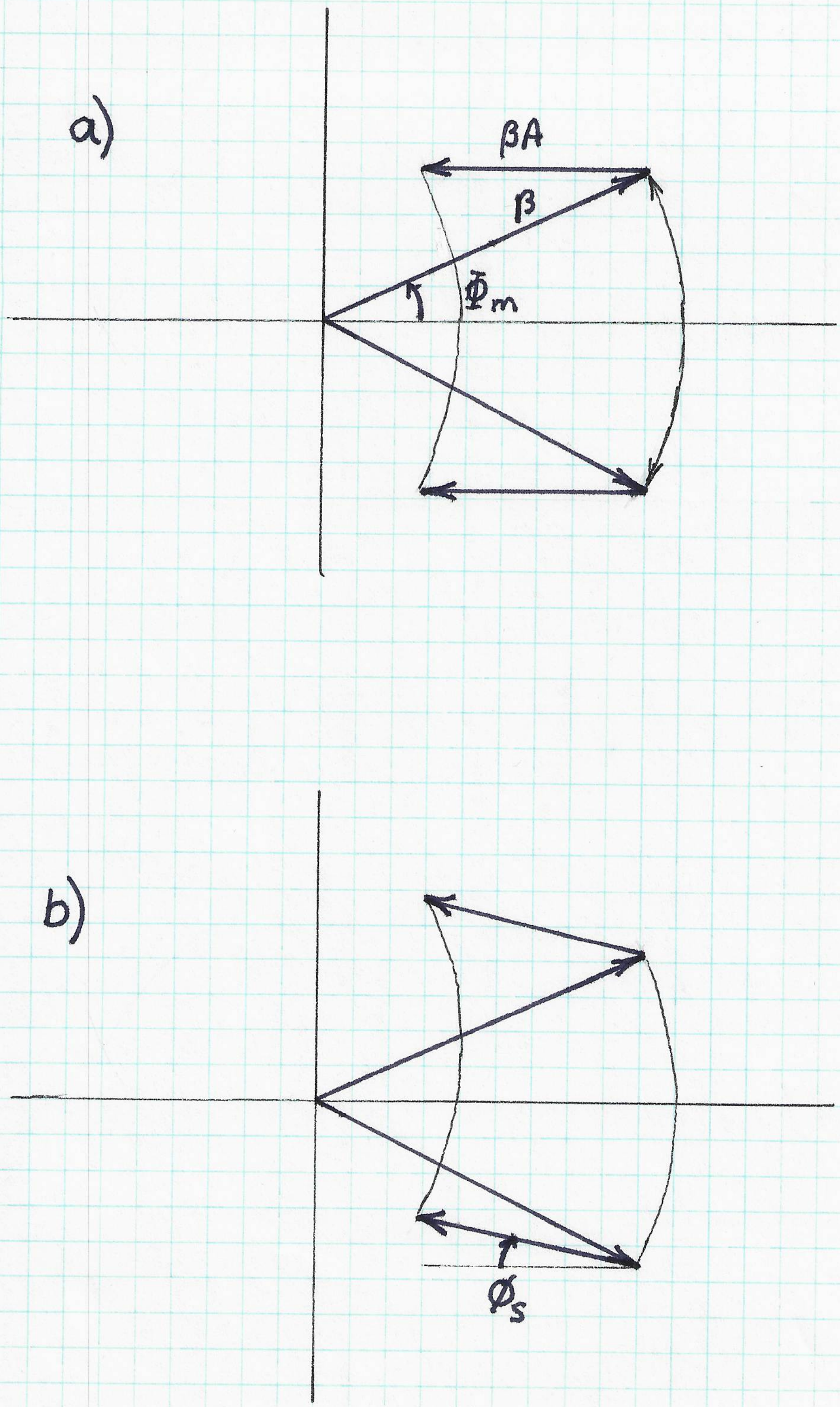
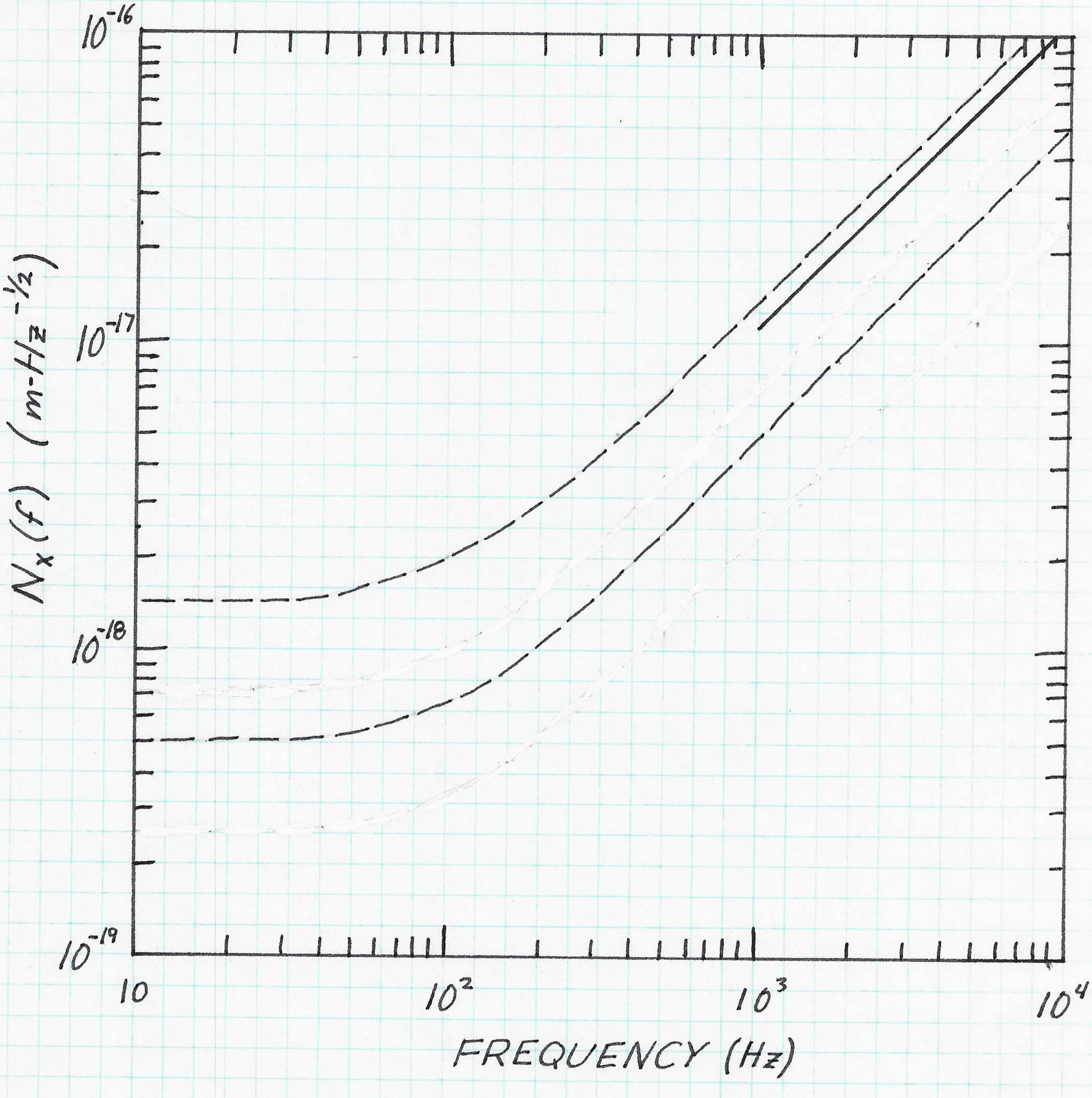


FIGURE 2



10^{-16}
10^{-17}
10^{-18}
10^{-19}
$N_x(f)$ $(m\text{-}Hz^{-1/2})$
10
10^2
10^3
10^4
FREQUENCY (Hz)

FIGURE 3