Memo: THERMAL121788.TEX

From: RW (DEC 1, 1988 hand written, DEC 30, 1988 Typed)

To : ALTHOUSE

Concerning: Thermal Considerations for Ligo Tubes

SUMMARY

The memo discusses a variety of phenomena driven by temporal and spatial temperature variations that could occur in the LIGO facilities. The effect on the vacuum has been described in an earlier memo. The topics covered are:

1) The strain near the earth's surface driven by temperature waves. The conclusion is that there are no serious problems providing that a uniform model of the earth's elastic properties applies. The experience with laser strain seismometers confirms the conclusion, however the issue should be revisited if the near surface geology at a site is very heterogeneous. A useful reference is the strain due to the inevitable solid earth tide which is larger than the predicted thermal strain.

2) The thermal distortions of structures not firmly coupled to the elastic earth will obey the usual expansion equations.

3) Thermal diffusion into the ground is presented in graphical form both as a transfer function and in terms of a step response. The diffusion length for seasonal temperature changes has the largest amplitude. The diffusion lengths for non convecting air and water are included for estimation purposes that may be important in considering different cover designs.

4) The stress in the beam tubes as a function of thermal gradients and clamping technique is calculated. The thermal stresses and accompanying distortions are compared with those due to gravity, wind loading, atmospheric pressure and the fatique stress for multiple stress cycles.

5) The anticipated thermal gradient on the tubes due to solar illumination is calculated. The dominant cooling mechanism is radiation, convection plays a smaller role.

6) The largest thermally driven strain using simple tube supports is bending of the tubes due to thermal gradients. This must be taken seriously. In addition, the stresses for clamped supports are too large and care must be taken in making sure that tube supports do not become clamped accidentally.

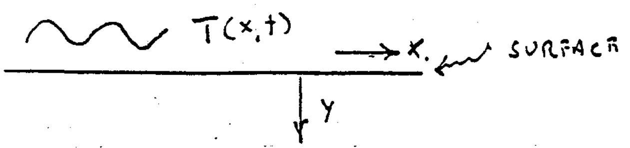
7) The problem of thermal expansion driving stick/slip motion of the tubes and the associated impulsive noise that propagates down the tubes is not addressed in this memo.

Thermo Elastic Distortions of the Ground

Reference: Berger, J., Journal Geophys Res. 80, 274 1975. Model assumes uniform Poisson's ratio and Young's modulus for the ground.

Berger solves 2-dimensional problem of Earth driven by temperature waves with angular frequency ω and wavevector k as independent variables.

Model:



$$T(x,t) = T_0 e^{i(kx-\omega t)}$$

Definitions of Symbols:

$\tau = \frac{2\pi}{\omega}$

β= Thermal Expansion Coefficient $\sim 10^{-5}\ K$ (Gravel, Rock, Sand)

K_{th}=Thermal Conductivity $1.6 \times 10^{-2} \longrightarrow 6 \times 10^{-3} Watts/cmK$

C_v= Specific Heat $1 \longrightarrow 0.6 joules/gmK$

ρ =Density $.9 \longrightarrow 3.0 gms/cm^3$

κ=Thermal Diffusivity $K_{th}/c_v\rho \sim 1.0 \times 10^{-2} \pm 3 \times 10^{-3} cm^2/sec$

Typical Value: adopt $1 \times 10^{-2} cm^2/sec$

σ=Poisson's ratio $0.20 \longrightarrow 0.26$ adopt .25

Thermal Diffusion into Ground

$$T(y,x,t) = T_o e^{-\gamma y} e^{i(kx-\omega t)}$$

$$\gamma = K\left[1 + \frac{i\omega}{\kappa k^2}\right]^{1/2}$$

In our limit, unless we bury to a depth, $y \sim \frac{2\pi}{k}$, we will always be near the thermal boundary layer.

$$\frac{\omega}{\kappa k^2} \gg 1 \qquad \gamma \cong (1+i)[\frac{\omega}{2\kappa}]^{\frac{1}{2}} \qquad |\gamma| \approx \left(\frac{2\pi}{\tau\kappa}\right)^{\frac{1}{2}}$$

Typical values of γ

$\lvert\gamma\rvert \frac{1}{cm}$	τ
3.2	1 min
4.1×10^{-1}	1 hour
8.4×10^{-2}	1 day
3.16×10^{-2}	1 week
5.8×10^{-3}	1 month
1.7×10^{-3}	1 year

Scale of Waves – Temporal and Spatial

Phenomena	λcm	τ	$\frac{k}{\gamma}$
Clouds + Rain	$10^4 \rightarrow 10^7 cm$	min→ hrs	$< 1.5 \times 10^{-3}$
Solar Heating	10^7 *(Bounded by ocean)* → 3×10^8 *(Continent)* → 3×10^9 *(Earth)*	1*day*	$< 8 \times 10^{-6}$
Season	10^9	1 year	$< 4 \times 10^{-6}$

For all reasonable cases $\frac{k}{\gamma} \ll 1$

Limiting forms of thermo elastic 2 dimensional solutions are:

Strain Along Horizontal

$$e_{xx}(y,x,t) \simeq \left(\frac{1+\sigma}{1-\sigma}\right) \frac{k}{\gamma} \{(2(1-\sigma)-ky) \quad e^{-ky} - \frac{k}{\gamma} \quad e^{-\gamma y}\} \quad \beta T_o e^{i(kx-\omega t)}$$

Strain Along Vertical

$$e_{yy}(y,x,t) \simeq \left(\frac{1+\sigma}{1-\sigma}\right) \{\frac{-k}{\gamma} \quad (2\sigma - ky) \quad e^{-ky} + e^{-\gamma y}\} \quad \beta T_o e^{i(kx-\omega t)}$$

Shear Strain

$$e_{xy}(y,x,t) \simeq i\left(\frac{1+\sigma}{1-\sigma}\right) \frac{k}{\gamma} \{(1-ky) \quad e^{-ky} - e^{-\gamma y}\}\beta T_o e^{i(kx-\omega t)}$$

Tilt of Surface At Depth y

$$\Omega_y(y,x,t) \simeq -i\left(\frac{1+\sigma}{1-\sigma}\right) \frac{k}{\gamma} \{((1-2\sigma)-ky)\, e^{-ky} + e^{-\gamma y}\}\beta T_o e^{i(kx-\omega t)}$$

Assuming Further That $ky \ll 1$(within the thermal boundary layer)

$$e_{xx} \approx \frac{5}{2}\frac{k}{\gamma}\beta T_o \qquad ky \ll \frac{3}{2}$$

$$e_{yy} \approx \frac{5}{3}\left[e^{-\gamma y} - \frac{1}{2}\frac{k}{\gamma}\right]\beta T_o \qquad \text{crossing point} \qquad y = -\frac{ln(\frac{1}{2}\frac{k}{\gamma})}{\gamma}$$

$$e_{xy} \approx i\frac{5}{3}\frac{k}{\gamma}\left[1-e^{-\gamma y}\right]\beta T_o$$

$$\Omega y \approx -i\frac{5}{3}\frac{k}{\gamma}\left[\frac{1}{2}+e^{-\gamma y}\right]\beta T_o$$

Specific Examples for 4 km Arms

<u>Clouds and Rain</u>

Assume $\lambda \approx 4Km = 4\times 10^5 cm \quad \tau \approx 1\text{hour} \quad \Delta T \approx 10^o K$

$$\frac{k}{\gamma} \simeq 3.8\times 10^{-5} \qquad \gamma \approx 4.1\times 10^{-1} cm^{-1}$$

$$|\varepsilon_{xx}| = 1\times 10^{-8} \qquad \Delta x \approx 3.8\times 10^{-3} cm$$

$$|\varepsilon_{yy}|_{y=0} = 1.7\times 10^{-4} \qquad \Delta y_{(over\ length\ 1/\gamma)} \approx 4.2\times 10^{-4} cm \qquad \varepsilon_{yy} = 0 \quad y = 26cm$$

$$|\Omega_y|_{(y>2.4)} \approx 3.2\times 10^{-9} radians$$

Solar Heating

$$\text{Assume} \quad \lambda \approx 5 \times 10^7 cm \qquad \tau = 1\text{day} \qquad \Delta T \approx 30^o K$$

$$\frac{k}{\gamma} \approx 1.5 \times 10^{-6} \qquad \gamma \approx 8.4 \times 10^{-2} cm^{-1}$$

$$|\varepsilon_{xx}| = 3.8 \times 10^{-6} \qquad \Delta x \approx 4.5 \times 10^{-4} cm$$

$$|\varepsilon_{yy}|_{y=0} = 5 \times 10^{-4} \qquad \Delta y_{(over\ length\ 1/\gamma)} \approx 6 \times 10^{-3} cm \qquad \varepsilon_{yy} = 0 \quad y = 168cm$$

$$|\Omega_y|_{y > 12cm} \approx 3.8 \times 10^{-10} radians$$

Seasons

$$\text{Assume} \quad \lambda \approx 3 \times 10^8 cm \qquad \tau \approx 1\text{year} \qquad \Delta T \approx 30^o K$$

$$\frac{k}{\gamma} \approx 1.2 \times 10^{-5} \qquad \gamma \approx 1.7 \times 10^{-3} cm^{-1}$$

$$|\varepsilon_{xx}| = 9 \times 10^{-9} \qquad \Delta x \approx 3.6 \times 10^{-3} cm$$

$$|\varepsilon_{yy}|_{y=0} = 5 \times 10^{-4} \qquad \Delta y_{(over\ length\ 1/\gamma)} \approx 0.3cm \qquad \varepsilon_{yy} = 0 \quad y = 694cm$$

$$|\Omega_y|_{y=0} = 4.5 \times 10^{-9} radians$$

$$|\Omega_y|_{y=\gamma^{-1} \sim 588cm} = 3 \times 10^{-9} radians$$

Summary

1) The earth, providing it has homogeneous elastic and thermal constants, distorts little under thermal loading. Crudely, the thermal distortion (horizontal) is reduced relative to a free body at the surface by the ratio of the thermal diffusion distance divided by the driving thermal wavelength.

2) The vertical motions are not constrained by elasticity but the thermal strain occurs only over the thermal diffusion length so that the total thermal length change remains small. The vertical motions are generally the largest. The vertical strain crosses zero at approximately the thermal diffusion depth which depends on the period of the excitation.

3) We are lucky that the thermal distortions are as small as they are since shallow burial does not reduce them greatly. The thermal stress at the surface propagates into the depth through the elasticity to a distance approximately equal to λ of the thermal surface wave. It is no wonder that the heating of the sides of mountains can be seen in the constant temperature cavities deep in tunnels.

4) As a consequence of 3) the reduction of thermal distortions acting on the elasticity of the earth are not a reason to bury the LIGO. Such reasons must come from the temperature fluctuations acting on the apparatus directly.

5) A useful comparison is the unavoidable earth tide strain.

$$\varepsilon \ (12, 24hrs) \approx 10^{-7} \qquad \Delta x = 4 \times 10^{-2} cm \qquad 4km$$

This is larger than the thermal horizontal strain.

Thermal Diffusion into the Ground

Dominant transport into ground is thermal diffusion governed by the transport equation.

$$\nabla^2 T = \frac{c_v \rho}{K_{th}} \ \frac{\partial T}{\partial T} = \frac{1}{\kappa} \frac{\partial T}{\partial T}$$

c_v is the heat capacity
ρ is the density
K_{th} is the thermal conductivity (no convection)
κ is the thermal diffusivity

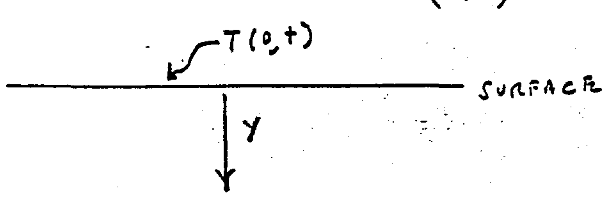
Typical Numbers

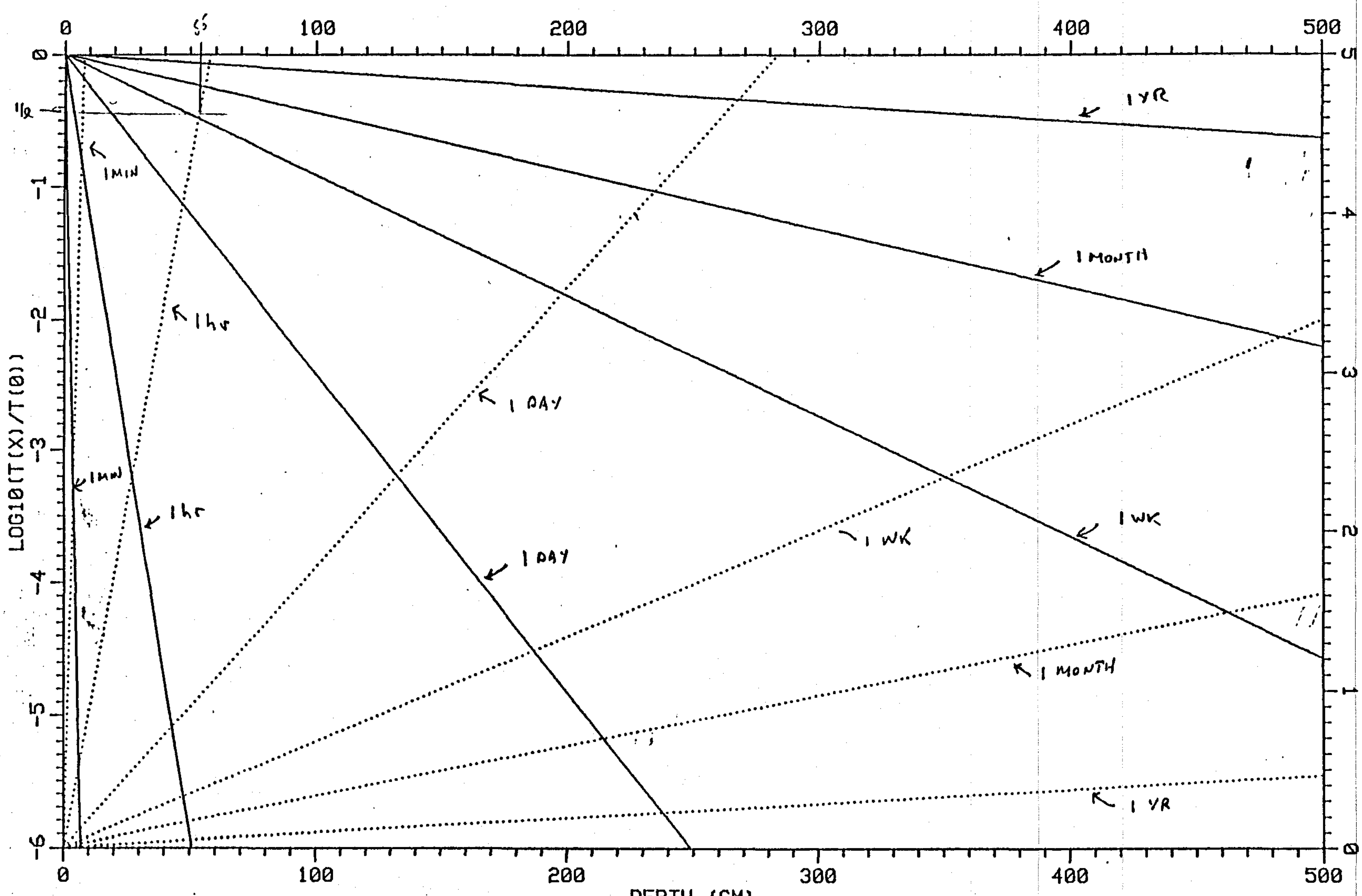
	$K_{th}(Watts/cmK)$	$c_V(Joules/gmK)$	$\rho(gram/cm^3)$	$\kappa\left(\frac{cm^2}{sec}\right)$
Gravel, rock	1.6×10^{-2}	8.4×10^{-1}	2.8	7×10^{-3}
Air*	1.7×10^{-4}	1	1.25×10^{-3}	1.36×10^{-1}
Water*	6×10^{-3}	4.2	1	1.43×10^{-3}

* non convecting

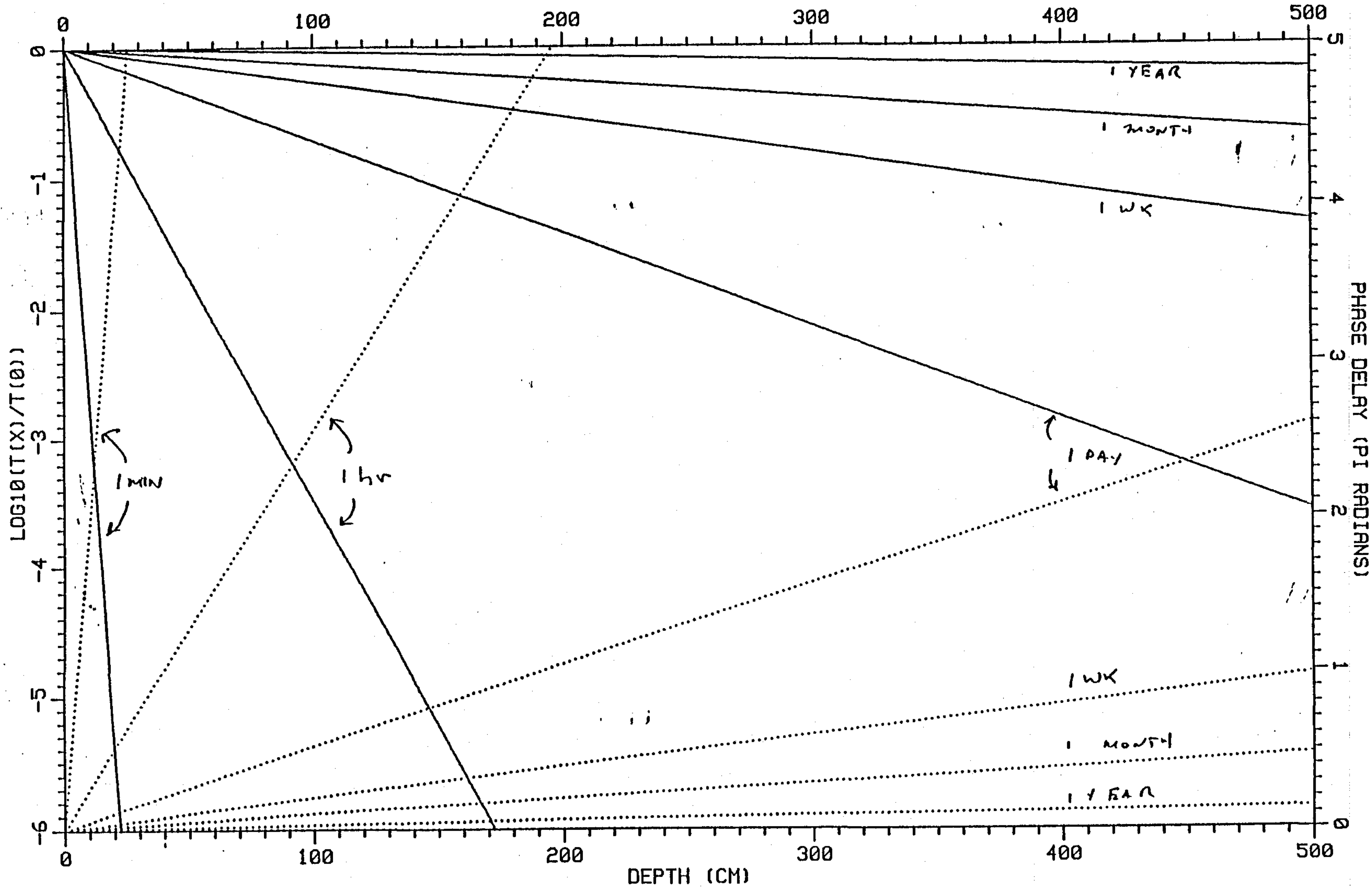
One dimensional solution to the diffusion equation is

$$T(0,t) = T_\circ \ cos\left(\frac{2\pi t}{\tau}\right) \quad \longleftarrow \text{ Surface Driving Temperature}$$





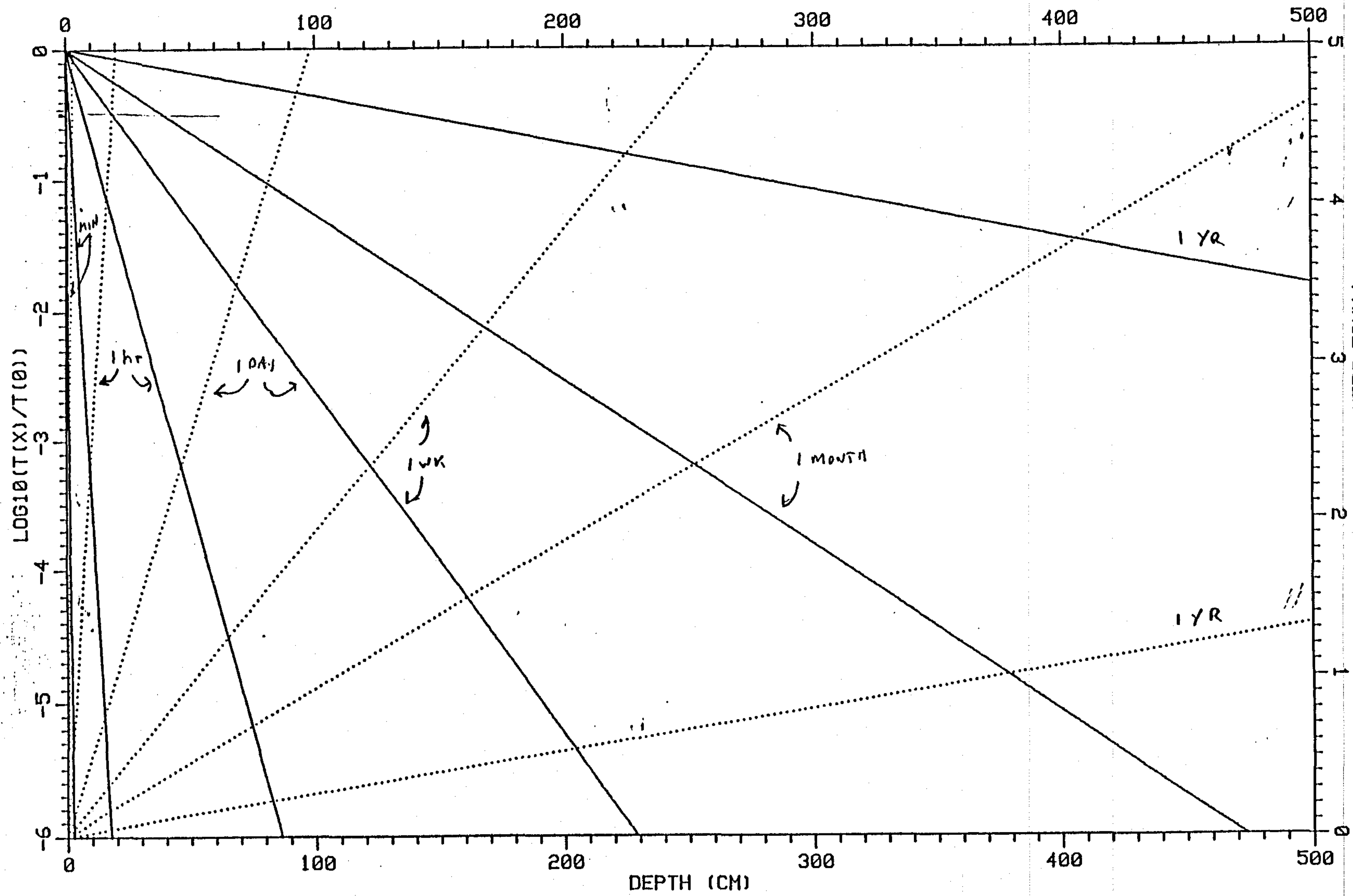
TRANSFER FUNCTION OF THERMAL DIFFUSION IN SOIL



TRANSFER FUNCTION OF THERMAL DIFFUSION IN AIR

NON CONVECTING

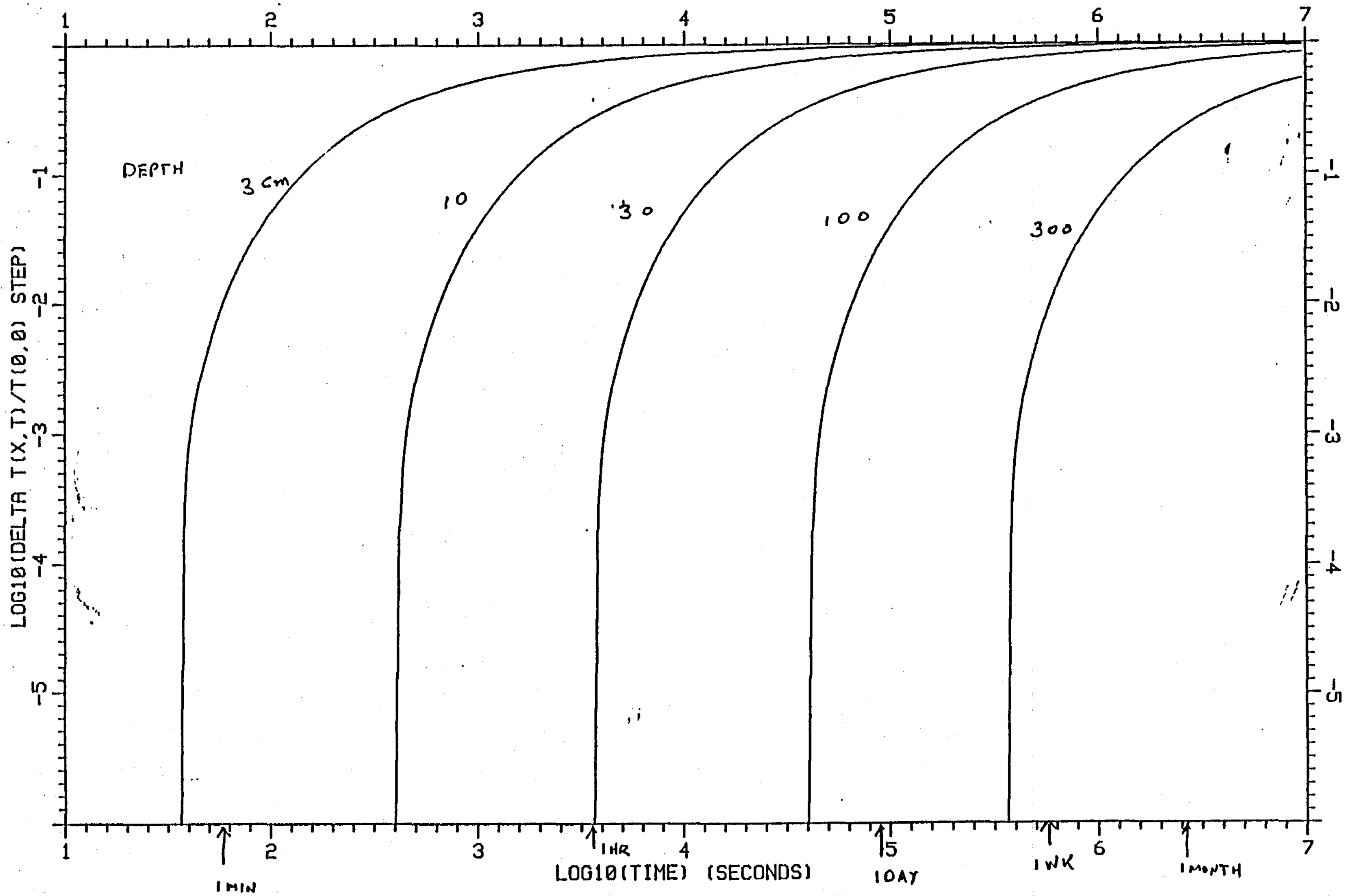
$\kappa = 1.36 \times 10^{-1}$



TRANSFER FUNCTION OF THERMAL DIFFUSION IN WATER

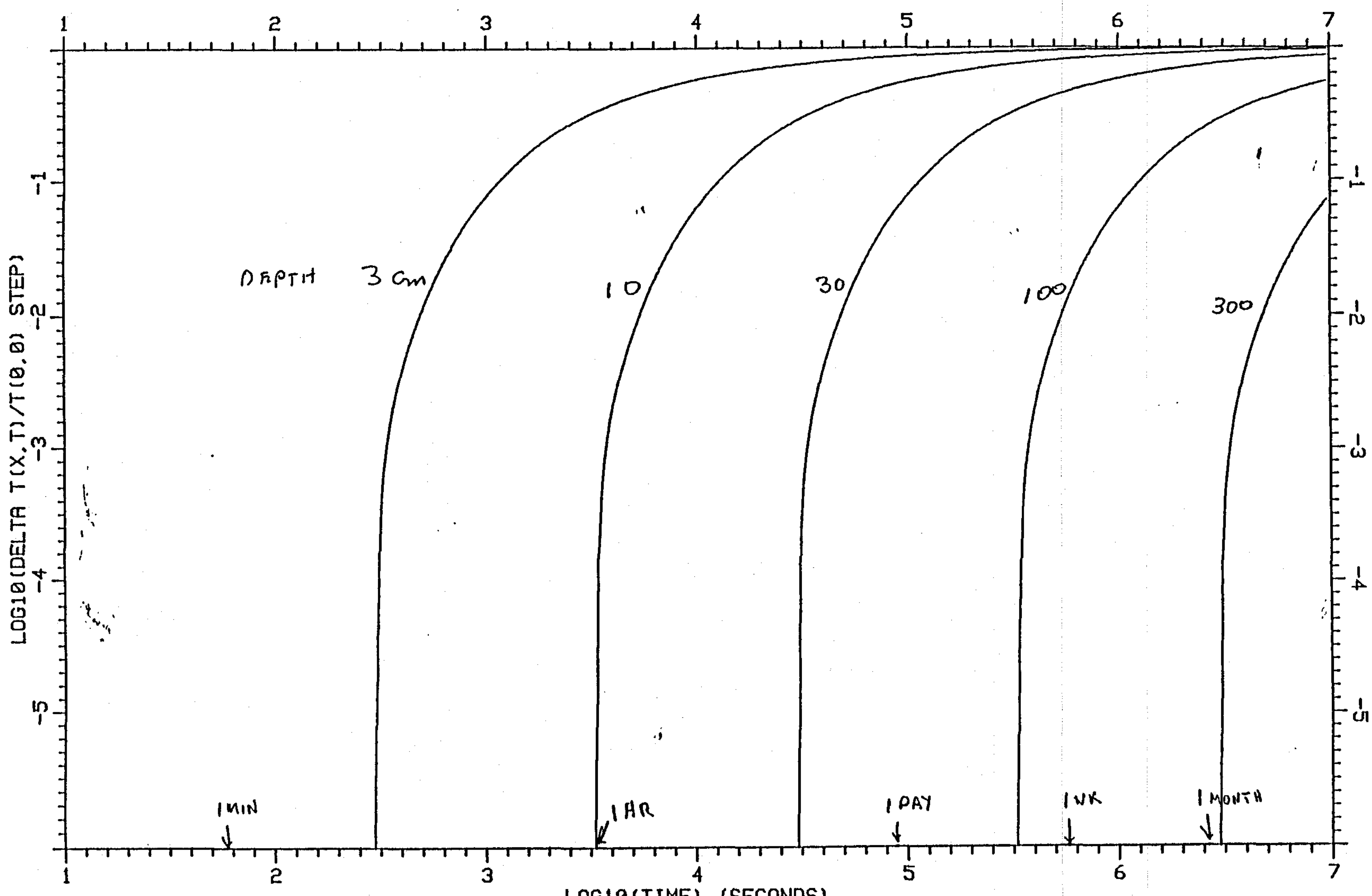
NON CONVECTING

$\kappa = 1.43 \times 10^{-3} \frac{sec}{cm^2}$



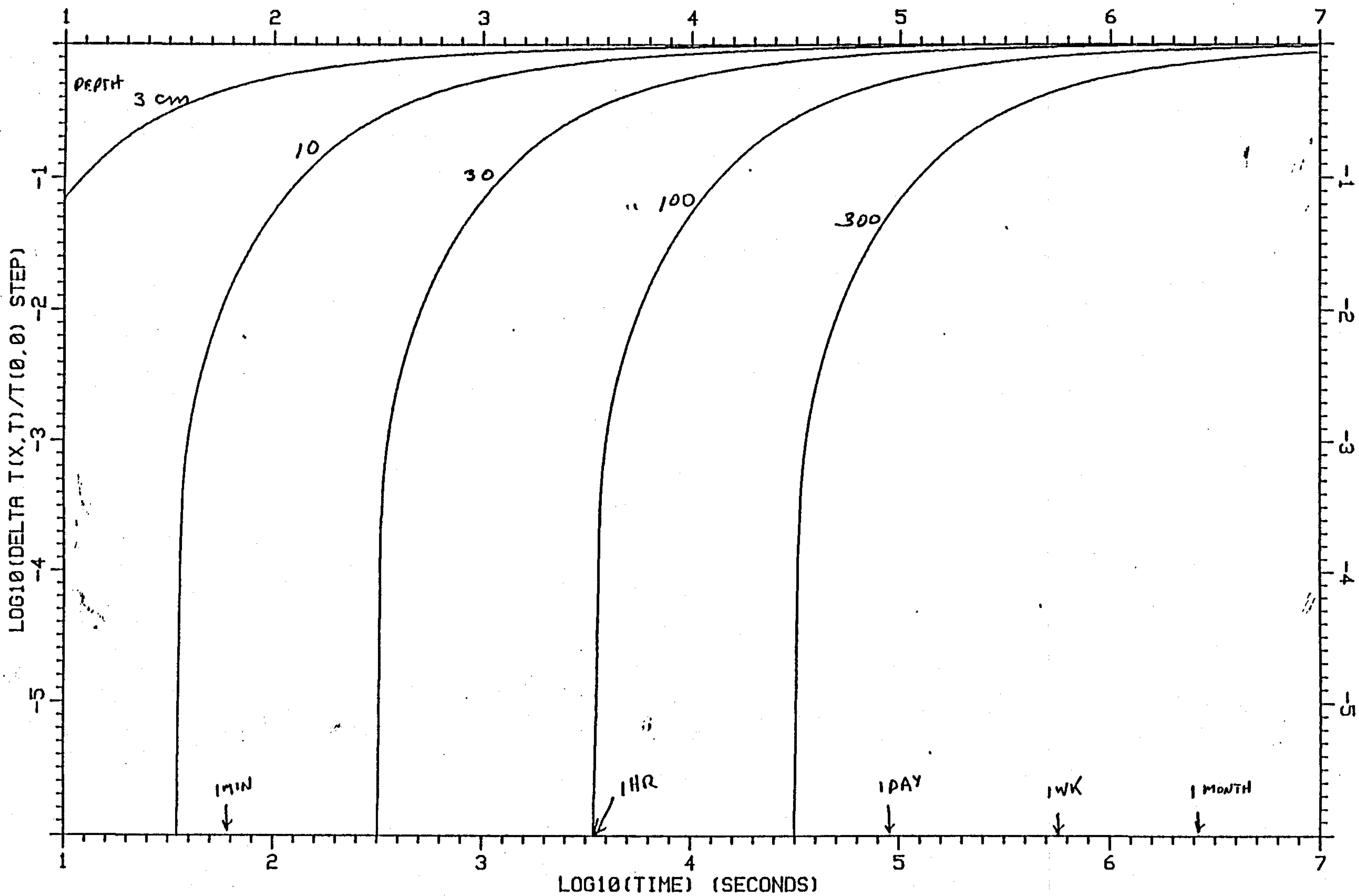
THERMAL STEP RESPONSE OF SOIL

$\kappa = 1.2 \times 10^{-2} \frac{sec}{cm^2}$



THERMAL STEP RESPONSE OF NON CONVECTING WATER

$\mathcal{K} = 1.43 \times 10^{-3} \frac{sec}{cm^2}$



THERMAL STEP RESPONSE OF NON CONVECTING AIR

$\tau = 1.36 \times 10^{-1} \frac{sec}{cm^2}$

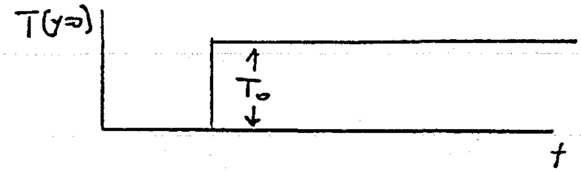
Response Transfer Function

$$T\left(y,t\right) = T_{\circ} e^{-\left(\frac{\pi}{\kappa\tau}\right)^{1/2} y} \quad cos\frac{2\pi}{\tau}\left(t - y\left(\frac{\tau}{4\pi\kappa}\right)^{1/2}\right)$$

See Plots

Step Response Function

If surface makes a step temperature change



$T\left(y = 0,\, t\right)$

Response to the step is

$$\frac{T\left(0,\,0\right) - T\left(y,\,t\right)}{T\left(0,\,0\right)} = \frac{2}{\sqrt{\pi}} \int_{\circ}^{y/2(\kappa\, t)^{1/2}} e^{-\lambda^2} d\,\lambda$$

Here expressed as the difference between the surface temperature and temperature at depth y and time t, normalized to step amplitude.

See Plots

Stresses on the tubes and deformations of the tubes

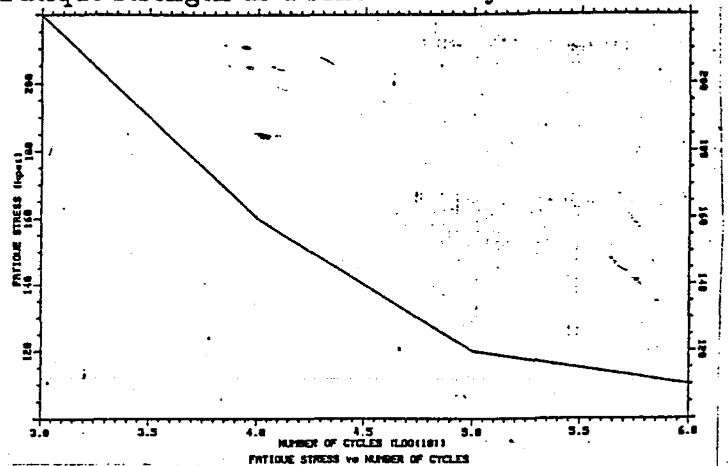
Young's Modulus $(10C) = 26000kpsi \Rightarrow 1.77 \times 10^{12} dynes$

$\rho = 7.8 \rightarrow 8 \;\; grams/cm^3$

<u>Properties of 304L Steel</u>

State	Temp (C)	yield stress	
Annealed	10	35 *kpsi*	$2.38 \times 10^9 dynes/cm$
Not Annealed	10	130 *kpsi*	8.8×10^9
		tensile strength	
Annealed	10	105 *kpsi*	7.14×10^9
Annealed	38	87 *kpsi*	5.92×10^9
Not Annealed	10	170 *kpsi*	1.16×10^{10}
Not Annealed	38	150 *kpsi*	1.02×10^{10}
Weld Strength in Tension			
	10	90 *kpsi*	6.12×10^9
	38	70 *kpsi*	4.76×10^9

Fatique strength as a function of cycles of flexure stress for Low carbon 304 at $21C$



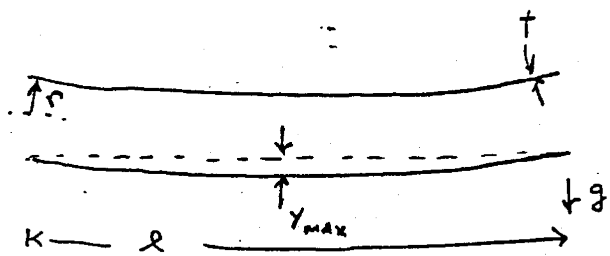
Weld fatigue about
0.5 parent metal

Number of Thermal Cycles in 20 years $\cong 10^4$

JPL numbers:

$$\ell = 70' \text{ between simple supports} \Rightarrow 2.13 \times 10^3 \text{cm}$$
$$r = 24'' \Rightarrow 61cm \text{ radius of tube}$$
$$t = 3/16'' \Rightarrow .476cm \text{ thickness of tube}$$
$$\mu = 3.06 \times 10^3 gm/cm \text{ linear mass density}$$

Gravitational loading <u>Tube Alone</u>



$$y_{max} = \frac{5}{384} \quad \frac{\mu g \ell^4}{\pi r^3 \, t \, y} = 0.64cm$$

$\sigma_{max}\left(\frac{\ell}{2}\right) = \frac{\mu g \ell^2}{8\pi r^2 t} =$ $\quad 3.1 \times 10^8 \quad dynes/cm^2$

bending moment — compression at bottom, tension at top

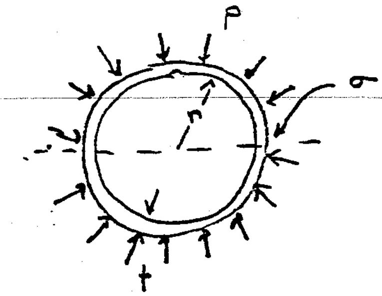
Adding 4" insulation (10cm) with aluminium cover 1/32″ thick

$\rho_{\text{insulation}}$	$=3lbs/ft^3$	$\Rightarrow 4.8 \times 10^2 \ gms/cm^3$
$\mu_{\text{insulation}}$	$=\rho 2\pi r t_{insul}$	$= 1.84 \times 10^2 gms/cm$
μ_{Al}	$=$	$9 \times 10' gms/cm$
$\mu_{\text{total insulation}}$		$2.74 \times 10^2 gms/cm$

$\mu_{\text{insulation}}/\mu_{\text{tube}} \approx 9 \times 10^{-2}$ Not serious

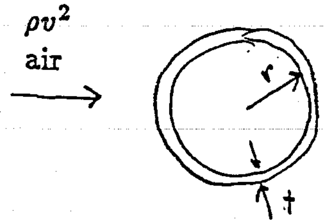
Stresses due to pressure from atmosphere

$$\sigma_{\text{compressional}} = \frac{Pr}{t} = 1.3 \times 10^8 \ dynes/cm^2$$



Wind stresses compare to μg

ρv^2
air



$$\frac{F}{\ell} = 2r\rho_{air}v^2 r$$

$\rho_{air} = 1.25 \times 10^{-3} gm/cm^3$
$v(10mph) = 4.5 \times 10^2 cm/sec$
$v(100mph) = 4.5 \times 10^3 cm/sec$

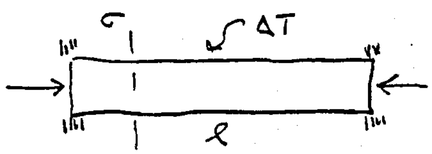
$\sigma(\ell/2) = \frac{\rho_{air} v^2 \ell^2}{4\pi r t}$

10 MPH → $3.2 \times 10^6 \ dynes/cm^2$, $y\left(\frac{\ell}{2}\right) = 6.61 \times 10^{-3} cm$

100 MPH → $3.2 \times 10^8 \ dynes/cm^2$, $y\left(\frac{\ell}{2}\right) = 0.64 \ cm$

Thermal stresses and deformation

a) Clamped ends (by accident due to corrosion of supports)



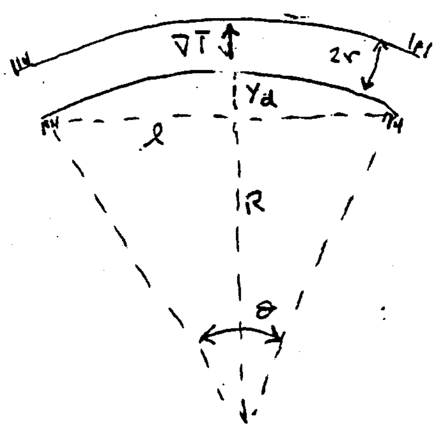
$\beta_{ss} = 1.6 \times 10^{-5}/K$
$c_v = 4.2 \times 10^{-1} \ Joules/gm \ K$
$K_{th} = 1.5 \times 10^{-1} \ Watts/cm \ K$

Uniform temperature change

$$\sigma_{\text{compression/tension}} = Y\beta\Delta T = 2.8 \times 10^7 \ dynes/cm^2 \ \Delta T(^\circ K)$$

(must look at buckling)

b) Fixed supported ends, transverse thermal gradiant



$$S = R\theta \qquad \ell = R\sin\theta$$

$$\frac{\Delta\ell}{\ell} = \frac{R\theta - R\sin\theta}{R\theta} = 1 - \frac{\theta + \frac{\theta^3}{6}}{\theta} = \frac{\theta^2}{6}$$

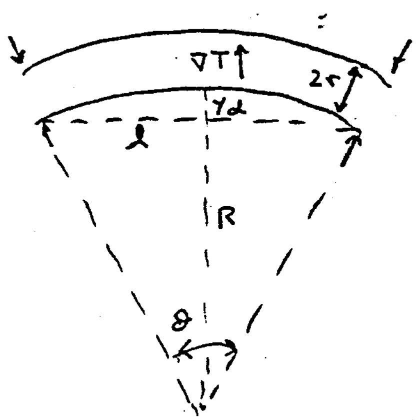
$$\frac{\theta^2}{6} = 2\beta\ \nabla Tr$$

$$R \cong \frac{\ell}{\theta} = \frac{\ell}{(12\beta\nabla Tr)^{1/2}}$$

$$y_d = R(1-cos\theta) = R\frac{\theta^2}{2}$$
$$= \ell(3\beta\nabla Tr)^{1/2}$$

For $1°K = \nabla T2r$
y_d=14.8cm
σ_{th}=7.2 × 10^9 $dynes/cm^2$ much too large

c) Transverse gradient simple supports



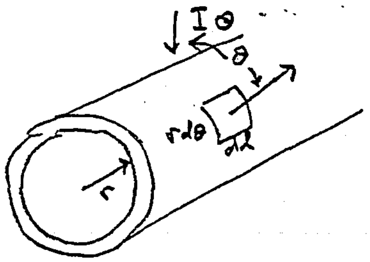
$$\frac{(R+r)\theta-(R-r)\theta}{R\theta} = \beta\nabla T2r$$

$\frac{1}{R}=\beta\nabla T$ $\quad y_d=R\frac{\theta^2}{2}$ $\quad \theta=\frac{\ell}{2R}$

$y_d=\frac{\ell^2}{8R}=\frac{\beta\nabla T\ell^2}{8}$

∇T=1K/120cm $\quad y_d$=.6cm/K_{diff} $\quad \sigma\,(\ell/2)$=3 × 10^8 $dynes/cm^2$

Solar illumination



$I_{\odot}=7\times10^{-2}\ watts/cm^2$

$<\varepsilon>=$ Average emissivity of $SS \approx 0.5$

$$P_{\odot}= <\varepsilon> 2\int_0^{\pi/2} I_{\odot}\cos\theta dA = 2rI_{\odot}\ell <\varepsilon>$$

$$J_{\odot}= <\varepsilon> I_{\odot}\cos\theta$$

$$\frac{P_{\odot}}{\ell} \simeq 4.3\ watts/cm$$

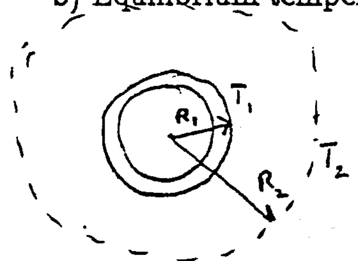
a) Thermal inertia of tube / length

$$\frac{dP}{d\ell} = c_v\ \mu\ \frac{dT}{dt}$$

From solar illumination

$$dT/dt \cong \frac{I_{\odot}<\varepsilon>}{c_v\rho\pi t} = 7\times10^{-3}\ K/sec \Rightarrow 25\ K/hr$$

b) Equilibrium temperature to static air $t \longrightarrow \infty$



$$J(r) = K_{th}\frac{dT}{dr} = \frac{P}{2\pi\ell r}$$

$$T_1 - T_2 = \frac{P}{\ell 2\pi K_{th}}\ lnR_2/R_1$$

Sample numbers

$K_{th\ (non\ convecting\ air)}=1.7\times10^{-4}$

$R_1=60cm$

$R_2=150cm$ (10ft diameter)

$T_2 - T_1 \approx 3700°K$ clearly too large (as expected, conduction cannot dominate)

Radiative equilibrium

$T_2 > T_1$ Assume T_1=300°K
σ=5.6 × 10^{-12} $\frac{W}{cm^2 k^4}$

$$T_2 = \left(\frac{\frac{P_\odot}{\ell}}{r 2\pi\sigma < \varepsilon >} + T_1^4 \right)^{1/4} = 332°K$$

$$T_2 - T_1 \approx 32°K$$

Convective Equilibrium

Under what regime does convective transport dominate?

Convection dominates when grashof # $G \geq 10^3$

$G = \frac{gr^3 \Delta T}{\nu^2 T}$

For LIGO tube at 300°K

$G \cong 3.6 \times 10^7 \ \Delta T$

convection could dominate

r = Tube radius, convection cell size
g = gravity 980cm/sec^2
ΔT = Difference in tube temperature and ambient air temperature
ν = kinematic viscosity of air 1.44 × 10^{-1} cm^2/sec at 300K
η = viscosity of air 1.8 × 10^{-4} poise $\frac{g}{cmsec}$
K_{th} = thermal convectivity of air =1.7 × 10^{-4} w/Kcm =1.7 × $10^3 ergs/secKcm$
c_p = heat capacity of air 1.7 × 10^7 $ergs/Kgm$

To determine convective flow

Prandtl # = $\eta c_p/K_{th}$=1.06= P
Handbook empirical relation for convective transport

$$J = \frac{K_{th}}{\pi r} \Delta T \left[\frac{GP^2}{2.4 + 4.88P^{1/2} + 4.953P} \right]^{1/4}$$

For our tubes in air at $300° K$

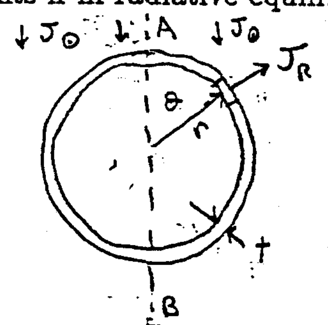
$$J = 3.75 \times 10^{-5} \Delta T^{5/4} \qquad watts/cm^2$$

Convective equilibrium in solar illumination

$$J = \left(\frac{P_\odot}{\ell}\right)\frac{1}{\pi r} \qquad \Delta T = \left[\frac{\left(\frac{P_\odot}{\ell}\right)}{\pi r 3.75 \times 10^{-5}}\right]^{4/5} = 167° K$$

<u>System mostly constrained by reradiation</u>

Gradients if in radiative equilibrium

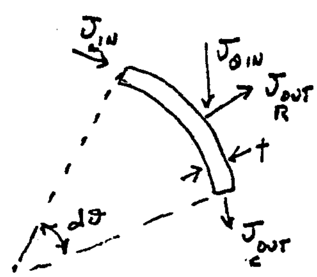


Note symmetry about AB, problem becomes one dimensional

linearize reradiation , T_p (excess overambient)

$$\frac{dP}{\ell} = r d\theta < \varepsilon > 4T_\circ^3 T_p$$

Take a point on circumference



Between $o < \theta < \pi/2$

$\frac{dP_\odot}{\ell} = I_\odot r d\theta cos\theta$

Equilibrium of piece $r d\theta$

Power out $\sigma 2 < \varepsilon > 4T_\circ^3 T_p r d\theta$ Radiation out

$\frac{\partial T}{r\partial\theta}(\theta + d\theta)K_{th}t$ Conduction out

Power in $< \varepsilon > I_\odot r d\theta cos\theta$ Solar in

$\frac{\partial T}{r\partial\theta}(\theta)K_{th}t$ Conduction in

Differential equation

$$< \varepsilon > I_\odot cos\theta r d\theta + \frac{\partial T}{r\partial\theta}(\theta)K_{th}t - 8 < \varepsilon > \sigma T_\circ^3 T r d\theta - \frac{\partial T}{r\partial\theta}(\theta + d\theta)K_{th}t = 0$$

$$-\frac{K_{th}t}{r}\frac{\partial^2 T}{\partial\theta^2} = < \varepsilon > I_\odot cos\theta r - 8 < \varepsilon > \sigma T_\circ^3 T r \qquad (1)$$

Differential equation

$$\pi/2 < \theta < \pi \qquad \text{no solar power in}$$

$$\frac{K_{th}t}{r}\frac{\partial^2 T}{\partial\theta^2} = -8 < \varepsilon > \sigma T_\circ^3 T r \qquad (2)$$

Solution: Try $T(\theta) = T cos\theta \qquad 0 < \theta < \pi/2$

$$\frac{\partial^2 T}{\partial\theta^2} = \frac{< \varepsilon > I_\odot cos\theta r^2}{K_{th}t} - \frac{8 < \varepsilon > \sigma T_\circ^3 r^2}{K_{th}t}T$$

$$-cos\,\theta T = \left[\frac{< \varepsilon > I_\odot r^2}{K_{th}} - \frac{8 < \varepsilon > \sigma T_\circ^3 r^2}{K_{th}}T\right] cos\theta$$

$$T = \frac{\frac{< \varepsilon > I_\odot r^2}{K_{th}t}}{\left[\frac{8 < \varepsilon > \sigma T_\circ^3 r^2}{K_{th}t} - 1\right]} = \frac{1820}{29.7 - 1} = 63.6$$

$I_\odot = 7 \times 10^{-2} w/cm^2$

$< \varepsilon > = .5$

$K_{th\ ss} = 1.5 \times 10^{-1}\ watts/cmK$

$T_\circ = 300\ K$

$\sigma = 5.5 \times 10^{-12}\ w/cm^2 K^4$

$t = .476\ cm$

Clearly linearization is not correct (at most a factor of 2 off)

$$T(\theta) = 63.6\ cos\theta \qquad o < \theta < \pi/2$$

Solution in lower part $\pi/2 < \theta < \pi$

$$T(\theta) = A\ cos\ \left(\frac{8 < \varepsilon > \sigma T_\circ^3 r^2}{K_{th}t}\right)^{1/2}\theta + B\ sin\left(\frac{8 < \varepsilon > \sigma T_\circ^3 r^2}{K_{th}t}\right)^{1/2}\theta$$

Scale length $\approx 5.5 radians^{-1}$ so the gradient is equal to the maximum temperature rise.