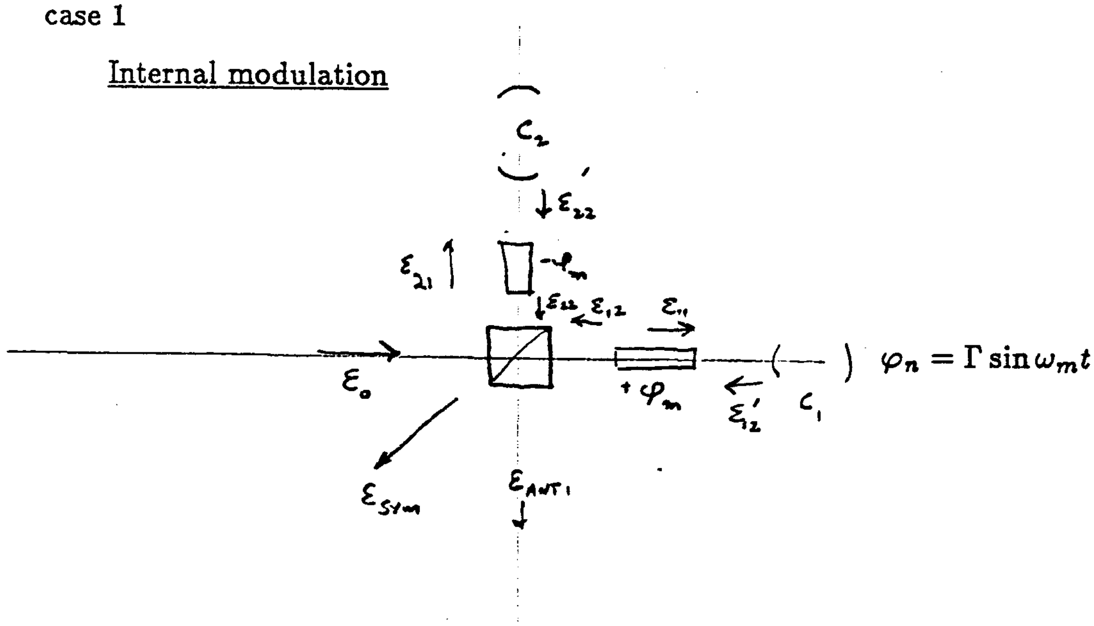
SHOT NOISE IN TWO BEAM INTERFEROMETERS

and modulation techniques in recombined systems

case 1

<u>Internal modulation</u>



Consider c_1 and c_2 cavities for any storage scheme

The cavities are characterized by the relation of input to output fields

$\left(\frac{E_{out}}{E_{in}}\right)_j = A_j \, e^{i\varphi_j}$

$\varphi_j = \varphi_{oj} + \frac{2\varphi_j}{dh_g} h_g$ where $\frac{\partial \varphi}{\partial h_g} = H_g$, the antenna transfer function to gravity wave amplitude

A_j = amplitude ratio assumed to be independent of h

Determine fields at interferometer outputs

Anti sym output

$$E_{\text{anti}} = E_o rt \left[A_2 \, e^{i(\varphi_2 - \varphi_m)} - A_1 \, e^{i(\varphi_1 + \varphi_m)} \right]$$

Sym output

$$E_{\text{sym}} = E_o \left[rr A_2 \, e^{i(\varphi_2 - \varphi_m)} + tt A_1 \, e^{i(\varphi_1 + \varphi_m)} \right]$$

Intensities at input and outputs

$$P_{\text{in}} = E_o^2 \qquad P_{\text{anti}} = E_{\text{anti}} E^*_{\text{anti}} \qquad P_{\text{sym}} = E_{\text{sym}} E^*_{\text{sym}}$$

Output intensities

$$P_{\text{anti}} = E_o^2 (rt)^2 \left[A_2^2 + A_1^2 - 2A_1A_2 \cos\left[(\varphi_2 - \varphi_1) - 2\varphi_m \right] \right]$$

$$P_{\text{sym}} = E_o^2 \left[r^4 A_2^2 + t^4 A_1^2 + 2r^2t^2 A_1 A_2 \cos\left[(\varphi_2 - \varphi_1) - 2\varphi_m \right] \right]$$

At anti sym port define contrast with $\varphi_m = 0$ (also no wavefront distortions or misalignments)

$$c = \frac{P_{\underset{\max}{\text{anti}}} - P_{\underset{\min}{\text{anti}}}}{P_{\underset{\max}{\text{anti}}} + P_{\underset{\min}{\text{anti}}}} = \frac{2A_1A_2}{A_1^2 + A_2^2}$$

Rewriting P_{anti} in terms of the contrast

$$P_{\text{anti}} = E_0^2 \left[(rt)^2 \left[A_1^2 + A_2^2 \right] \right] \left[1 - c \; \cos\left[(\varphi_2 - \varphi_1) - 2\varphi_m \right] \right]$$

Introduce phase sensitivity and the phase modulation

$$P_{\text{anti}} = E_o^2 \left[(rt)^2 \left[A_1^2 + A_2^2 \right] \left[1 - c \cos\left[(\varphi_{02} - \varphi_{01}) + \left(\frac{\partial \varphi_2}{dh_g} - \frac{d\varphi_1}{\partial h_g} \right) - 2\Gamma \sin \omega_m t \right] \right] \right]$$

Assume output is on dark fringe $\qquad \varphi_{02} - \varphi_{01} = N2\pi$

$$\text{Expand} \cos(\quad) = \cos\left[\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{d\varphi_1}{\varphi h_g} \right) h_g - 2\Gamma \sin \omega_m t \right]$$

$$= \cos\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right) h_g \cos(2\Gamma \sin \omega_m t) + \sin\left(\frac{\partial \varphi_2}{\partial h} - \frac{\partial \varphi_1}{\partial h} \right) h \sin(2\Gamma \sin \omega_m t)$$

Assume $\quad h_g \ll 1 \quad \Gamma < 1$

$$\cos(\quad) = J_o(2\Gamma) + \left[\left(\frac{\partial \varphi_2}{\varphi h_g} - \frac{\partial \varphi_1}{\partial h_g} \right) h_g \right] 2J_1(2\Gamma) \sin \omega_m t$$

In this approximation light power in DC term and at sub carrier DC term

$$P_{\text{anti}}(\omega = 0) = E_0^2 \left[(rt)^2 [A_1^2 + A_2^2] \right] \left(1 - cJ_0(2\Gamma) \right)$$

Light power at sub carrier

$$P(\omega = \omega_m) = 2E_0^2 r^2 t^2 [A_1^2 + A_2^2] c J_.(2\Gamma) \left[\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right] h_g \sin \omega_m t$$

Shot noise spectral density expressed as fluctuating power at detector

$P^2(f) = \frac{2<P>h\nu}{\eta}$

$<P>$ average power on photo detector

η quantum efficiency of photodetector

$h\nu$ photon energy

Noise spectral density at anti sym port

$$P^2(f) = \frac{2E_0^2 (rt)^2 [A_1^2 + A_2^2] \big(1 - cJ_0(2\Gamma)\big) h\nu}{\eta}$$

Signal power

↓gravity wave spectrum

$$P_{\text{sig}}^2(f) = 4\big[E_0^2 r^2 t^2 [A_1 + A_2^2]\big]^2 c^2 J_1^2(2\Gamma) \left[\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right]^2 h_g^2(f) \left(\frac{1}{2} \right)$$

↑sub carrier power = $<\sin^2 \omega_m t>$

Setting shot noise spectral density to signal power spectral density gives the condition

$$h_g^2(f)_{\text{noise}} = \left(\frac{h\nu}{\eta E_0^2 r^2 t^2 [A_1^2 + A_2^2]} \right) \left(\frac{1 - cJ_0(2\Gamma)}{c^2 J_1^2(2\Gamma)} \right) \frac{1}{\left[\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right]^2}$$

Define $P_{\text{anti max}} = E_0^2 r^2 t^2 [A_1^2 + A_2^2] = P_{\text{in}} RT [A_1^2 + A_2^2]$

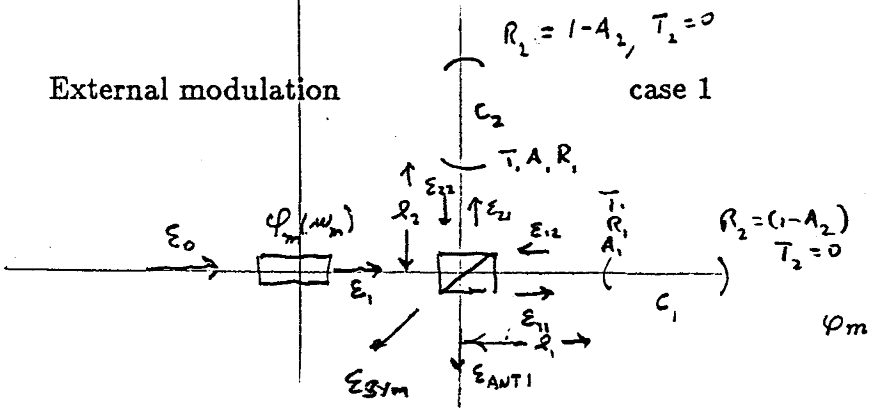
<u>Special cases</u> $\underset{x \to 0}{J_0(x)0} = 1 - x^2/4$ $\quad \underset{x \to 0}{J_1(x)} = x/2$

$$c \to 1 \qquad \frac{1 - cJ_0(2\Gamma)}{c^2 J_1^2(2\Gamma)} \Rightarrow 1 \qquad R = T = 1/2 \qquad A_1 = A_2 = 1$$

$h_g(f) = \frac{\left(\frac{2h\nu}{\eta P_{\text{in}}} \right)^{1/2}}{\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right)}$

$h_g(f) = \frac{\varphi(f)}{H_g(f)} \qquad \varphi(f) = \left(\frac{2h\nu}{\eta P_{\text{in}}} \right)^{1/2}$ limiting best case

External modulation case 1



$$\varphi_m(\omega) = \Gamma \sin \omega_m t$$

Use only the side bands sensitive to cavity response to gravity wave

$$E_1 = \Re\Big[E_0\ e^{i\omega t + \varphi_m(\omega_m)}\Big] \cong E_0 \Re\Big[J_0(\Gamma)\ e^{i\omega t} - \overset{\omega_-}{J_1(\Gamma) e^{i(\omega-\omega_m)t}} + \overset{\omega_+}{J_1(\Gamma)\ e^{i(\omega+\omega_m)t}} + \dots$$

The 3 separate frequency fields are carried through the system separately

<u>Cavity system</u>

$$\left(\frac{E_{\text{refl}}}{E_{\text{inc}}}\right)_j = \alpha_j(\Delta\omega_j)\ e^{i\varphi_j(\Delta\omega_j)}$$

$$\varphi_j(\Delta\omega_j) = \varphi_{0j}(\Delta\omega_j) + \frac{d\varphi_j}{\partial h_g}(\Delta\omega_j) h_g \qquad j \text{ designates cavity}$$

$$E_{\text{anti}}(\omega) = E_0 \Re\Big\{ J_0(\Gamma) rt \Big[\alpha_2(\Delta\omega_2)\ e^{i(\varphi_2(\Delta\omega_2)+2k\ell_2)} - \alpha_1(\Delta\omega_1)\ e^{i\varphi_2(\Delta\omega_1)+k_+\ell_1}\Big] e^{i\omega t}\Big\}$$

$$E_{\text{anti}}(\omega+) = E_0 \Re\Big\{ J_1(\Gamma) rt \Big[\alpha_2(\Delta\omega_2+)\ e^{i(\varphi_2(\Delta\omega_2+)+2k_+\ell_2} - \alpha_1(\Delta\omega_1+)\ e^{i(\varphi_1(\Delta\omega_1+)+2k_+\ell_1}\Big] e^{i(\omega t+\omega_m t)}\Big\}$$

$$E_{\text{anti}}(\omega-) = -E_0 \Re\Big\{ J_1(\Gamma) rt \Big[\alpha_2(\Delta\omega_2-)\ e^{i(\varphi_2(\Delta\omega_2-)+2k_-\ell_2} - \alpha_1(\Delta\omega_1-)\ e^{i\varphi_1(\Delta\omega_1-)+2k_-\ell_1}\Big] e^{i(\omega t-\omega_m t)}\Big\}$$

Simplify by taking perfect case $\qquad r = \frac{1}{\sqrt{2}} \qquad t = \frac{1}{\sqrt{2}}$

Almost tuned cavities

$$\Delta\omega_2 \cong 0 \cong \Delta\omega_1 = \delta\varphi_j \qquad \alpha_2(\Delta\omega_2) = \alpha_1(\Delta\omega_1) = \alpha(0) = \left(1 - \frac{4(A_1 - A_2)}{T_1}\right)^{1/2}$$

$$\Delta\omega_2+ = \Delta\omega_1+ = \Delta\omega+ \qquad \alpha_2(\Delta\omega_2+) = \alpha_1(\Delta\omega_2+) = \alpha(+) = (1-A_1)^{1/2}$$

$$\Delta\omega_2- = \Delta\omega_1- = \Delta\omega- \qquad \alpha_2(\Delta\omega_2-) = \alpha_1(\Delta\omega_1-) = \alpha(-) = (1-A_1)^{1/2}$$

$$\varphi_2(\Delta\omega_2) = \pi + \delta\varphi_1 + \frac{d\varphi_2}{\partial h_g} h_g \qquad \varphi_1(\Delta\omega_1) \cong \pi + \delta\varphi_2 + \frac{d\varphi_1}{\partial h_g} h_g$$

$$\varphi_2(\Delta\omega_+) \cong \varphi_1(\Delta\omega_+) = \frac{1}{2} T_1 \qquad \longleftarrow \text{assume subcarrier far outside}$$

$$\varphi_2(\Delta\omega_-) \cong \varphi_1(\Delta\omega_-) = -\frac{1}{2} T_1 \qquad \text{resonance of cavity}$$

Rewriting the fields at antisymmetrical output

↙- sign from $e^{i\pi}$ at resonance of cavities

$$E_{\text{anti}}(\omega) = -E_0\left\{ J_0(\Gamma) rt\alpha(0)\left[e^{i\left(\frac{\partial\varphi_2}{\partial h_g}h_g+\delta\varphi_2+2k\ell_2\right)} - e^{i\left(\frac{\partial\varphi_1}{dh_g}h_g+\delta\varphi_1+2k\ell_1\right)}\right]\right\}e^{i\omega t}$$

$$E_{\text{anti}}(\omega+) = E_0\left\{ J_1(\Gamma) rt\alpha(+)\left[e^{i\left(\frac{1}{2}T_1+2k_+\ell_2\right)} - e^{i\left(\frac{1}{2}T_1+2k_+\ell_1\right)}\right]\right\}e^{i\omega_+ t}$$

$$E_{\text{anti}}(\omega_-) = -E_0\left\{ J_1(\Gamma) rt\alpha(-)\left[e^{i\left(-\frac{1}{2}T_1+2k_-\ell_2\right)} - e^{i\left(-\frac{1}{2}T_1+2k_-\ell_1\right)}\right]\right\}e^{i\omega_+ t}$$

↖from $J_{-1}(\Gamma) = -J_1(\Gamma)$

Limiting case

1) if $\omega+$ and $\omega-$ fall far outside the F.P. resonance even if $\delta\varphi_1 \neq \delta\varphi_2$

and the loss in both cavities is the same and $\ell_1 = \ell_2$

$$E_{\text{anti}}(\omega_+) = E_{\text{anti}}(\omega-) = 0 \qquad E_{\text{anti}}(\omega) \to 0 \quad h_g \to 0$$

What are the detected modulation products if $\ell_1 \neq \ell_2$

$$I_{\text{anti}} = \Big(E(\omega) + E(\omega+) + E(\omega-)\Big)\Big(E(\omega) + E(\omega+) + E(\omega-)\Big)^*$$

DC term comes from $\sum E(\omega_i)E(\omega_i)^*$

$$I(0) = 2E_0^2(rt)^2J_0^2(\Gamma)\alpha^2(0)\left[1-\cos\left[\left(\frac{\partial\varphi_2}{\partial h_g}-\frac{\partial\varphi_1}{\partial h_g}\right)h_g + (\delta\varphi_2-\delta\varphi_1)+2k(\ell_2-\ell_1)\right]\right]$$

$$+2E_0^2(rt)^2J_1^2(\Gamma)\alpha^2(+)\left[1-\cos\left[2k_+(\ell_2-\ell_1)\right]\right] \quad \text{upper side band}$$

$$+2E_0^2(rt)^2J_1^2(\Gamma)\alpha^2(-)\left[1-\cos\left[2k_-(\ell_2-\ell_1)\right]\right] \quad \text{lower side band}$$

Complex algebra check on sidebands

Trick: use sym and anti sym sum and difference of complex factors

$$e^{i\alpha} \pm e^{i\beta} = e^{i\frac{\alpha+\beta}{2}}\left[e^{i\frac{(\alpha-\beta)}{2}} \pm e^{i\frac{(\alpha-\beta)}{2}}\right]$$

$$= e^{i\frac{\alpha+\beta}{2}} \quad 2\cos\frac{(\alpha-\beta)}{2} \qquad + \text{sign}$$

$$= e^{i\frac{\alpha+\beta}{2}} \quad 2i\sin\frac{(\alpha-\beta}{2} \qquad - \text{sign}$$

Use $E(\omega) = (e^{iA} - e^{iB})e^{i\omega t}$ $\quad E(\omega+) = (e^{ic} - e^{iD})e^{i(\omega+\omega_m)t}$

$$E(\omega-) = (e^{i-c'} - e^{iD'})e^{i}(\omega - \omega_m)t$$

Then terms at ω_m in output intensity have form $E(\omega)\ E^*(\omega+) + E^*(\omega)\ E(\omega+)$

The products are

$$\left(e^{i(C-A)} + e^{i(D-B)}\right) - \left(e^{i(C-B)} + e^{i(D-A)}\right)e^{i\omega_m t} \qquad + \text{complex conjugate}$$

$$\left(Z_1 \quad - \quad Z_2\right)e^{i\omega_m t} \qquad + \text{complex conjugate}$$

$$Z_1 = e^{i\frac{\left((C+D)-(A+B)\right)}{2}}\left[2\cos\left[\frac{(C-A)-(D-B)}{2}\right]\right]$$

$$Z_2 = e^{i\frac{\left((C+D)-(A+B)\right)}{2}}\left[2\cos\left[\frac{(C-B)-(D-A)}{2}\right]\right]$$

$$Z_1 - Z_2 = 2\ e^{i\frac{\left[(C+D)-(A+B)\right]}{2}}\left[\cos\left[\frac{(C-D)-(A-B)}{2}\right] - \cos\left[\frac{(C-D)+(A-B)}{2}\right]\right]$$

$$Z_1 - Z_2 = -4\ e^{i\frac{\left[C+D)-(A+B)\right]}{2}}\sin\left(\frac{C-D}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Finally

$$Z = (Z_1 - Z_2)\ e^{i\omega_m t} + \text{complex conjugate}$$

$$= -8\sin\left(\frac{C-D}{2}\right)\sin\left(\frac{A-B}{2}\right)\left[\cos\left(\frac{(C+D)-(A+B)}{2} + \omega_m t\right)\right]$$

End Result <u>$\omega+$ sidband beat</u>

$$C - D = 2k_+(\ell_2 - \ell_1)$$

$$A - B = \left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g}\right) h + (\delta\varphi_2 - \delta\varphi_1) + 2k(\ell_2 - \ell_1)$$

$$C + D = T_1 + 2k_+(\ell_2 + \ell_1)$$

$$A + B = (\delta\varphi_2 + \delta\varphi_1) + 2k(\ell_2 + \ell_1) \qquad \left(\frac{\partial \varphi_2}{\partial h_g} + \frac{\partial \varphi_1}{\partial h_g}\right) = 0 \quad \text{from g wave polarization}$$

$$I(\omega+) = -8E_0^2(rt)^2\alpha(0)\alpha(+)J_0(\Gamma)J_1(\Gamma) \quad *$$

$$\sin K_+(\ell_2 - \ell_1) \sin\left[\frac{1}{2}\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g}\right) h + \frac{1}{2}(\delta\varphi_2 - \delta\varphi_1) + k(\ell_2 - \ell_1)\right] \cos(\psi_+ - \omega_m t)$$

$$\text{Where} \qquad \psi_+ = \frac{1}{2}(\delta\varphi_2 + \delta\varphi_1) + (k - k_+)(\ell_2 + \ell_1) - \frac{T_1}{2}$$

End result <u>$\omega-$ side band beat</u>

$$C' - D' = 2k_-(\ell_2 - \ell_1)$$

$$A - B = \text{(same as above)}$$

$$C' + D' = -T_1 + 2k_-(\ell_2 + \ell_1)$$

$$A + B = \text{(same as above)}$$

$$I(\omega_-) = +8E_0^2(rt)^2\alpha(0)\alpha(-)J_0(\Gamma)J_1(\Gamma) \quad *$$

$$\sin k_-(\ell_2 - \ell_1) \sin\left[\frac{1}{2}\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g}\right) h + \frac{1}{2}(\delta\varphi_2 - \delta\varphi_1) + k(\ell_2 - \ell_1)\right] \cos(\psi_- + \omega_m t)$$

$$\text{Where} \qquad \psi_- = \frac{1}{2}(\delta\varphi_2 + \delta\varphi_1) + (k - k_-)(\ell_2 + \ell_1) + \frac{T_1}{2}$$

Combined beat outputs assuming $\alpha(+) = \alpha(-) = \alpha(\pm)$

$$I(\omega+) + I(\omega-) = 16E_0^2(rt)^2\alpha(0)\alpha(\pm)J_0(\Gamma)J_1(\Gamma) \quad *$$

$$\sin\left[\frac{1}{2}\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g}\right) h + \frac{1}{2}(\partial\varphi_2 - \partial\varphi_1) + k(\ell_2 - \ell_1)\right] \quad *$$

$$\left[\cos k_0(\ell_2 - \ell_1) \sin k_m(\ell_2 - \ell_1) \cos\frac{1}{2}(\delta\varphi_2 + \delta\varphi_1) \cos\left(\omega_m t + \frac{T_1}{2} + k_m(\ell_1 + \ell_2)\right)\right.$$

$$\left. - \sin k_0(\ell_2 - \ell_1) \cos k_m(\ell_2 - \ell_1) \sin\frac{1}{2}(\delta\varphi_2 + \delta\varphi_1) \sin\left(\omega_m t + \frac{T_1}{2} + k_m(\ell_1 + \ell_2)\right)\right]$$

Note: Critical that for either SSB or DSB detection

$\underline{\ell_2 - \ell_1 \neq 0}$ for this modulation which puts further AM noise and frequency stabilization conditions on the antenna if this modulation is used.

Take simplified case

Assume cavities both in resonance $\delta\varphi_1 = \delta\varphi_2 = 0$

1) To make first order sensitive to GW requires $k_0(\ell_2 - \ell_1) = m\pi$ This means that for DSB detection one must use $\cos\omega_m t$ term.

2) $\sin k_m(\ell_2 - \ell_1) = 1$ therefor $k_m(\ell_2 - \ell_1) = n\pi/2$

If $(\ell_2 - \ell_1)$ is to be a minimum $n = \pm 1$

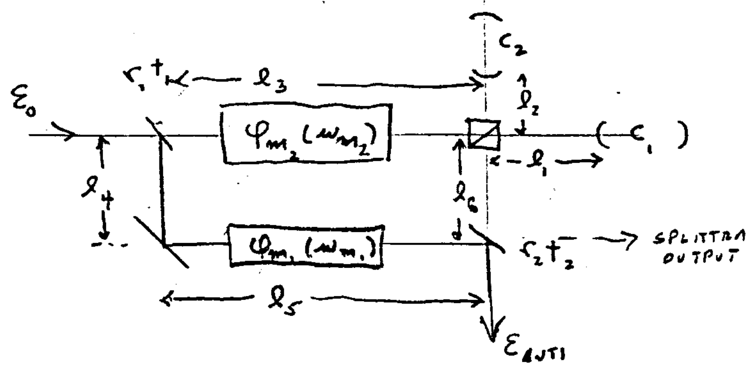
Which says that $\frac{\omega_m}{c}\Delta\ell = \pi/2$ $\Delta\ell = \frac{c}{4f_m}$

For example with $10MHz = f_m$

$$\Delta\ell = \frac{3 \times 10^{10}}{4 \times 10^7} = 3/4 \times 10^3 cm = 750cm = 7.5 \text{ meters}$$

This modulation scheme does not seem practical.

External modulation in separate loop and in line modulator



For cavity locking

$$\varphi_{m_2}(\omega_{m_2}) = \Gamma_2 \sin\omega_{m_2} t$$
$$\varphi_{m_1}(\omega_{m_1}) = \Gamma_1 \sin\omega_{-}m_1 t$$

Use input splitter as reference plane

Beam returning from cavities at exit pupil of anti sym detector

$$E_1 = E_0 t_1 t r t_2 A_1(\Delta\omega_1)\ e^{i\varphi_{c_1}(\Delta\omega_1)}\ e^{ik2\ell_1}\ e^{ik(\ell_3+\ell_6)}\ e^{\varphi_{m_2}(t)}$$
$$E_2 = -E_0 t_1 r t t_2 A_2(\Delta\omega_2)\ e^{i\varphi_{c_2}(\Delta\omega_2)}\ e^{ik2\ell_2}\ e^{ik(\ell_3+\ell_6)}\ e^{i\varphi_{m_2}(t)}$$

Reference beam

$$E_R = E_0 r_1 r_m r_2\ e^{i\varphi_{m_1}(t)}\ e^{ik(\ell_4+\ell_5)}$$

Beams at exit from cavities

from π at cavity resonance

$$E_{\text{anti}}(\omega) = -J_0(\Gamma_2) E_0 t_1 t r t_2 \alpha(0)\left[e^{\left(\frac{\partial\varphi_2}{\partial h_g} h_g + \delta\varphi_2 + 2k\ell_2\right)} - e^{i\left(\frac{\partial\varphi_1}{\partial h_g} hg + \delta\varphi_1 + 2k\ell_1\right)}\right] *$$
$$e^{i\left(\omega t + k(\ell_3+\ell_6)\right)}$$

$$E_{\text{anti}}(\omega+_2) = J_1(\Gamma_2) E_0 t_1 t r t_2 \alpha(+_2)\left[e^{i\left(\frac{1}{2}T_1 + 2k_+\ell_2\right)} - e^{\left(\frac{1}{2}T_1 + 2k_+\ell_1\right)}\right] *$$
$$e^{i\left((\omega+_2)t + k_{+_2}(\ell_3+\ell_6)\right)}$$

$$E_{\text{anti}}(\omega-_2) = -J_1(\Gamma_2) E_0 t_1 t r t_2 \alpha(-_2)\left[e^{i\left(-\frac{1}{2}T_1 + 2k_-\ell_2\right)} - e^{i\left(-\frac{1}{2}T_1 + 2k_-\ell_1\right)}\right] *$$

from $J_{-1}(\Gamma) = -J_1(\Gamma)$

$$e^{i\left((\omega-_2)t + k_{-_2}(\ell_3+\ell_6)\right)}$$

Reference beam

$$E_{\text{ref}}(\omega) = E_0 r_1 r_m r_2 J_0(\Gamma_1)\ e^{i\omega t + ik(\ell_4+\ell_5)}$$
$$E_{\text{ref}}(\omega+_1) = E_0 r_1 r_m r_2 J_1(\Gamma_1)\ e^{i(\omega+_1)+ik_+(\ell_4+\ell_5)}$$
$$E_{\text{ref}}(\omega-_1) = -E_0 r_1 r_m r_2 J_1(\Gamma_1)\ e^{i(\omega-_1)t+ik_-(\ell_4+\ell_5)}$$

The beat terms become

$$I_{\substack{\text{Anti sym}\\ \text{P.D.}}} = \left(E_{\text{anti}}(\omega) + E_{\text{anti}}(\omega{+}_2) + E_{\text{anti}}(\omega{-}_2) + E_{\text{ref}}(\omega) + E_{\text{ref}}(\omega{+}_1) + E_{\text{ref}}(\omega{-}_1)\right)\left(\quad cc \quad\right)$$

DC terms in the intensity $\quad \sum E_i(\omega_i)E_i^*(\omega_i) + E_{\text{anti}}(\omega)E_{\text{ref}}^*(\omega) + E_{\text{anti}}^*(\omega)E_{\text{ref}}(\omega)$

$$I(0) = 2E_0^2(t_1trt_2)^2\left\{J_0^2(\Gamma_2)\alpha^2(0)\left(1-\cos\left[\left(\frac{\partial\varphi_2}{\partial h_g}-\frac{\partial\varphi_1}{\partial h_g}\right)h_g + (\delta\varphi_2-\partial\varphi_1) + 2k(\ell_2-\ell_1)\right]\right)\right.$$

$$+ J_1^2(\Gamma_2)\alpha^2(+_2)\left[1-\cos\left[2k_{+2}(\ell_2-\ell_1)\right]\right]$$

$$\left.+ J_1^2(\Gamma_2)\alpha^2(-_2)\left[1-\cos\left[2k_{-2}(\ell_2-\ell_1)\right]\right]\right\} \qquad \uparrow \text{ from cavity beams}$$

$$+E_0^2(r_1r_mr_2)^2\left(J_0^2(\Gamma) + 2J_1^2(\Gamma_1)\right) \qquad \leftarrow \text{ from ref}$$

$\swarrow$ cross product ref with cavity

$$+4E_0^2r_1r_mr_2J_0(\Gamma_1)J_0(\Gamma_2)t_1trt_2\alpha(0)\sin\left[\frac{1}{2}\left(\frac{\partial\varphi_2}{\partial h_g}-\frac{\partial\varphi_1}{\partial h_g}\right)h_g + \frac{\delta\varphi_2-\delta\varphi_1}{2} + k(\ell_2-\ell_1)\right] \quad *$$

$$\sin\left[\left(\frac{\delta\varphi_1+\delta\varphi_2}{2} + k(\ell_1+\ell_2) + \frac{k}{2}(\ell_3+\ell_6) - (\ell_4+\ell_5)\right)\right]$$

Total $I(0)$ when balanced system $\delta\varphi_2 = \delta\varphi_1 = 0 \qquad \ell_2-\ell_1 = 0 \qquad \ell_3+\ell_6 = \ell_4+\ell_5$

$I(0) = E_0^2(r_1r_mr_2)^2\left[J_0^2(\Gamma_1) + 2J_1^2(\Gamma_1)\right]$ <u>The reference intensity</u> is always there!

Shot noise from this?

The terms at ω_1

$$I(\omega_1) = E_{\text{anti}}(\omega)\ E_{\text{ref}}^*(\omega{+}_1) + E_{\text{anti}}^*(\omega)\ E_{\text{ref}}(\omega{+}_1) \qquad \text{upper side band}$$

$$E_{\text{anti}}(\omega)\ E_{\text{ref}}^*(\omega{-}_1) = E_{\text{anti}}^*(\omega)\ E_{\text{ref}}(\omega{-}_1) \qquad \text{lower side band}$$

$$I(\omega_1)^+ = -4E_0^2J_0(\Gamma_2)J_1(\Gamma_1)\alpha(0)t_1trt_2r_1r_mr_2 \quad * \qquad \text{upper side band}$$

$$\sin\left[\frac{1}{2}\left(\frac{\partial\varphi_2}{\partial h_g}-\frac{\partial\varphi_1}{\partial h_g}\right)h + \frac{1}{2}(\delta\varphi_2-\partial\varphi_1) + k(\ell_2-\ell_1)\right] \quad *$$

$$\left[\sin\ \omega_{m_1}t\ \cos\left[k\Big((\ell_4+\ell_5) - (\ell_3+\ell_6-(\ell_2+\ell_1)\Big) - \frac{1}{2}(\delta\varphi_1+\delta\varphi_2)\right] + \right.$$

$$\left.\cos\ \omega_{m_1}t\ \sin\left[k\Big((\ell_4+\ell_5) - (\ell_3+\ell_6) - (\ell_2+\ell_1)\Big) - \frac{1}{2}(\delta\varphi_1+\delta\varphi_2)\right]\right]$$

$$I(\omega_1)^- = -4E_0^2 J_0(\Gamma_2)J_1(\Gamma_1)\alpha(0)t_1trt_2r_1r_mr_2 \quad * \qquad \text{lower side band}$$

$$\sin\left[\frac{1}{2}\left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_2}{\partial h_g}\right)h + \frac{1}{2}(\delta\varphi_2 - \delta\varphi_1) + k(\ell_2 - \ell_1)\right] \quad *$$

$$\left[\sin\,\omega_{m_1}t\,\cos\left[k\Big((\ell_3+\ell_6) - (\ell_4+\ell_5) + (\ell_2+\ell_1)\Big) + \frac{1}{2}(\delta\varphi_1 + \delta\varphi_2)\right] + \right.$$

$$\left.\cos\,\omega_{m_1}t\,\sin\left[k\Big((\ell_3+\ell_6) - (\ell_4+\ell_5) + (\ell_2+\ell_1)\Big) + \frac{1}{2}(\delta\varphi_1 + \delta\varphi_2)\right]\right]$$

Total DSB signal

$$I(\omega_1) = -8E_0^2 J_0(\Gamma_2)J_1(\Gamma_1)\alpha(0)t_1trt_2r_1r_mr_2 \quad *$$

$$\sin\left[\frac{1}{2}\left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_1}{\partial h_g}\right)h_g + \frac{1}{2}(\delta\varphi_1 - \delta\varphi_2) + k(\ell_2 - \ell_1)\right] \quad *$$

$$\cos\left[k\Big((\ell_4+\ell_5) - (\ell_3+\ell_6) - (\ell_2+\ell_1)\Big)\right] \quad \sin\,\omega_m t$$

Note: The terms $k(\ell_4+\ell_5) - k(\ell_3+\ell_6)$ do not cancel as written. Sloppy arithmetic let $k_t = k_- = k$ so small correction of order $k_m(\ell_4+\ell_5)$ is required in cos term

Limiting case of balanced system DSB GW sensitivity

$$I(\omega_1) = -4E_0^2 J_0(\Gamma_2)J_1(\Gamma_1)\alpha_0 t_1trt_2r_1r_mr_2 \left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_1}{\partial h_g}\right) hg\ \sin\,\omega_m t$$

$$I_{\min}(0) = E_0^2(r_1r_mr_2)^2\left[J_0^2(\Gamma_1) + 2J_1^2(\Gamma_1)\right] \simeq E_0^2(r_1r_mr_2)^2$$

Bessel function sum rule
Also ϕ mod does not
alter amplitude

Shot noise limit is interesting in this case since one has an additional degree of freedom in the splitter properties and the PC1 modulation depth - best case would put blocking filter for PC2 modulation in front of detector (although if system completely balanced there is no intensity for any of the PC2 modulation carriers at the detector.)

Assume 100% contrast internal to interferometer

Shot noise

$$P^2(f) = \frac{2 < P > h\nu}{\eta} \quad = \quad \frac{2E_0^2(r_1 r_m r_2)^2 h\nu}{\eta}$$

Signal power

$$\underset{\text{signal}}{P^2}(f) = 16E_0^2 J_0^2(\Gamma_2) J_1^2(\Gamma)\alpha^2(0)(t_1 t r t_2 r_1 r_m r_2)^2 \left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_1}{\partial h_g}\right)^2 h_g^2(f)\underbrace{\tfrac{1}{2}}_{<\sin^2 \omega_m t>}$$

Setting shot noise spectral density to signal power spectral density gives the condition

$$h_g^2(f) = \frac{1}{4}\left(\frac{h\nu}{\eta E_0^2 J_0^2(\Gamma_2)\alpha^2(0)(tr)^2}\right)\left(\frac{1}{\left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_1}{\partial h_g}\right)^2}\right) \frac{1}{J_1^2(\Gamma_1)(t_1 t_2)^2}$$

Limit is given by the largest value of $J_1^2(\Gamma) \sim 1/3 \qquad J_1(1.84) = 0.5819$

$$t_1 t_2 \to 1 \quad (tr^2) = 1/4 \quad J_0(\Gamma_2) \to 1 \quad \alpha^2(0) \to 1$$

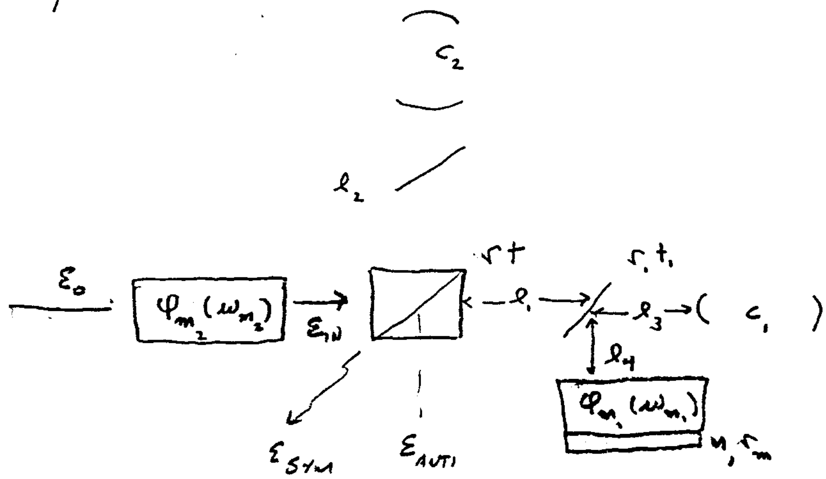
$$\underset{\min}{\varphi(f)} = h_g(f)\left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_1}{\partial h_g}\right) = \left(\frac{h\nu}{\eta J_1^2(\Gamma_{\max})P_{\text{in}}}\right)^{1/2} = \left(\frac{3h\nu}{\eta P_{\text{in}}}\right)^{1/2}$$

$(3/2)^{1/2} = 1.22\times$ worse than internal modulation or another way to state it requires $1.5\times$ more power to give the same shot noise as internal modulation.

Case with square wave modulation may reduce the difference

Another thing to consider is a single side band modulator in the auxiliary arm. This reduces signal by 2 but shot noise by more, so that one could check to see if this recovers the $\sqrt{3/2}$

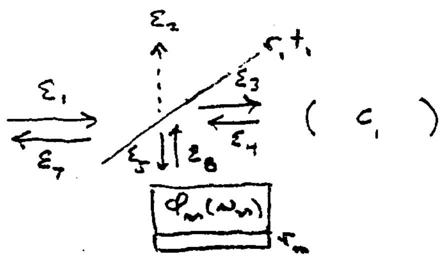
FOX/LI auxiliary cavity modulation



$$\varphi_{m_1}(\Gamma_2) = \Gamma_2 \sin\ \omega_{m_2} t$$
$$\varphi_{m_1}(\Gamma_1) = 2\Gamma_1 \sin\ \omega_{m_1} t$$

Γ_1 Two passes assume in phase

Solve for fields at $BS2$ use center of reflecting surface as reference



For initial calculation assume E_1 is all in the unshifted carrier of $PC2$ $\qquad E_1 = tJ_0(\Gamma_2)E_0\ e^{ik\ell_1}$

The $PC1$ produces an unshifted carrier and two sidebands (neglect higher $J_n(\Gamma_2)$). The auxiliary cavity conditions are different for the carrier and the sidebands. For example, E_2 can be made 0 for the carrier but not for the sidebands since E_1 does not contain the sidebands. (Infact the auxiliary cavity can be looked at as suppressed carrier DSB generator at E_2 (This may change with recycling when E_1 will contain these sidebands, (one step at a time)).

For unshifted carrier

$$E_2 = t_1 E_6 - r_1 E_1, \quad E_3 = t_1 E_1 + r_1 E_6, \quad E_5 = r_1 E_4, \quad E_7 = t_1 E_4$$

Active terms (A)

$$E_6 = r_m E_5 J_0(\Gamma_1)\ e^{i2k\ell_4} \qquad E_4 = -E_3 A_1(0)\ e^{i\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} h_g + 2k\ell_3}$$

assume cavity near resonance

Condition for making $E_2 = 0$

$$\frac{E_6}{E_1} = \frac{r_1}{t_1} \qquad \text{do not impose at beginning}$$

Output unshifted carrier beam solve equations (A) –

$$E_7 = \frac{-tJ_0(\Gamma_2)\ e^{ik\ell_1} t_1^2 A_1(0)\ e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} h_g + 2k\ell_3\right)} E_0}{\left(1 + r_1^2 r_m J_0(\Gamma_1) A_1(0)\ e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} h_g + 2k(\ell_3+\ell_4)\right)}\right)}$$

condition that no unshifted carrier exit in E_2 if $r_1^2 + t_1^2 = 1$

$$1 = -A_1(0) r_m J_0(\Gamma_1)\ e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_2}{\partial h_g} h_g + 2k(\ell_3+\ell_4)\right)}$$

Not possible since $A_1(0) \leq 1 \quad J_0(\Gamma_1) < 1$ with any modulation

But closest condition occurs when

$$\delta\varphi_1 + 2k(\ell_3 + \ell_4) \quad = \quad (2n+1)\pi \qquad n = 0\cdots$$

Loss at E_2

$$E_2 = -r_1 E_0 \left[\frac{A_1(0) J_0(\Gamma_1)\ e^{i\Omega} + 1}{1 + r_1^2 A_1(0) J_0(\Gamma_1)\ e^{i\Omega}}\right] \qquad \Omega = \delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} + 2k(\ell_3 + \ell_4)$$

If $r_1^2 \ll 1 \quad 10^{-3} - 10^{-2}$ expected in experiment

$${}_{\text{lost}}I_{2\,\text{min}} \cong r_1^2 E_0^2 \Big[1 - A_1(0) J_0(\Gamma_1)\Big]^2$$

without compensation in other arm limits the contrast

$\Omega = \pi$

Rewriting the output beam in more useable form for recombination

$$E_7 = -A\left(\frac{1}{1 + a\ e^{i\varphi}}\right) = -A\left(\frac{1 - a\ e^{i\varphi}}{1 - a^2}\right) \simeq -A(1 - a\ e^{-i\varphi})$$

since $a \ll 1$

$$E_7(\omega) = -tt_1^2 J_0(\Gamma_2) A_1(0) E_0\ e^{ik\ell_1}\left[e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} h_g + 2k\ell_3\right)} - r_1^2 r_m J_0(\Gamma_1) A_1(0)\ e^{-i2k\ell_4}\right] e^{i\omega t}$$

The auxiliary cavity and the output side bands

diagram is different. Sidebands generated in Pockel's cell make a interconversion series if $r_1^2 \ll 1$ need only single pass through cell, error made is of order r_1^2 maximum

(This calculation must be made again if answers depend on r_1^4)

The interconversion terms will sum as:

Unshifted carrier

$$E_7(\omega) = -\left(tJ_0(\Gamma_2)E_0\ e^{ik\ell_1}\right)\left(t_1^2 A_1(0)\ e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g}h_g + 2k\ell_3\right)}\right)\ *$$

↑phase π phase shift on resonance

$$\left(1 - r_1^2 r_m J_0(\Gamma_1)A_1(0)\ e^{-i2k\ell_4}\left(1 + \sum_n \left[r_1^2 r_m J_0(\Gamma_1)A_1(0)\right]^n\ e^{-in2k\ell_4} + \right.\right.$$

↑multiple passes unshifted carrier

$$\left.\left. 2\sum_m (t^2 A^2(0) r^4 r_m^2 J_1^2 A(\pm))^m\ e^{im_1\psi_\pm}\right)\quad e^{i\omega t}$$

interconversion term

Sidebands

$$E_7(\omega_+) = -\left(tJ_0(\Gamma_2)E_0\ e^{ik\ell_1}\right)\left(t_1^2 A_1(0)\ e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g}h_g + 2k\ell_3\right)}\right)\ *$$

$$\left(A_1(+)r_1^2 r_m J_1(\Gamma_1)\ e^{i(k+k_+)\ell_4}\ e^{ik_+2\ell_3}\ e^{i\frac{1}{2}T_1}\right)$$

$$\left[1 + \sum_q \left[r_1^4 r_m^2 J_0(\Gamma_1)J_1(\Gamma_1)A_1(0)A_1(+)\right]^q\ e^{iq\psi_+} + \begin{pmatrix}\text{double conversions}\\ \text{multiple pass}\end{pmatrix}\right] e^{i(\omega+\omega_{m_1})t}$$

multiple pass first order conversions

$$E_7(\omega_-) = +\left(tJ_0(\Gamma_2)E_0\ e^{ik\ell_1}\right)\left(t_1^2 A_1(0)\ e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g}h_g + 2k\ell_3\right)}\right)\ *$$

↑from $J_1 = -J_{-1}$

$$\left(A_1(-)r_1^2 r_m J_1(\Gamma_1)\ e^{i(k+k_-)\ell_4}\ e^{ik_-2\ell_3}\ e^{-i\frac{1}{2}T_1}\right)$$

$$\left[1 + \sum_q \left[r_1^4 r_m^2 J_0(\Gamma_1)J_1(\Gamma_1)A_1(0)A_1(-)\right]^q\ e^{iq\psi_-} + (\text{double conv.})\right]\ e^{i(\omega-\omega_{m_1})t}$$

multiple first order conversions

Pull terms together from both sides of the interferometer at the anti sym output in lowest order needed in r_1^2

Beam from arm 1

$$E_1(\omega) = -\Big(trJ_0(\Gamma_2)E_0\ e^{i2k\ell_1}\Big)\Big(t_1^2A_1(0)\ e^{i\left(\delta\varphi_1+\frac{\partial\varphi_1}{\partial h_g}h_g+2k\ell_3\right)}\Big)\ e^{i\omega t}$$

$$E_1(\omega+) = -\Big(trJ_0(\Gamma_2)E_0\ e^{i(k+k_+)\ell_1}\Big)\Big(t_1^2A_1(0)\ e^{i\left(\delta\varphi_1+\frac{\partial\varphi_1}{\partial h_g}h_g+2k\ell_3\right)}\Big)\ *$$
$$\Big(A_1(+)r_1^2r_mJ_1(\Gamma_1)\ e^{(k+k_+)\ell_4}\ e^{i2k_+\ell_3}\ e^{i\frac{1}{2}T_1}\Big)e^{i(\omega+\omega_{m_1}t)}$$

$$E_1(\omega-) = \Big(trJ_0(\Gamma_2)E_0\ e^{i(k+k_-)\ell_1}\Big)\Big(t_1^2A_1(0)\ e^{i\left(\delta\varphi_1+\frac{\partial\varphi_1}{\partial h_g}h_g+2k\ell_3\right)}\Big)\ *$$
$$\Big(A_1(-)r_1^2r_mJ_1(\Gamma_1)\ e^{i(k+k_-)\ell_4}\ e^{i2k_-\ell_3}\ e^{-i\frac{1}{2}T_1}\Big)e^{i(\omega-\omega_{m_1})t}$$

+sidebands from PC2

Beam from arm 2

$$E_2(\omega) = \Big(trJ_0(\Gamma_2)E_0\ e^{i2k\ell_2}\Big)\Big(A_2(0)\ e^{i\left(\delta\varphi_2+\frac{\partial\varphi_2}{\partial h_g}h_g\right)}\Big)e^{i\omega t}$$

+sidebands from PC2

Intensity at antisym output

$$I_{\text{anti}} = \Big(E_1(\omega)+E_1(\omega+)+E_1(\omega-)+E_2(\omega)\Big)\Big(\quad cc\quad\Big)$$

DC term

$$I(0) = (tr)^2J_0^2(\Gamma_2)E_0^2\Big\{\Big[t_1^4A_1^2(0)+A_2^2(0)+t_1^4A_1^2(0)r_1^4r_m^2J_1^2(\Gamma_1)\big[A_1^2(+)+A_1^2(-)\big]\Big]$$
$$-2A_1(0)A_2(0)t_1^2\ \cos\Big[2k(\ell_1+\ell_3-\ell_2)+(\partial\varphi_2-\partial\varphi_1)+\Big(\frac{\partial\varphi_2}{\partial h_g}-\frac{\partial\varphi_1}{\partial h_g}\Big)h_g\Big]\Big\}$$

Combine the direct cavity reflections first   Assume $A_1(0) = A_2(0) = A(0)$

Keep t^2 term

First rewrite $E_1(\omega) + E_2(\omega)$

↙modulator asym term

$$E_1(\omega)+E_2(\omega) = E_0trJ_0(\Gamma_2)A(0)(1-t_1^2)\ e^{i2k\ell_2}\ e^{i\left(\delta\varphi_2+\frac{\partial\varphi_2}{\partial h_g}h_g\right)}\ e^{i\omega t}$$
$$+E_0trt_1^2J_0(\Gamma_2)A(0)\Big[e^{i2k\ell_2}\ e^{i\left(\delta\varphi_2+\frac{\partial\varphi_2}{\partial h_g}h_g\right)}-e^{i2k(\ell_1+\ell_3)}\ e^{i\left(\delta\varphi_1+\frac{\partial\varphi_2}{\partial h_g}h_g\right)}\Big]e^{i\omega t}$$

↖true interference term

Now evaluate cross terms

For ease of algebra redefine factors

$$E_1(\omega) + E_2(\omega) = A_u\ e^{i\psi}\ e^{i\varphi_2} + A_s\Big[e^{i(\psi_2+\varphi_2)} - \ e^{i(\psi_1+\varphi_1)}\Big]\ e^{i\omega t}$$

$$A_u = trJ_0(\Gamma_2)A(0)(1-t_1^2)E_0 \qquad A_s = trJ_o(\Gamma_2)A(0)t_1^2E_0$$

$$\psi = 2k\ell_2 \qquad \psi_2 = 2k\ell_2 \qquad \psi_1 = 2k(\ell_1+\ell_3) \qquad \varphi_2 = \delta\varphi_2 + \frac{\partial\varphi_2}{\partial h_g}h_g \qquad \varphi_1 = \delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g}h_g$$

$$E_1(\omega+) = -A_\pm\ e^{(\psi_+ + \varphi_1)}\ e^{i(\omega+\omega_{m_1})t}$$

$$A_\pm = trJ_0(\Gamma_2)t_1^2r_1^2r_mA(0)A(\pm)J_1(\Gamma_1)E_0 \qquad \text{where} \quad A(\pm) = (1-A_1)^{1/2}$$

$$\psi_+ = (k+k_+)(\ell_1+\ell_4) + 2(k+k_+)\ell_3 + \tfrac{1}{2}T_1 \qquad A(0) + \left(1 - \tfrac{4(A_1+A_2)}{T_1}\right)$$

$$E_1(\omega_-) = A_\pm\ e^{i(\psi_- + \varphi_1)}\ e^{i(\omega+\omega_{m_1})t}$$

$$\psi_- = (k+k_-)(\ell_1+\ell_4) + 2(k+k_-)\ell_3 - \tfrac{1}{2}T_1$$

Intensity terms at ω_{m_1} from upper sideband

$$I(\omega_{1_+}) = \Big[(E_1\omega) + E_2(\omega)\Big]\Big[E_1(\omega+)\Big]^* + cc$$

$$I(\omega_{1_+}) = -A_uA_\pm\Big[e^{i(\psi+\varphi_2)-(\psi_+ + \varphi_1)}\ e^{-i\omega_m t} + e^{-i(\psi+\varphi_2)+(\psi_+ + \varphi_1)}\ e^{i\omega_m t}\Big]$$

$$- A_sA_\pm\Bigg[\Big[e^{i(\psi_2+\varphi_2)-(\psi_+ + \varphi_1)} - e^{i(\psi_1+\varphi_1)-(\psi_+ + \varphi_1)}\Big]e^{-i\omega_m t}$$

$$+ \Big[e^{-i(\psi_2+\varphi_2)+(\psi_+ + \varphi_1)} - e^{-i(\psi_1+\varphi_1)+(\psi_+ + \varphi_1)}\Big]e^{+i\omega_m t}\Bigg]$$

Lower sideband

$$I(\omega_{1_-}) = \quad A_uA_\pm\Big[e^{i(\psi+\varphi_2)-(\psi_- + \varphi_1)}\ e^{i\omega_m t} + e^{-i(\psi+\varphi_2)+(\psi_- \varphi_1)}\ e^{-i\omega_m t}\Big]$$

$$+ A_sA_\pm\Bigg[\Big[e^{i(\psi_2+\varphi_2)-(\psi_- + \varphi_1)} - e^{i(\psi_1+\varphi_1)-(\psi_- + \varphi_1)}\Big]e^{e\omega_m t}$$

$$+ \Big[e^{-i(\psi_2+\varphi_2)+(\psi_- \varphi_1)} - e^{-i(\psi_1+\varphi_1)+(\psi_- + \varphi_1)}\Big]E^{-i\omega_m t}\Bigg]$$

Intensity from upper sideband $\Delta\varphi = \varphi_2 - \varphi_1$

$$I(\omega_{1_+}) = -2A_sA_\pm\Big[\cos\Big[(\psi_2-\psi_+)+\Delta\varphi-\omega_{m_1}t\Big]-\cos\Big[(\psi_1-\psi_+)-\omega_{m_1}t\Big]\Big]$$
$$-2A_uA_\pm\Big[\cos\Big[(\psi-\psi_+)+\Delta\varphi-\omega_{m_1}t\Big]\Big]$$

Intensity from lower sideband

$$I(\omega_{1_-}) = 2A_sA_\pm\Big[\cos\Big[(\psi_2-\psi_-)+\Delta\varphi+\omega_{m_1}t\Big]-\cos\Big[(\psi_1-\psi_-)+\omega_{m_1}t\Big]$$
$$2A_uA_\pm\Big[\cos(\psi-\psi_-)+\Delta\varphi+\omega_{m_1}t\Big]\Big]$$

Separate time dependent terms

$\cos(\alpha\pm\beta) = \cos\ \alpha\ \ \cos\ \beta \pm \sin\ \alpha\ \ \sin\ \beta$

The modulation terms that sum will be those associated with

$\sin\ \omega_{m_1}t$ *and* $\sin(\psi+\Delta\varphi)$

$$\begin{aligned}I(\omega_1) = \quad & 2A_sA_\pm\Big[\Big(\sin\big[(\psi_2-\psi_+)+\Delta\varphi\big]+\sin\big[(\psi_1-\psi_-)+\Delta\varphi\big]\Big)\sin\omega_{m-1}t \\ & -\Big(\cos\big[(\psi_2-\psi_+)+\Delta\varphi\big]-\cos\big[(\psi_2-\psi_-)+\Delta\varphi\big]\Big)\cos\omega_{m_1}t\Big] \\ & -2A_sA_\pm\Big[\Big[\sin(\psi_1-\psi_-)+\sin(\psi_1-\psi_+)\sin\omega_{m_1}t \\ & -\Big[\cos(\psi_1-\psi_-)-\cos(\psi_1-\psi_+)\Big]\cos\omega_{m_1}t\Big]\end{aligned}$$

↑ interesting term, only involves input mirror position and not cavity resonance

$$\begin{aligned}& +2A_uA_\pm\Big[\Big(\sin\big[(\psi-\psi_-)+\Delta\varphi\big]+\sin\big[(\psi-\psi_+)+\Delta\varphi\big]\Big)\sin\omega_{m_1}t \\ & +\Big(\cos\big[(\psi-\psi_-)+\Delta\varphi\big]-\cos\big[(\psi-\psi_+)+\Delta\varphi\big]\Big)\cos\omega_{m_1}t\Big]\end{aligned}$$

Look at balanced case and shot noise

Look only at leading cavity phase dependent term

$$I_{\text{signal}}(\omega) = 2A_s A_\pm \left[\sin\left[(\psi_2 - \psi_+) + \Delta\varphi\right] + \sin\left[(\psi_2 - \psi_-) + \Delta\varphi\right]\right] \sin\omega_{m_1} t$$

Can be rewritten

$$\sin(\psi_2 - \psi_+) + \Delta\varphi = \sin(\psi_2 - \psi_+)\cos\Delta\varphi + \cos(\psi_2 - \psi_+)\sin\Delta\varphi$$

$$\sin(\psi_2 - \psi_-) + \Delta\varphi = \sin(\psi_2 - \psi_-)\cos\Delta\varphi + \cos(\psi_2 - \psi_-)\sin\Delta\varphi$$

$$I_{\text{signal}}(\omega) = 2A_s A_\pm \Big[\sin\Delta\varphi\left[2\cos\frac{1}{2}(2\psi_2 - (\psi_+ + \psi_-))\cos\frac{1}{2}(\psi_+ - \psi_-)\right] + \cos\Delta\varphi\left[2\sin\frac{1}{2}(2\psi_2 - (\psi_+ + \psi_-))\cos\frac{1}{2}(\psi_+ - \psi_-)\right]\Big] \sin\omega_{m_1} t$$

For maximum fringe sensitivity

$$\cos\frac{1}{2}(2\psi_2 - (\psi_+ + \psi_-)) = 1 \qquad \text{and} \qquad \cos\frac{1}{2}(\psi_+ - \psi_-) = 1$$

Require

$$\frac{1}{2}\left[2\psi_2 - (\psi_+ + \psi_-)\right] = \frac{1}{2}\left[2k\ell_2 - 4k(\ell_1 + \ell_4 + 2\ell_3)\right] = k\ell_2 - 2k(\ell_1 + \ell_4 + 2\ell_3) = n2\pi$$

$$\tfrac{1}{2}(\psi_+ - \psi_-) = \tfrac{1}{2}\left[2k_m(\ell_1 + \ell_4) + 8k_m\ell_3 + T_1\right] = n2\pi$$

done with $n = 0$

if $T_1 \ll 1$

$k_m\ell_i \ll 1$

System will not work on white light fringe

but it is only λ adjustment not λ_m

A system with FOX/LI cavities on both cavities could be symmetric eliminating the unbalanced A_u term and could operate at the white light fringe. It remains to see it one can get information in quadrature for each front mirror individually - later calcultion will try this.

Assuming that near balance has been achieved

$$I_{\text{signal}}(\omega) = 4A_s A_\pm \sin\Delta\varphi \ \sin\omega_{m_1} t$$

$$= 4(tr)^2 J_0^2(\Gamma_2) A^2(0) t_1^4 r_1^2 r_m \ A(\pm) J_1(\Gamma_1) E_0^2 \sin\omega_{m_1} t \left(\frac{\partial\varphi_2}{\partial h_g} - \frac{\partial\varphi_1}{\partial h_g}\right) h_g$$

The average intensity

$$I(0) = (tr)^2 J_0^2(\Gamma_2) E_0^2 \left\{ t_1^4 A_1^2(0) + A_2^2(0) + t_1^4 A_1^2(0) r_1^4 r_m^2 J_1^2(\Gamma_1) \left[A_1^2(+) + A_1^2(-) \right] - 2A_1(0) A_2(0) t_1^2 \cos(\psi) \right\}$$

$$\psi = 2k(\ell_1 + \ell_3 - \ell_2) + (\delta\varphi_2 - \delta\varphi_1) + \left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right) h_g$$

Assume $\psi = 0$ $\quad A_1(0) = A_2(0) = A(0) \simeq A_1(+) = A_1(-)$ a very good cavity

$$I(0) \cong (tr)^2 J_1^2(\Gamma_2) E_0^2 A^2(0) \left[(1 + t_1^4 - 2t_1^2) + t_1^4 r_1^4 r_m^2 J_1^2(\Gamma_1) \right]$$

Shot noise

$$P^2(f) = \frac{2 \overset{\swarrow I(0)}{<P>} h\nu}{\eta}$$

Signal power

$$P_{\text{sig}}^2(f) = 16(tr)^4\ J_0^4(\Gamma_2)\ A^4(0) t_1^8 r_1^4 r_m^2\ A^2(\pm)\ J_1^2(\Gamma)\ E_0^4 \overset{\swarrow <\sin^2 \omega_{m_1} t>}{\tfrac{1}{2}} \left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right)^2 h_g^2(f)$$

Set noise power to signal power solve for min $h_g^2(f)$

$$h_{g_{\text{shot}}}(f) = \left(\frac{h\nu}{\eta 4 (tr)^2 J_0^2(\Gamma_2) A^2(0) A^2(\pm) E_0^2} \right)^{1/2} \left\{ \frac{\left[1 + t_1^4 - 2t_1^2 + t_1^4 r_1^4 r_m^2 J_1^2(\Gamma_1) \right]^{1/2}}{t_1^4 r_1^2 r_m J_1(\Gamma)} \right\} \frac{1}{\left(\frac{\partial \varphi_2}{\partial h_g} - \frac{\partial \varphi_1}{\partial h_g} \right)}$$

$\nwarrow g(t_1 J_1(\Gamma)$

Look at optimization of $g\big(t_1, J_1(\Gamma)\big)$

Most likely optimization occurs at $\underset{\max}{J_1(\Gamma)} \cong 1/\sqrt{3}$ $\quad \leftarrow$ true checked 1/28/88
good to 10%

Let $r_m = 1$ find best value of $g(t_1 J_{1_{\max}}(\Gamma))$

$$r_1^2 + t_1^2 = 1 \qquad\qquad T = t^2$$

$$g(T J_{1_{\max}}(\Gamma)) = \frac{\sqrt{3}\left[1 + T^2 - 2T + \frac{T^2(1-T)^2}{3} \right]^{1/2}}{T^2(1-T)}$$

Values	T	$g(T_1 J_{\max}$
	.9999	2.0000
	.999	2.0035
	.990	2.0355
	.900	2.4098
	.800	2.9811

again as in loop modulator
sq. wave will make limit smaller

With perfect cavity $\quad (tr)^2 = 1/4 \quad J_0(\Gamma_2) = 1 \quad A(0) = A^2(\pm) = 1$

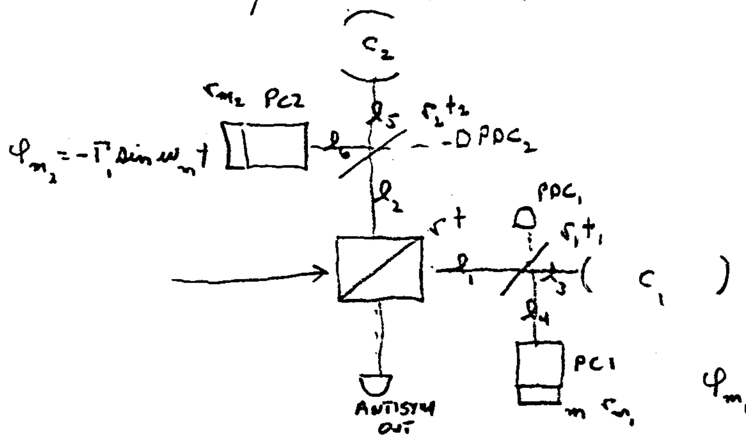
$$\varphi(f) = \left(\frac{4h\nu}{\eta p(_{\text{in}})}\right)^{1/2}$$ worse than internal modulation by 2 in power

and worse than external link system
by $2\Big/3/2 = 4/3$ in power

Remains to determine if sym system improves the situation, intuitively it should.

(Trouble is that J_1 cannot be made larger so the double modulation schemes win)

Double FOX/LI auxiliary cavity modulator



Drive modulators at same frequency
180^o out of phase

PDC_2 for cavity signal 2
PDC_1 for cavity signal 1

$\varphi_{m_1} = \Gamma \sin \omega_m t$

Follow prior calculation for single FOX/LI system.

For simplicity assume $r_2 = r_1 \qquad t_1 = t_2 \qquad r_{m_1} = r_{m_2}$

Beams at anti sym output

$$E_1(\omega), \quad E_1(\omega_{+_1}), \quad E(\omega_{-_1}), \quad E_2(\omega), \quad E_2(\omega_{+_2}), \quad E_2(\omega_{-_2})$$

$$E_1(\omega) = -trE_0 \; e^{i2k\ell_1} \left(t_1^2 A_1(0) \; e^{i\left(\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} h_g + 2k\ell_3\right)} \right) e^{i\omega t}$$

$$E_2(\omega) = trE_0 \; e^{i2k\ell_2} \left(t_1^2 A_2(0) \; e^{i\left(\delta\varphi_2 + \frac{\partial\varphi_2}{\partial h_g} h_g + 2k\ell_5\right)} \right) e^{i\omega t}$$

$$E_1(\omega) + E_2(\omega) = trE_0 t_1^2 \Big[A_2(0) \; e^{i2k(\ell_2+\ell_5)} \; e^{i\delta\varphi_2 + \frac{\partial\varphi_2}{\partial h_g} h_g} - A_1(0) \; e^{i2k(\ell_1+\ell_3)} \; e^{i\delta\varphi_1 + \frac{\partial\varphi_1}{\partial h_g} h_g} \Big]$$

Simplify by letting $A_2(0) = A_1(0) = A(0)$

$$\psi_2 = 2k(\ell_2 + \ell_5) \quad \varphi_2 = \delta\varphi_2 + \frac{\partial\varphi_2}{\partial h_g} h_g$$

$$\psi_1 = 2k(\ell_1 + \ell_3) \quad \varphi_1 = \delta\varphi_1 + \frac{\partial\varphi_1}{h_g} h_g$$

$$A_s = trt_1^2 A(0) E_0$$

$$A_1(\pm) = trt_1^2 r_1^2 r_m A(0) A(\pm) J_1(\Gamma_1) E_0$$

$$\psi_{+_1} = (k + k_+)(\ell_1 + \ell_4) + 2(k + k_+)\ell_3 + \frac{1}{2} T_1$$

$$\psi_{-_1} = (k + k_-)(\ell_1 + \ell_4) + 2(k + k_-)\ell_3 - \frac{1}{2} T_1$$

$$\psi_{+_2} = (k + k_+)(\ell_2 + \ell_6) + 2(k + k_+)\ell_5 + \frac{1}{2} T_1$$

$$\psi_{-_2} = (k + k_-)(\ell_2 + \ell_6) + 2(k + k_-)\ell_5 - \frac{1}{2} T_1$$

$$E_1(\omega) + E_2(\omega) = A_s \left[e^{i(\psi_2 + \varphi_2)} - e^{i(\psi_1 + \varphi_1)} \right] e^{i\omega t}$$

$$E_1(\omega_{+_1}) = -A_1(\pm) \; e^{i(\omega + \omega_{m_1})t}$$

$$E_1(\omega_{-_1}) = A_1(\pm) \; e^{i(\psi_{-_1} + \varphi_1)} e^{i(\omega - \omega_{m_1})t}$$

$$E_2(\omega_{+_2}) = -A_1(\pm) \; e^{i(\psi_{+_2} + \varphi_2)} \; e^{i(\omega + \omega_{m_1})t}$$

$$E_2(\omega_{-_2}) = A_1(\pm) \; e^{i(\psi_{-_2} + \varphi_2)} \; e^{i(\omega - \omega_{m_1})t}$$

DC terms in this approximation good to r_1^2

$$I(0) = A_s^2\left[2-2\ \cos\left[(\psi_2-\psi_1)+(\varphi_2-\varphi_1)\right]\right]$$
$$-A_\pm^2\left[4+\ 2\cos\left[(\psi_{+_1}-\psi_{+_2})+(\varphi_2-\varphi_1)\right]+2\cos\left[(\psi_{-_2}-\psi_{-_1})+(\varphi_2-\varphi_1)\right]\right]$$

AC terms

Upper sideband

$$I(\omega)_+ = -A_s A_\pm\Bigg\{\left[e^{i(\psi_2-\psi_{+_1})+i(\varphi_2-\varphi_1)} - e^{i(\psi_1-\psi_{+_1})} + e^{i(\psi_2-\psi_{+_2})}\right.$$
$$\left.- e^{i(\psi_1-\psi_{+_2})+i(\varphi_1-\varphi_2)}\right]e^{-i\omega_m t} + \left[e^{-i(\psi_2-\psi_{+_1})-i(\varphi_2-\varphi_1)} - e^{-i(\psi_1-\psi_{+_1})}\right.$$
$$\left.+ e^{-i(\psi_2-\psi_{+_2})} - e^{-i(\psi_1-\psi_{+_2})-i(\varphi_1-\varphi_{+_2})}\right]e^{i\omega_m t}\Bigg\}$$

Lower sideband

$$I(\omega)_- = A_s A_\pm\Bigg\{\left[e^{i(\psi_2-\psi_1)+i(\varphi_2-\varphi_1)} - e^{(\psi_1-\psi_{-_1})} + e^{i(\psi_2-\psi_{-_2})}\right.$$
$$\left.- e^{i(\psi_1-\psi_{-_2})+i(\varphi_1-\varphi_2)}\right]e^{i\omega_m t} + \left[e^{-i(\psi_2-\psi_{-_1})-i(\varphi_2-\varphi_1)} - e^{i(\psi_1-\psi_{-_1})}\right.$$
$$\left.+ e^{i(\psi_2-\psi_{-_2})} - e^{-i(\psi_1-\psi_{-_2})-i(\varphi_1-\varphi_2)}\right]e^{i\omega_m t}\Bigg\}$$

Check arithmetic in limit of $\psi_1 = \psi_2 \quad \psi_{+_1} = \psi_{+_2} \quad \psi_{-_1} = \psi_{-_2}$ (total balance)

$I(\omega) = I(\omega_+) + I(\omega_-) = -8A_s A(\pm)\ \sin\Delta\varphi\ \sin\omega t$ OK

Shot noise in this limit when $\delta\varphi_1 - \delta\varphi_2 = 0$

$I(0) = 8A_\pm^2$

$\underset{\text{shot}}{P^2(f)} = \frac{2\ I(0)h\nu}{\eta}$ $\underset{\text{signal}}{P^2(f)} = 64\ A_s^2 A_\pm^2\ \sin^2\Delta\varphi\left(\frac{1}{2}\right)$

Shot noise equals signal power

$\varphi^2(f) = -\frac{h\nu}{\eta}\frac{16\ A_\pm^2}{32\ A_s^2 A_\pm^2} = \frac{h\nu}{2\eta A_s^2} = \frac{h\nu}{2(rt)^2\eta t_1^2 A_0^2 E_0^2}$

If $(rt)^2 = 1/4$

$\varphi^2(f) = \frac{2\ h\nu}{\eta t_1^2 A_0^2 E_0^2}$ which within a factor of $1/t_1^2$ is as good as the original internal modulator

The system is insensitive to Γ as long as $J_2(\Gamma)$ is small and one may neglect terms of order r_1^2

Use of FOX/LI system as controller

One can achieve in either single or double FOX/LI system a small amount of fringe control from the second order term in the modulator

From page 14 Either output beam from the FOX/LI cavity

has terms that look like at cavity resonance

$$E_{\text{out}_{FOX/LI}} = A\ e^{ik\ell_1}\left[e^{i2k\ell_3} - r_1^2 r_m J_0(\Gamma_1)A(0)\ e^{-i2k\ell_4}\right]e^{i\omega t}$$

Now ℓ_4 can be made variable by putting servo signals on the Pockel's cell

What is the ϕ shift possible? $J_0(\Gamma_1) = 1 \quad r_m = 1 \quad A(0) \sim 1$

And what is the AM generated?

$$\left(\frac{E_{\text{out}_{FOX/LI}}}{A\ e^{ik\ell_1 + \ell_3}}\right) = \left[1 - r_1^2\ e^{-i2k(\ell_4+\ell_3)}\right] = |Z|\ e^{i\psi}$$

$$|Z| = \left[1 - 2r_1^2\ \cos 2k(\ell_4+\ell_3) + r_1^4\right]^{1/2}$$

$$\psi = \tan^{-1}\frac{r_1^2\ \sin 2k(\ell_4+\ell_3)}{1 - r^2\ \cos 2k(\ell_4+\ell_3)}$$

Look at derivatives

AM terms $\quad \frac{d|Z|}{d\ell_4} = \frac{2kr_1^2\ \sin\left(2k(\ell_3+\ell_4)\right)}{\left[1 - 2r_1^2\ \cos 2k(\ell_4+\ell_3) + r_1^4\right]^{1/2}} \cong 2kr_1^2\ \sin\left(2k(\ell_3+\ell_4)\right)$

Phase terms

$$\frac{d\psi}{d\ell_4} \cong \underset{\substack{\text{to order} \\ r_1^2}}{2kr_1^2\ \cos 2k(\ell_4+\ell_3)}$$

The amount of phase control $\quad \Delta\psi_{\max} = 2kR_1$

If $\quad 2k(\ell_4+\ell_3) = n\pi$

Cannot control much more than $\quad 2k\Delta\ell_4 \sim \pi/4$ without losing control and generating AM

max $\Delta\psi = r_1^2\pi/4 \sim R_1\pi/4$ radians $\qquad$ Note: Could go to $10\% = R$ without more than 1.05 penalty in shot noise

max AM $\quad \frac{\Delta A}{A} \sim R_1 \qquad \downarrow^{\text{for} R_1 = 10^{-2}} \quad 5^o for R = 10^{-1}$

So a limited amount, say $1/2^o$, of phase change in each FOX/LI system is possible. Amount is traded against shot noise increase by $\left(\frac{1}{1-R_1}\right)^{1/2}$ and increased system loss

Exit port of FOX/LI - How big is the cavity discriminant?

The beams incident on the cavity detectors in lowest order

page 14 E_2

Unshifted carrier: assume cavity off resonance by $\delta\varphi_1$

$$E_2(\omega) \cong -r_1 E_{\text{in}}\left[A_1(\Delta\omega_1)J_0(\Gamma_1)\ e^{i\delta\varphi_1+2k(\ell_3+\ell_4)}+1\right]e^{i\omega t}$$

Upper sideband

$$E_2(\omega_+) \cong t_1^2 A_1(\Delta\omega_1) r_1 E_{\text{in}}\ e^{i\delta\varphi_1} r_m J_1(\Gamma_1)\ e^{i(2k\ell_3+k\ell_4+k_+\ell_4)}\Big[1+ r_1^2 A(+)\ e^{iT_{1/2}} r_m J_0(\Gamma_1)\ e^{i2k_+(\ell_3+\ell_4)}\Big]e^{i(\omega+\omega_m)t}$$

Lower sideband

$$E_2(\omega_-) \cong -t_1^2 A_1(\Delta\omega_1) r_1 E_{\text{in}}\ e^{i\delta\varphi_1} r_m J_1(\Gamma_1)\ e^{i(2k\ell_3+k\ell_4+k_+\ell_4)}\Big[1+ r_1^2 A(-)\ e^{-iT/2} r_m J_0(\Gamma_1)\ e^{i2k_-(\ell_3+\ell_4)}\Big]e^{i(\omega-\omega_m)t}$$

Assume for ease of calculation to determine if a discriminant exists that $2k(\ell_3+\ell_4)\cong\pi$ $\quad 2k_-(\ell_3+\ell_4)\sim\pi$, $\quad 2k_+(\ell_3+\ell_4)\sim\pi$

$$E_2(\omega) \sim -r_1 E_{\text{in}}\left[1-A(\Delta\omega_1)J_0(\Gamma_1)\ e^{i\delta\varphi_1}\right]e^{i\omega t}$$

$$E_2(\omega_+) \sim -t_1^2 E_{\text{in}} A(\Delta\omega_1) r_1 r_m J_1(\Gamma_1)\ e^{i\delta\varphi_1}\left[1-r_1^2 A(+)\ e^{iT/2} r_m J_0(\Gamma_1)\right]e^{i(\omega+\omega_m)t}$$

$$E_2(\omega_-) \sim t_1^2 E_{\text{in}} A(\Delta\omega_1) r_1 r_m J_1(\Gamma_1)\ e^{i\delta\varphi_1}\left[1-r_1^2 A(-)\ e^{-iT/2} r_m J_0(\Gamma_1)\right]e^{i(\omega-\omega_m)t}$$

Look first only at lowest order terms at cavity photodetector

$$I(\omega) = E_{\text{in}}^2 4J_1(\Gamma_1) r_1^2 t_1^2 A(\Delta\omega) r_m\ \sin\delta\varphi_1\ \sin\omega_m t$$
$$+\ \text{terms of order } r_1^2 \text{ smaller}$$

So the cavity discriminator uses sidebands generated after reflection from cavity

Must check that the intensity is sufficient for adequate reduction of Townes width for frequency noise
Power lost is primarily into the sidebands. The Bessel sum rule keeps it honest

$$I_{2_{\text{lost}}} = r_1^2 E_{\text{in}}^2\left[1-A_1(0)J_0(\Gamma_1)\right]^2$$

Shot noise in cavity control loop is E_{in}^2 (↙ main splitter $\genfrac{}{}{0pt}{}{R_1}{1/2}\ \genfrac{}{}{0pt}{}{T}{1/2}$) is $E_0^2 r^2 \sim \frac{1}{2}E_0^2$

$$P^2(f) = \frac{2I_{\text{lost}}h\nu}{\eta} \qquad \underset{\text{signal}}{P^2(f)} = 16J_1^2(\Gamma)(r_1^2 t_1^2)^2 A^2(0) r_m^2 \varphi^2(f)\frac{1}{2}E_{\text{in}}^4$$

$$\underset{\text{shot}}{\varphi^2(f)} = \frac{h\nu}{\eta}\left(\frac{1}{2E_0^2 r_m^2 r_1^2 t_1^4}\right)\frac{\left(1-A_1(0)J_0(\Gamma)\right)^2}{J_1^2(\Gamma)A_1^2(0)}$$

This sets limit on how small r_1^2 can be

$$\underset{\Gamma\to 0}{\text{optimum } \Gamma \text{ minimize}} \Rightarrow \frac{1-A_1(1-\Gamma^2/4)}{\Gamma/2\ A_1} = \frac{1-A_1+A_1\Gamma^2/4}{A_1\Gamma/2}$$

Tradeoffs in choosing R_1

1) Shot noise gets poorer by $\left(\frac{1}{1-R}\right)^{1/2}$ in amplitude
2) Frequency discriminator loop improves $R^{1/2}$
3) Phase control dynamic range increases as R

Advantages

Get minimum shot noise. Retain best contrast
Cavity interrogation done as by product
No further compensation plates required
Balancing can be done electrically
Lower power in Pockel's cells
Cancellation of odd $(n = 000, J_n)$ sidebands at sym port, helps recycling

Disadvantages

Relative to inline Pockel's cells: Small control range, AM terms out of balance
Relative to outside loop modulator: 1 additional position servo
single FOX/LI: 1 additional position servo

Questions remaining:

How best to get information on ℓ_i other than cavity

To do:

Finish reduction of anti sym output when system out of balance