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LIGO Optics Vibration Levels Equivalent to Shot Noise in the Advanced Detectors

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Abstract

The laser beam entering the arms of a LIGO interferometer is highly dewiggled by the mode cleaner cavity(ies) in the beam conditioning part of the system. In order to take full advantage of this purified beam, one has to avoid rewiggling it by vibrating optical components. On the basis of simple geometric considerations, the present note gives estimates of the optical component vibration levels which are equivalent to the shot noise in the advanced detectors.

Less straightforward effects like a beam scanning an uneven mirror surface, field inhomogeneity in a Faraday rotator coupling to the beam via Faraday glass movements, variations of the RF modulation index due to Pockels cell tilt and a whole variety of polarization effects related to the interaction of the beam with a vibrating birefringent medium have not been considered.

1 Beam Wiggle Levels Equivalent to Shot Noise

The levels of beam wiggle, equivalent to shot noise in the advanced detectors, are given in in Table 1, for the paths between the recycling mirror and the input to the 4 *km* arms (lines 6-9) and at the input to the recycling

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*following discussions with F. Raab and P. Saulson mirror (lines 14-17). The derivation of the numbers for each entry is discussed below. The theory and formulæ are contained in Appendices 1[1], 2[2], 3[3].

- Line 1 contains the frequencies for which the wiggle levels have been estimated. 10 Hz is considered the lowest frequency at which the seismic isolation can be made good enough to allow a substantial sensitivity; 50 Hz is the frequency with the highest expected sensitivity for bursts; 215 Hz is the lower limit of the frequency band where one expects the highest sensitivity for periodic signals; and 1000 Hz is an arbitrarily chosen higher frequency. These data are taken from Appendix A of the December 1987 Proposal.
- Line 2 indicates the type of signal corresponding to each column.
- Line 3 lists the spectral density of strain noise equivalent to the highest sensitivities expected from the advanced later stage detectors, as described in Appendix A of the December 1987 Proposal.
- Line 4 contains the optimized storage times corresponding to the sensitivities in Line 3. For burst signals, $\tau = (4\pi f)^{-1}$. For periodic signals, τ is half the period of the signal.
- Line 5 gives the finesse values corresponding to the storage times on Line 4: $\mathcal{F} = \pi c\tau/l$.
- Line 6 lists the shot noise equivalent spectral density of time varying higher order transverse mode amplitudes[4], in the presence of a static higher order mode amplitude of 0.3. ϵ's are estimated by using $\delta l/l = \delta\nu/\nu$, Eq. (8) (Appendix 1) for $N = 1$ and $\mathcal{F}$ from Line 5.
- Lines 7-9 translate ϵ from Line 6 into angular wiggle, lateral displacement of the beam and beam pulsation, by use of Eqs. (5,3,7) (Appendix 3), respectively. The appropriate beam radius is 1.8 cm.

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[1] Appendix 1: Alex Abramovici, *Do Wiggle Effects Depend on Mode Cleaner Length?*

[2] Appendix 2:Alex Abramovici, *Variable Transmission Mirror for Recycling*

[3] Appendix 3:Dana Z. Anderson, *Alignment of Resonant Optical Cavities*, Appl. Optics **23**, 2944 (1984)

[4] at the input of the 4 km cavities

1	f	Hz	10	50	215	1000
2	Source		Burst	Burst	Periodic	Periodic
3	$\delta l(f)/l$	$\mathrm{Hz}^{-1/2}$	$2 \cdot 10^{-24}$	$4.6 \cdot 10^{-25}$	$2.15 \cdot 10^{-25}$	$2.15 \cdot 10^{-25}$
4	Storage time	ms	8	1.6	2.3	0.5
5	Finesse		1,875	375	548	118
6	$\epsilon(f)$	$\mathrm{Hz}^{-1/2}$	$4.7 \cdot 10^{-7}$	$4.3 \cdot 10^{-9}$	$4.3 \cdot 10^{-9}$	$2 \cdot 10^{-10}$
7	Angular wiggle	$\mathrm{rad}\cdot\mathrm{Hz}^{-1/2}$	$4.3 \cdot 10^{-12}$	$3.9 \cdot 10^{-14}$	$3.9 \cdot 10^{-14}$	$1.8 \cdot 10^{-15}$
8	Lateral beam displacement	$\mathrm{m}\cdot\mathrm{Hz}^{-1/2}$	$8.5 \cdot 10^{-9}$	$7.7 \cdot 10^{-11}$	$7.7 \cdot 10^{-11}$	$3.6 \cdot 10^{-12}$
9	Beam pulsation	$\mathrm{Hz}^{-1/2}$	$4.7 \cdot 10^{-7}$	$4.3 \cdot 10^{-9}$	$4.3 \cdot 10^{-9}$	$2 \cdot 10^{-10}$
10	Coupler transmission	%	0.33	1.68	1.15	5.32
11	Loss in 4 *km* cavities	%	11.4	2.35	3.4	0.75
12	Finesse of recycling cavity		27.6	133.7	91.9	419
13	Recycling cavity dewiggling factor		15.7	76	52.3	238.6
14	$\epsilon(f)$	$\mathrm{Hz}^{-1/2}$	$7.4 \cdot 10^{-6}$	$3.3 \cdot 10^{-7}$	$2.2 \cdot 10^{-7}$	$4.8 \cdot 10^{-8}$
15	Angular wiggle	$\mathrm{rad}\cdot\mathrm{Hz}^{-1/2}$	$6.7 \cdot 10^{-11}$	$3 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$4.4 \cdot 10^{-13}$
16	Lateral beam displacement	$\mathrm{m}\cdot\mathrm{Hz}^{-1/2}$	$1.3 \cdot 10^{-7}$	$5.9 \cdot 10^{-9}$	$4 \cdot 10^{-9}$	$8.6 \cdot 10^{-10}$
17	Beam pulsation	$\mathrm{Hz}^{-1/2}$	$7.4 \cdot 10^{-6}$	$3.3 \cdot 10^{-7}$	$2.2 \cdot 10^{-7}$	$4.8 \cdot 10^{-8}$

Table 1: Beam wiggle levels equivalent to shot noise, inside the recycled system (lines 6-9) and at the input to the recycling mirror (lines 14-17). The other lines contain the data from which the wiggle levels have been derived, as explained in the text.

- Line 10 lists the transmission of the 4 km cavity input coupler corresponding to the finesse $\mathcal{F}$ given in Line 5, by using $T_c \sim 2\pi/\mathcal{F}$ (mirror losses are disregarded).

- Line 11 contains the loss L_I which the light experiences by interacting with the 4 km cavities, calculated by using the transmissions from Line 10, a sum of mirror losses $L_c + L_h = 100$ *ppm* and: $L_I = 1 - [(T_c - L_c - L_h)/(T_c + L_c + L_h)]^2$, where "c" stands for "coupler" and "h" stands for "high reflector".

- Line 12 lists the finesse of the recycling cavity (defined in Appendix 2): $\mathcal{F}_R = 2\pi/(\Theta + L_I)$, where Θ is the transmission of the recycling mirror which is assumed to have negligible loss. The recycling factor is maximized by $\Theta = L_I$, thus $\mathcal{F}_R = \pi/L_I$.

- Line 13: the dewiggling effect of the recycling cavity is estimated by use of $\mathcal{F}_R$ from Line 12 and Eq. (10) (Appendix 1), for $N = 1$. It is assumed that the ratio between the recycling mirror curvature and the distance between this mirror and the 4 km cavity input mirrors is the same as for the big cavities. i. e. 5:4.

- Line 14: multiplying the ϵ's from Line 6 by the dewiggling factors from Line 13 yields the shot noise equivalent higher order mode amplitudes at the input to the recycling mirror.

- Lines 15-17: the higher order mode amplitude ϵ from Line 14 is converted to angular wiggle, lateral beam displacement and beam pulsation by use of Eqs. (5,3,7) (Appendix 3). These are the values which should not be exceeded at the input to the recycling mirror.

2 Shot Noise Equivalent Component Vibration Levels

Optical component motion impresses or is equivalent to a certain amount of wiggle on the laser beam. The shot noise equivalent vibration[5] levels

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[5] *translation and rotation or tilt*

for various components located downstream from the main mode cleaner are detailed in this section. Component motions which generate differential noise in the two arms should be limited to amounts matching the shot noise equivalent beam wiggle given in Table 1. If the interferometer arms are perfectly matched in storage time, component motions which generate common mode noise should not be of concern. Real interferometer arms can not be perfectly matched, however, and therefore some noise may result from this kind of component motion. The position taken here will be that the arms can be matched within 10%. As a consequence, the limits on component motion generating common mode noise will be required to match the wiggle specifications of Table 1, relaxed by a factor of 10.

The shot noise equivalent movements for a component are estimated under the assumption that the other components are noise free. Also, a signal to noise ratio equal to 1 is assumed. If all N components were to contribute an equal amount of noise, the total noise level would be $\sqrt{N}$ times higher and the numbers given in Tables 2,3 should be divided by $\sqrt{N}$.

As is the case at any point along the beam which is not at the beam waist, beam tilt also results in lateral displacement of the waist. For distances from the waist in the tens of meters[6], the resulting waist displacement is way below the shot noise equivalent value, as long as the beam tilt does not exceed its shot noise equivalent level, according to Lines 7,8 and Lines 15,16 from Table 1[7].

2.1 Components Inside the Recycled System

The shot noise equivalent motions of components inside the recycled system are given in Table 2. Explanations concerning individual lines are given below.

- Lines 1,2 are the same as in Table 1.
- Line 3 contains the shot noise equivalent motions for the far test mass, assumed to have a curvature $r = 5\ km$.

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[6] for the geometry considered here - flat input mirror and far mirror with $r = 5\ km$ - the main cavity waist is located at the input mirror

[7] henceforth to be denoted Lines 1.7,8 and Lines 1.15,16

1	f	Hz	10	50	215	1000
2	Source		Burst	Burst	Periodic	Periodic
3	Far test mass	a	$8 \cdot 10^{-21}$	$1.8 \cdot 10^{-21}$	$8.6 \cdot 10^{-22}$	$8.6 \cdot 10^{-22}$
		p	$8.5 \cdot 10^{-9}$	$7.7 \cdot 10^{-11}$	$7.7 \cdot 10^{-11}$	$3.6 \cdot 10^{-12}$
		α	$1.7 \cdot 10^{-12}$	$1.5 \cdot 10^{-14}$	$1.5 \cdot 10^{-14}$	$7.2 \cdot 10^{-16}$
4	Close test mass	a	$8 \cdot 10^{-21}$	$1.8 \cdot 10^{-21}$	$8.6 \cdot 10^{-22}$	$8.6 \cdot 10^{-22}$
		α	$4.3 \cdot 10^{-12}$	$3.9 \cdot 10^{-14}$	$3.9 \cdot 10^{-14}$	$1.8 \cdot 10^{-15}$
5	Deflection mirror	a,p	$4.8 \cdot 10^{-18}$	$2.1 \cdot 10^{-19}$	$1.5 \cdot 10^{-19}$	$3.2 \cdot 10^{-20}$
		α	$2.1 \cdot 10^{-12}$	$1.9 \cdot 10^{-14}$	$1.9 \cdot 10^{-14}$	$9 \cdot 10^{-16}$
6	Compensation plate	α	$2.7 \cdot 10^{-7}$	$2.5 \cdot 10^{-9}$	$2.5 \cdot 10^{-9}$	$1.2 \cdot 10^{-10}$
7	Beam splitter	a,p	$4.8 \cdot 10^{-18}$	$2.1 \cdot 10^{-19}$	$1.5 \cdot 10^{-19}$	$3.2 \cdot 10^{-20}$
		α	$2.1 \cdot 10^{-12}$	$1.9 \cdot 10^{-14}$	$1.9 \cdot 10^{-14}$	$9 \cdot 10^{-16}$
8	Recycling mirror	a	$4.8 \cdot 10^{-17}$	$2.1 \cdot 10^{-18}$	$1.5 \cdot 10^{-18}$	$3.2 \cdot 10^{-19}$
		p	$8.5 \cdot 10^{-9}$	$7.7 \cdot 10^{-11}$	$7.7 \cdot 10^{-11}$	$3.6 \cdot 10^{-12}$
		α	$8.5 \cdot 10^{-10}$	$7.7 \cdot 10^{-12}$	$7.7 \cdot 10^{-12}$	$3.6 \cdot 10^{-13}$
9	Ground motion	a,p	$4 \cdot 10^{-10}$	$1.6 \cdot 10^{-11}$	$8.7 \cdot 10^{-13}$	$4 \cdot 10^{-14}$
		α	$4 \cdot 10^{-10}$	$1.6 \cdot 10^{-11}$	$8.7 \cdot 10^{-13}$	$4 \cdot 10^{-14}$
10	Residual motion	a	$4 \cdot 10^{-14}$	$2.6 \cdot 10^{-18}$	$4 \cdot 10^{-22}$	$4 \cdot 10^{-26}$
		p	10^{-12}	$6.4 \cdot 10^{-17}$	10^{-20}	10^{-24}
		α	$4 \cdot 10^{-14}$	$2.6 \cdot 10^{-18}$	$4 \cdot 10^{-22}$	$4 \cdot 10^{-24}$

Table 2: Shot noise equivalent movements of components inside the recycled system. The shorthands in Column 3 are: a - axial movements ($\mathrm{m} \cdot \mathrm{Hz}^{-1/2}$), p - transverse movements ($\mathrm{m} \cdot \mathrm{Hz}^{-1/2}$), α - horizontal and vertical tilt ($\mathrm{rad} \cdot \mathrm{Hz}^{-1/2}$). For more details see the text.

The axial motion entry corresponds to the strain sensitivity from Line 1.3.

The transverse motion entry corresponds to the transverse motion spec for the beam (Line 1.8).

The angular motion of this test mass results in a lateral displacement of the resonator axis $x_p = \alpha r$. α is evaluated by using x_p from Line 1.8.

- Line 4 contains the shot noise equivalent motions of the close test mass, assumed to be flat.

  The axial motion entry corresponds again to the strain sensitivity from Line 1.3.

  This being a flat mirror, its transverse motion is irrelevant, except for yet to be evaluated effects related to surface uneveness.

  Tilt of this mass has approximately the same effect as the beam tilting by the same angle (angular wiggle), therefore α is taken equal to the angular wiggle spec (Line 1.7).

- Line 5:

  Translation of the deflection mirror causes a change in the optical path and/or a lateral displacement of the deflected beam, depending on whether the movement is parallel or transverse to the incident beam. The more stringent limit on deflection mirror movement results from the change in optical path and is evaluated by noting that the interferometer is $\mathcal{F}/\pi$ (i. e. the number of bounces in the main cavities) times less sensitive to deflection mirror motion than it is to test mass movements. The shot noise equivalent deflection mirror translation is thus evaluated by using the value derived for test mass motion (Line 2.3a) and $\mathcal{F}$ from Line 1.5.

  Tilt of the deflection mirror by α causes the beam to tilt by 2α, therefore the shot noise equivalent mirror tilt is half the angular beam wiggle spec (Line 1.7).

- Line 6: If the compensation plate is plano-parallel and homogeneous, its translation will not generate noise. Tilt by a small angle α moves the beam sideways by $x_p = d\alpha(n-1)/n$, where d is the thickness of the plate and n is the index of refraction. α is evaluated by using $n = 1.45$ (fused silica), $d = 10\ cm$ and Line 1.8.

- Line 7:

  When the beam splitter moves along one of the beams, the beam in one arm is displaced laterally while the the optical path in the other arm is changed. The more stringent limit on beam splitter movement results from the latter effect and is the same as for the deflection mirrors.

  The shot noise equivalent beam splitter tilt should be the same as for the deflection mirrors.

- Line 8:

  Axial movement of the recycling mirror changes the phase of the light reaching the main cavities. unlike the beam splitter, however, the recycling mirror generates noise which is common mode to the two arms. Therefore, the requirement on recycling mirror translation is ten times more lenient than the corresponding one for the beam splitter (Line 2.7a,p).

  With the main cavity geometry assumed here, the recycling mirror has to be curved, therefore its transverse motion results in lateral beam displacement, which should not exceed the lateral beam displacement spec (Line 1.8).

  The shot noise equivalent recycling mirror tilt is estimated as in the case of the far test mass. A reasonable value for the curvature is $10\ m$.

- Line 9: This line shows the ground motions, as measured ar Cherryfield and Edwards. The angular motion is determined by dividing the linear motion by an appropriate correlation length. $1\ m$ was considered here, corresponding e. g. to a $1\ km/s$ speed of sound in the ground, at $1\ kHz$.

- Line 10: it is assumed that the components are on top an optical table floated on air mounts, with $1\ Hz$ resonances in all directions and also for rotation (tilt). Moreover, it is assumed that the components are suspended from 1 Hz pendulums, such that the tilt frequency is also 1 Hz. The pendulum wires are assumed to stretch 1 cm, which corresponds to a vertical frequency of 5 Hz. As a consequence, all motions are damped as f^{-4} above 1 Hz, except for vertical transverse motions, which are damped as f^{-2} between 1 and 5 Hz and as f^{-4} above 5 Hz. Wire resonances which are expected to occur at ~ 1 kHz are disregarded here, as are undamped resonances in the table and the supports. Cross coupling between various degrees of freedom has also been disregarded.

2.2 Components Before the Recycling Mirror

The shot noise equivalent motions of components between the main mode cleaner and the recycling mirror are given in Table 3.

This part of the interferometer includes two telescopes. It should be kept in mind that a telescope with magnification $\mathcal{M}$ increases lateral beam displacement $\mathcal{M}$ times and decreases beam tilt $\mathcal{M}$ times. This will be used to refer all beam motions to a point at the recycling mirror, so that they can be compared with the contents of Lines 1.15,16.

A few explanations follow.

- Lines 1,2 in Table 3 are the same as in preceding tables.

- Lines 3,4: The mode matching telescope has to magnify the beam from $w = 6$ mm[8], to $w = 18$ mm (magnification $\mathcal{M}_{mm} = 3\times$). For an aperture of F:5 and an output aperture of 15 cm, the focal length of the small (input) lens is $F_3 = 25$ cm, while the large (output) lens has $F_4 = 75$ cm.

  Beam jitter due to tilts of the lens is second order in the tilt angle and will be neglected.

  Beam pulsation due to fluctuations in lens spacing is a second order effect and will also be neglected.

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[8] required to fill well the 20 mm aperture of the optics in this section, for reasonable power density

1	f	Hz	10	50	215	1000
2	Source		Burst	Burst	Periodic	Periodic
3	Large mode matching lens	p	$5 \cdot 10^{-11}$	$2.3 \cdot 10^{-12}$	$1.5 \cdot 10^{-12}$	$3.3 \cdot 10^{-13}$
4	Small mode matching lens	p	$5 \cdot 10^{-11}$	$2,3 \cdot 10^{-12}$	$1.5 \cdot 10^{-12}$	$3.3 \cdot 10^{-13}$
5	Pick-off	α	10^{-5}	$4.4 \cdot 10^{-7}$	$3 \cdot 10^{-7}$	$6.4 \cdot 10^{-8}$
6	Faraday rotator	α	$1.1 \cdot 10^{-6}$	$4.9 \cdot 10^{-8}$	$3.3 \cdot 10^{-8}$	$7.2 \cdot 10^{-9}$
7	Polarizer	α	$5.2 \cdot 10^{-6}$	$2.4 \cdot 10^{-7}$	$1.6 \cdot 10^{-7}$	$3.4 \cdot 10^{-8}$
8	Pockels cell	α	$1.2 \cdot 10^{-6}$	$5.6 \cdot 10^{-8}$	$3.8 \cdot 10^{-8}$	$8.2 \cdot 10^{-9}$
9	Wave plate	α	$6.1 \cdot 10^{-5}$	$2.8 \cdot 10^{-6}$	$1.9 \cdot 10^{-6}$	$4 \cdot 10^{-7}$
10	Output lens	p	$2 \cdot 10^{-11}$	$9 \cdot 10^{-13}$	$6 \cdot 10^{-13}$	$1.3 \cdot 10^{-13}$
11	Input lens	p	$2 \cdot 10^{-11}$	$9 \cdot 10^{-13}$	$6 \cdot 10^{-13}$	$1.3 \cdot 10^{-13}$
12	Mode cleaner mirror	p	$2.4 \cdot 10^{-8}$	$1.1 \cdot 10^{-9}$	$7.1 \cdot 10^{-10}$	$1.6 \cdot 10^{-10}$
		α	$1.3 \cdot 10^{-9}$	$5.9 \cdot 10^{-11}$	$3.9 \cdot 10^{-12}$	$8.6 \cdot 10^{-12}$
13	Ground motion	a,p	$4 \cdot 10^{-10}$	$1.6 \cdot 10^{-11}$	$8.7 \cdot 10^{-13}$	$4 \cdot 10^{-14}$
		α	$4 \cdot 10^{-10}$	$1.6 \cdot 10^{-11}$	$8.7 \cdot 10^{-13}$	$4 \cdot 10^{-14}$
14	Residual motion	a	$4 \cdot 10^{-14}$	$2.6 \cdot 10^{-18}$	$4 \cdot 10^{-22}$	$4 \cdot 10^{-26}$
		p	10^{-12}	$6.4 \cdot 10^{-17}$	10^{-20}	10^{-24}
		α	$4 \cdot 10^{-14}$	$2.6 \cdot 10^{-18}$	$4 \cdot 10^{-22}$	$4 \cdot 10^{-24}$

Table 3: Shot noise equivalent movements of components before the recycling mirror. The shorthands in Column 3 are: p - transverse movements (m$\cdot$Hz$^{-1/2}$), α - horizontal and vertical tilt (rad$\cdot$Hz$^{-1/2}$). For more details see the text.

Angular beam wiggle induced by lateral movement of the large lens is $\alpha = x_p/F_4$. This and the wiggle spec (Line 1.15) yield the transverse vibration entry for this lens.

The angular wiggle generated by transverse movements of the small lens, seen at the recycling mirror, is the same as for the large lens.

- Line 5: The pick-offs are plano-parallel plates at 45 deg. When the pick-off tilts by a small angle α, the beam is displaced by a small amount x_p. For fused silica ($n = 1.45$), $x_p = 0.448 d\alpha$, where d is the thickness of the plate. At the recycling mirror, $x_p = 0.448 d\alpha\mathcal{M}_{mm}$. The entry for α is derived by comparing this x_p with the spec on Line 1.16, and assuming $d = 10\ mm$.

- Line 6: Small tilts of the Faraday rotator result in a beam displacement $x_p = d\alpha\mathcal{M}_{mm}(n-1)/n$ where n is the index of refraction of the Faraday glass. α is obtained by comparing x_p with the spec on Line 1.16, for $n = 1.5$ and $d = 12\ cm$.

- Line 7: The polarizer tilt entry is estimated as for the previous item, for $n = 1.5$[9] and $d = 2.5\ cm$.

- Line 8: The entry for Pockels cell tilt is estimated as above, except that incidence is at the Brewster angle. For a Pockels cell made of ADP and neglecting small polarization effects, $x_p \sim 0.35 d\alpha\mathcal{M}_{mm}$ where $d = 10\ cm$.

- Line 9: The beam displacement caused by wave plate tilt is $x_p = d\alpha\mathcal{M}_{mm}(n-1)/n$, with $n = 1.548$ (quartz) and $d = 2\ mm$.

- Lines 10,11: The beam expansion telescope increases the beam radius from $1.2\ mm$[10] to $6\ mm$, so that $\mathcal{M}_e = 5$. For a clear aperture of $2\ cm$ at F:5, the lenses have $F_1 = 2\ cm$ and $F_2 = 10\ cm$.

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[9] one possible value for for calcite polarizers

[10] corresponding to a mode cleaner with length $l_m = 12.5\ m$, made of mirrors with radii $r_m = 18\ m$

The angular beam wiggle induced by transverse motion of the output lens is $\alpha = x_p/(F_2\mathcal{M}_{mm})$, which is then compared with Line 1.15 in order to derive the the shot noise equivalent transverse motion.

The transverse motion entry for the input lens is the same as for the output lens.

- Line 12: When one of the mode cleaner mirrors moves sideways or tilts, the line connecting the centers of the mode cleaner mirrors changes direction, with a corresponding change in the pointing of the output beam.

  When a mode cleaner mirror moves by x_p in the transverse plane, the beam reaching the recycling mirror changes direction by an angle $\alpha_w = x_p/[(2r_m - l_m)\mathcal{M}_e\mathcal{M}_{mm}]$. The transverse motion entry is obtained by comparing α_w with the wiggle spec on Line 1.15.

  The shot noise equivalent mode cleaner mirror tilt α is obtained as above, except that in this case $\alpha_w = \alpha r_m/[(2r_m - l_m)\mathcal{M}_e\mathcal{M}_{mm}]$.

- Lines 13,14: same as Lines 9,10 in Table 2.