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#65

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by R. Weiss
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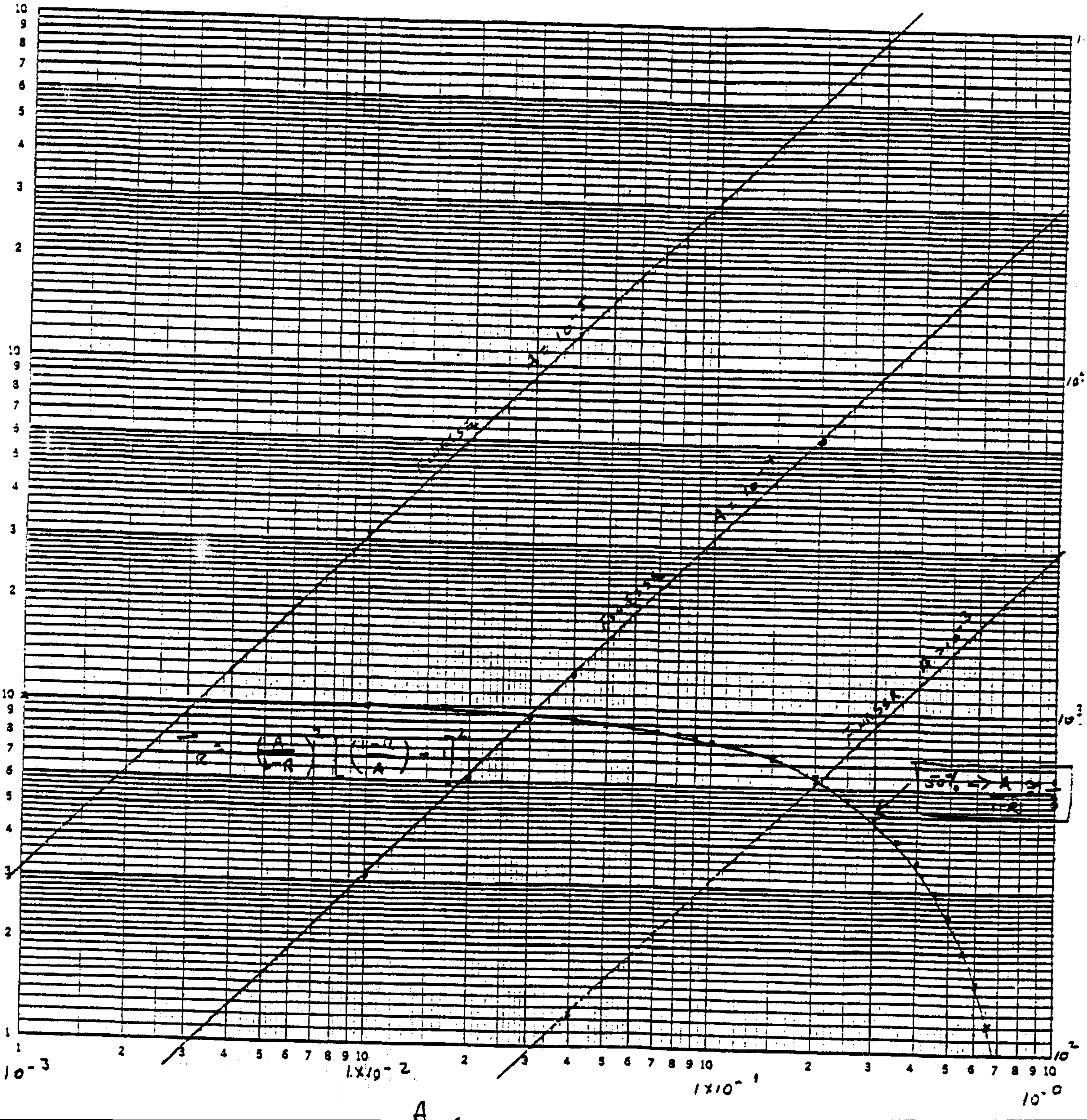
Finesse and transmission of a symmetric Fabry-Perot cavity

$R_1 = R_2 \qquad A_1 = A_2 \qquad \left({}_{R_1, A_1} \quad {}_{R_2, A_1} \right)$

$\text{Finesse} = \frac{\nu_{f_0}}{\Delta\nu(1/2\text{power})} = \frac{\pi R^{1/2}}{1-R} \qquad \nu_{f_0} = \frac{c}{2\ell} = 150 MHz/\text{meter}$



Fabry-Perot properties

Assumptions:

1) A single radial mode is excited
2) All calculations made at input which is a wavefront with $R \to \infty$; for example a plane mirror of a hemispherical cavity.

Plan of the calculation

1) Determine Green's function (useful for transient analysis later)
2) Determine response to sinusidal excitation
3) Show familiar intensity formula
4) Phase and amplitude of electric field reflection coefficient

Amplitude coefficients $z = 0$ $\quad$ $z = \ell$ $\quad$ $r = \frac{n'-n}{n'+n}$

$\rightleftarrows r_1$ $\quad$ $\rightleftarrows -r_2$

$\leftrightarrows -r_1$ $\quad$ $\to\to t_2$

$\rightrightarrows t_1$, $\leftleftarrows t_1$ $\quad$ $t = \frac{2n}{n'+n}$

<u>Green's function</u>: An impulse applied at input at time t'

$\varepsilon_{\text{inc}}(t) = \varepsilon_o \delta(t')$

produces an exponentially decaying set of pulses at both $z = \ell$ (transmitted) and $z = 0$ (reflected) pulses

$$h_T(t) = \frac{E_T(t)}{E_o} = t_1 t_2 \delta\left(t - \frac{\ell}{c}\right) + t_1 r_2 r_1 t_2 \delta\left(t - \frac{3\ell}{c}\right) + t_1 r_2 r_1 r_2 r_1 t_2 \delta\left(t - \frac{5\ell}{c}\right) + \cdots$$

$$= t_1 t_2 \sum_{n=0}^{\infty} (r_2 r_1)^n \delta\left(t - \frac{(2n+1)\ell}{c}\right)$$

$$h_R(t) = \frac{E_R(t)}{E_o} = r_1 \delta(t) - t_1 t_1 r_2 \delta\left(t - \frac{2\ell}{c}\right) - t_1 r_2 r_1 r_2 t_1 \delta\left(t - \frac{4\ell}{c}\right)$$

$$- t_1 r_2 r_1 r_2 r_1 r_2 t_1 \delta\left(t - \frac{6\ell}{c}\right) + \cdots$$

$$= r_1 \delta(t) - t_1 t_1 r_2 \sum_{n=0}^{\infty} (r_1 r_2)^n \delta\left(t - \frac{(2n+2)\ell}{c}\right)$$

The response of the system for an arbitrary incident electric field is the convolution of the Green's function with the incident field. The electric field at $z = 0$ (only the reflected part) will be the convolution of the Green's function with the incident electric field at $z = 0$

t is now, t' is the time associated with the incident field and τ is the delay time $t' = t - \tau$

$$\underset{\text{field now}}{E_R(t)} = \int_{-\infty}^{o} h_R(\tau) E_{\text{inc}}(t-\tau)\,d\tau \qquad \text{reflected field at } z = 0, t$$

The transmitted field at $z = \ell$ is then

$$\underset{\text{field now}}{E_T(t)} = \int_{-\infty}^{o} h_T(\tau) E_{\text{inc}}(t-\tau)\,d\tau \qquad \text{transmitted field at } z = \ell, t$$

The transfer function for-reflection or transmission is the response for a sinusoidal incident field

$$E_{\text{inc}}(t') = 5_o \; e^{i\omega t'}$$

$$\frac{E_r(t)}{E_o} = e^{i\omega t}\Big[r_1 - t_1 t_1 r_2 \;\; e^{-i\frac{2\omega\ell}{c}} \sum_{n=0}^{\infty} (r_1 r_2)^n \;\; e^{-i\frac{2\omega n\ell}{c}}\Big]$$

Since $|(r_1 r_2)| < 1$ the series can be summed

$$= e^{i\omega t}\Big[r_1 - \frac{t_1 t_1 r_2 \; e^{-i\frac{2\omega\ell}{c}}}{(1 - r_1 r_2 \; e^{-i\frac{2\omega\ell}{c}})}\Big] = e^{i\omega t}\Big[\frac{r_1 - r_2(r_1^2 + t_1^2)\; e^{-i\frac{2\omega\ell}{c}}}{(1 - r_1 r_2 \; e^{-i\frac{2\omega\ell}{c}})}\Big]$$

Summing series for transmitted beam gives

$$\frac{E_T(t)}{E_o} = e^{i\omega t}\Big[\frac{t_1 t_2}{1 - r_1 r_2 \; e^{-i\frac{2\omega\ell}{c}}}\Big]$$

Check for familiar intensity formulae

power coefficients $T_i = t_i t_i^*$ $R_i = r_i r_i^*$

$$\frac{I_T}{I_{\text{inc}}} = \frac{E_T^* E_T}{E_o^2} = \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\cos\frac{2\omega\ell}{c}}$$

$$\frac{I_R}{I_{\text{inc}}} = \frac{E_R^* E_R}{E_o^2} = \frac{R_1 + (R_1 + T_1)^2 R_2 - 2\sqrt{R_1 R_2}(R_1 + T_1)\cos\frac{2\omega\ell}{c}}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\cos\frac{2\omega\ell}{c}}$$

If $\begin{matrix} R_1 + T_1 = 1 \\ R_1 + T_2 = 1 \end{matrix}$ $I_T + I_R = I_{\text{inc}}$

Amplitude coefficient for reflection cavity

$$\frac{E_R}{E_{\rm inc}}(x) = \frac{r_1 - r_1(r_1^2 + t_1^2)\, e^{-ix}}{(1 - r_1 r_2\, e^{-ix})} \qquad x = 2\omega\ell/c$$

$r_1 = |r_1|\, e^{i\varphi_1}$ $\qquad r_2 = |r_2|\, e^{i\varphi_2}$ $\qquad$ π Phase reversal has been taken into account already

Approximation

Let r_1 and r_2 be real and take account of the losses in the amplitude; T_1= front mirror transmission, A_1, and A_2= mirror loss

Approximate

$$r_1 \cong (1 - T_1 - A_1)^{1/2} \qquad r_2 = (1 - A_2)^{1/2}$$

Define $\quad A^2 = 1 - A_1 - T_1 \quad B^2 = (1 - A_2)(1 - A_1)^2 \quad C^2 = (1 - A_1 - T_1)(1 - A_2)$

$$\frac{E_R}{E_{\rm inc}}(x) = \left[\frac{A^2 - 2AB\cos x + B^2}{1 - 2C\cos x + C^2}\right] e^{i \underbrace{\left[\tan^{-1}\left(\frac{B\sin x}{A - B\cos x}\right) - \tan^{-1}\left(\frac{C\sin x}{1 - C\cos x}\right)\right]}_{\varphi}}$$

$$\frac{d\varphi}{dx} = \frac{B}{A^2 + B^2 - 2B\cos x}\left[A\cos x - B\right] - \frac{C}{1 + C^2 - 2C\cos x}\left[\cos x - C\right]$$

Special but important case

$A_1 \ll 1 \quad A_2 \ll 1 \quad T_1 < 1 \quad T_2 = 0$

near resonance

φ	x	
π	0	on resonance
$\pi \pm \pi/2$	$\pm \frac{T_1 + A_1 + A_2}{2}$	1/2 power pt
$\pi \pm 3\pi/4$	$\pm T_1$	
$2\pi \pm T_{1/2}$	$\pm \pi/2$	
$2\pi, 0$	$\pm \pi$	1/2 free spectral range

$$\frac{\partial\varphi}{dx}\Big|_{x=0} = \frac{B - CA}{(A - B)(1 - C)} \simeq -\frac{4}{T_1\left(1 - \frac{(A_1 + A_2)^2}{T_1}\right)}$$

$$\frac{\partial\varphi}{\partial x}\Big|_{x=\pi/2} \cong -T_{1/2}$$

$$\left|\frac{E_R}{E_{\rm inc}}\right|^2(x = 0) = 1 - \frac{4(A_1 + A_2)}{T_1} \qquad \left|\frac{E_R}{E_{\rm inc}}\right|^2(x = \pi/2) = 1 - A_1$$

1/2 Power points $\Delta x = \pm \frac{T_1 + A_1 + A_2}{2}$ radians

$$\left|\frac{E_R}{E_{inc}}\right|^2(\Delta x) = 1 - \frac{2(A_1+A_2)}{T_1}$$

phase $\varphi(\Delta x) = \pi \pm \frac{\pi}{2}$

Frequency and position sensitivity of reflection cavity

at $x = 0$

$$\Delta x = \frac{2\Delta\omega \ell_o}{c} + \frac{2\omega_o \Delta\ell}{c}$$

Frequency sensitivity $\Delta\ell = 0$ change in optical phase reflected/change in frequency

$$\left.\frac{\Delta\varphi}{\Delta f}\right|_o = \left.\frac{\partial\varphi}{dx}\right|_o \frac{dx}{df} = -\frac{16\pi}{T_1\left(1-(\frac{A_1+A_2}{T_1})^2\right)}\frac{\ell_o}{c}$$

In terms of energy storage time defined by a source internal to the cavity

Energy stored in cavity $= u\ell_o A_c$

A_c=beam cross section

ℓ_o=length of cavity

u=energy density

Power dissipated or emitted

$$\frac{dE}{dt} = -\frac{cuA_c}{2}(T_1 + A_1 + A_2)$$

$$E(t) = E(0) \quad e^{-t/\tau_{\text{store en}}}$$

$$\tau_{\text{store en}} = \frac{2\ell_o}{c(T_1+A_1+A_2)} = \frac{2\tau_{\text{trans}}}{(T_1+A_1+A_2)}$$

Using energy storage time

$$\tau_{\text{store field}} = 2\tau_{\text{store en}} = \frac{4\tau_{\text{trans}}}{(T_1+A_1+A_2)}$$

$$\frac{1}{T_1\left(1-(\frac{A_1+A_2}{T_1})^2\right)} = \frac{1}{2}\frac{\tau_s}{\tau_t}\frac{\left(1-\frac{1}{2}\frac{\tau_s}{\tau_t}(T_1+A_1+A-2)\right)}{\left(1-\frac{\tau_s}{\tau_t}(A_1+A_2)\right)}$$

another formulation of energy storage time

$$\frac{\Delta\varphi}{\Delta f} = -8\pi\ \tau_{\text{sen}}\frac{\left(1-\frac{1}{2}\frac{\tau_s}{\tau_t}(A_1+A_2)\right)}{\left(1-\frac{\tau_s}{\tau_t}(A_1+A_2)\right)}$$

$$\tau_{\text{sten}} = \frac{\tau_t}{(1-|r_1 r_2|)}$$

$A_1 + A_2 \ll T_1$

$$\frac{\Delta\varphi}{\Delta f} \cong -8\pi\ \tau_{\text{store en}} = -4\pi\ \tau_{\text{store field}}$$

$\frac{1}{2}$ power pt full width

$$\Delta f(\frac{1}{2}) = \frac{c}{2\ell_o}\frac{(T_1 + A_1 + A_2)}{2\pi} = \Delta\nu_{fs}\frac{(T_1 + A_1 + A_2)}{2\pi} \qquad \Delta\nu_{fs} = \text{free spectral range}$$

$$= \frac{1}{2\pi\ \tau_{\text{store en}}}$$

Position sensitivity

$$\frac{\Delta\varphi}{\Delta\ell} \cong \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\ell} = \frac{-16\pi f_o/c}{T_1\left(1-\left(\frac{A_1+A_2}{T_1}\right)^2\right)} = \frac{-16\pi}{\lambda_o T_1\left(1-\left(\frac{A_1+A_2}{T_1}\right)^2\right)}$$

In terms of the energy storage time

$$\frac{\Delta\varphi}{\Delta\ell} \cong \frac{-8\pi\tau_{\text{store en}}}{\lambda_o\tau_{\text{transit}}} \; \frac{\left(1-\frac{1}{2}\frac{\tau_s}{\tau_t}(A_1+A_2)\right)}{\left(1-\tau_s/\tau_t(A_1+A_2)\right)}$$

$A_1 + A_2 \ll T_1$

$$\frac{\Delta\varphi}{\Delta\ell} \cong \frac{-8\pi}{\lambda_o}\frac{\tau_{\text{store en}}}{\tau_{\text{transit}}} = -\frac{4\pi}{\lambda_o}\frac{\tau_{\text{store field}}}{\tau_{\text{transit}}}$$

Note: Maximum bandwidth for frequency or length control loops is determined by phase (φ) response going to $\varphi = \pi \pm \pi$. This corresponds to a frequency width of 1/2 $\Delta\nu_{fs}$.

$\Delta BW_{\text{max}} = \frac{c}{4\ell_o}$

For LIGO with 4km arms

$\Delta BW_{\text{max}} = 18.8KHz$

Further properties of cavities

DC transfer function of a Fabry-Perot cavity to a length displacement

Again assume input mirror has R_1 T_1 A_1

Back mirror has A_2 $R_2 = 1 - A_2$

$E_{\text{in}} = E_o\ e^{i\omega t} \longrightarrow$

$E_{\text{ref}} = E_o A(\Delta k)\ e^{i\delta(x)}\ (\longleftarrow \ell \longrightarrow)$

Phase change in reflection due to a change of length at resonance

$$x = 2\omega\ell/c$$

$$\left.\frac{d\varphi}{dx}\right|_{x=n2\pi} = -\frac{4}{T_1\left(1-(\frac{A_1+A_2}{T_1})^2\right)}$$

$$A^2(o) = 1 - \frac{4(A_1+A_2)}{T_1}$$

For a length change $\Delta\ell$

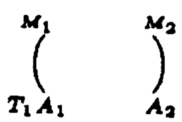
$$\Delta\varphi = \frac{d\varphi}{dx}\ \frac{dx}{d\ell}\Delta\ell = -\frac{8\omega}{cT_1\left(1-(\frac{A_1+A_2}{T_1})^2\right)} = -\frac{16\pi}{T_1\left(1-(\frac{A_1+A_2}{T_1})^2\right)}\ \frac{\Delta\ell}{\lambda}$$

$h = 2\frac{\Delta\ell}{\ell}$ strain to gravity wave amplitude

$\frac{\Delta\varphi}{h} = -\frac{8\pi}{T_1\left(1-(\frac{A_1+A_2}{T_1})^2\right)}\ \frac{\ell}{\lambda}$ for one cavity

Try to define storage time as when intensity drops by $1/e$

Begin with wave inside cavity



$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) \qquad u = \frac{1}{8\pi}(E^2 + B^2)$$
$$= \frac{c}{4\pi}E^2 \qquad\qquad = \frac{1}{4\pi}E^2$$

The energy in the cavity is made up of a standing wave with 1/2 the energy traveling $R \rightarrow L$ and the other $L \rightarrow R$

$$U_{\text{total}} = u_+ A\ell + u_- A\ell \qquad u_+ = u = u_-$$
$$U_{total} = 2uA\ell$$

flux out and into mirror 1 $S_+ A = v u A(T_1 + A_1)$

flux into mirror 2 $S_- A = cuA(A_2)$

$$\left.\frac{dU}{dt}\right|_{\substack{\text{inside}\\\text{cavity}}} = -(S_+A + S_-A) = -cuA[T_1 + A_1 + A_2] = -\frac{cU_{\text{total}}}{2A\ell}A[T_1 + A_1 + A_2]$$

$$\frac{d\cup}{\cup_{\text{total}}} = -\frac{c}{2\ell}(T_1 + A_1 + A_2)dt$$

$$\cup(t) = \cup_o \, d^{-\frac{c(t_1+A_1+A_2)}{2\ell}t} \qquad \tau_{ST_{\cup}} = \frac{2\ell}{c(T_1 + A_1 + A_2)} = \frac{\tau_t}{(1 - |r_1 r_2|)}$$

$$E^2(t) = E_o^2 \, e^{-\frac{c(T_1+A_1+A_2)}{2\ell}t}$$

$$E(t) = E_o \, e^{-\frac{c(T_1+A_1+A_2)}{4\ell}t} \qquad \tau_{ST_{\text{field}}} = \frac{4\ell}{c(T_1 + A_1 + A_2)}$$

With this definition of storage time

energy storage time $\qquad\qquad$ E field storage time

$$\frac{\Delta\varphi}{h} = -\frac{4\pi\tau_{S_{ENC}}c}{\lambda}\frac{\left(1-\frac{1}{2}\frac{\tau_s}{\tau_t}(A_1+A_2)\right)}{\left(1-\tau_s/\tau_t(A_1+A_2)\right)} = -\frac{2\pi\tau_{S_{\text{field}}}c}{\lambda}\,\frac{\left(1-\frac{1}{4}\frac{\tau_{SE}}{\tau_t}(A_1+A_2)\right)}{\left(1-\frac{1}{2}\frac{\tau_{SE}}{\tau_t}(A_1+A_2)\right)}$$

one cavity
h→DC

OPTICAL INHOMOGENEITY

Optical loss when averaged over wavefront

$A_{\frac{L}{wo}} = \left(1- < J_o(\Gamma_{\frac{rms}{beam}}) > \right) = \frac{\Gamma^2_{rms}}{4}$ for small phase shift

In terms of wavefront distortion in cm required to correct

φ_c contour of constant phase

average φ line

$$\Gamma_{rms} = \left(\frac{\int \delta x^2(r,\theta) A_{\text{mode dimension}} da}{\int A_{\text{mode dimension}} da} \right)^{1/2}$$

We do not know the distribution $\delta x^2(r,\theta)$. Optics industry gives average Δx over ill defined beam size.

Nevertheless using the average values gives some indication. Usually the specification is given in

$$\Delta x \Big/ \ell_{\text{path}} = Q$$

$A_L = \pi^2 \left(\frac{Q\ell_{\text{path}}}{\lambda}\right)^2$

Another way to specify is in teams of wave front distortion in units of the wavelength

$$\Delta x = f\lambda$$

$A_L = \pi^2(f^2)$

A_L	f	Q mm/cm $\ell = 10cm$ $\lambda = 5145A^\circ$	Δn
.1	1/10	$5nm/cm$	5×10^{-7}
.025	1/20	2.5	
.011	1/30	1.7	
.0062	1/40	1.25	
.0040	1/50	1	1×10^{-7}