New Folder Name Internal Scattering

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Internal Scattering in Fabry-Perot Cavities

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Interferometers are the most promising sensors for large gravity wave detectors. The coupling of the enclusure noise to the gravity wave signal in a Fabry-Perot interferometer is studied. The first part provides a theoretical framework, and in the second part two limiting cases are discussed as an illustration. The relevance of this noise source for large gravity wave interferometers is briefly discussed in the conclusion.

1. Introduction

Fabry-Perot interferometers are currently used in gravity wave detectors as extremely high quality resonators monitoring the length of interferometer arms. The fundamental limits to the precision of these measurements come from the photon counting noise (photon shot noise), from the

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thermal noise displacing the mirror surfaces and from the quantum noise (Heisenberg uncertainty). These limitations are well understood (Drever *et al.*1983) and the existing gravity wave detectors are designed so that this noise will be below the sensitivity threshold. However, the development of present gravity wave detectors with sensitivity of the order 10^{-18} $m/\sqrt{Hz}$ has witnessed a constant struggle with non-fundamental noise sources, i.e. noise sources which could in principle be eliminated and are present only due to one's inability to produce ideal components. In this article I will discuss one of the many "non-fundamental" noise sources – the scattering of light off the vacuum tank walls. It comes about as follows: the two mirrors in the interferometer are not perfect, so that some light is scattered off their surface and hits the walls of the vacuum tank. This light may again find its way into the interfering path if it either escapes through the other mirror and hits the photodiode or if it is scattered again into the main beam. These effects add to the interferometer noise only if the vacuum tank walls are moving and modulate the phase of the scattered light.

In the next paragraph I describe a model for the Fabry-Perot cavity. In paragraph 3 a perturbation theory, which includes the effect of moving walls is developed. In paragraph 4 the noise coupling to the interferometer signal is described. Finally in 5 and 6 these ideas are illustrated in two examples which are representative of two extreme situations - mirror walls and rough walls. The two examples were chosen for their mathematical simplicity, but they are considered to be important as representative of the strongest and weakest noise coupling, respectively.

2. The model

A Fabry-Perot interferometer is considered as an electromagnetic cavity enclosed by two end mirrors and the walls of a vacuum tank, which is a long cylinder with its axis coinciding with the optical axis common to the two mirrors. The two mirrors are the main components of the interferometer, since they provide boundary conditions for a Gaussian beam to form between them. The intensity in the Gaussian beam decays exponentially with the distance from the symmetry axis. This makes the coupling of the beam to the vacuum tank wall decay exponentially with the radius of the vacuum pipe. In this way one usually justifies the neglect of the vacuum tank boundary in solving the field equations inside the cavity. However, mirrors always scatter a small fraction of incident light. I consider the effect of this stray light by way of a perturbation analysis which starts with a perfect cavity enclosed by smooth nonabsorbing, (almost) nontransmitting mirrors and a nonabsorbing but possibly rough wall. I also assume that for the purpose of present estimates it suffices to describe the EM field inside the interferometer cavity by a scalar field Ψ obeying the wave equation:

$$\Delta\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \tag{1}$$

Since the perfect cavity is chosen to have nonabsorbing and nontransmitting mirrors, the field Ψ obeys the boundary condition:

$$\Psi|_{boundary} = 0 \tag{1a}$$

Here boundary includes the mirror surfaces and the walls of the vacuum tank. I will denote the stationary state solutions to this equation as:

$$\Psi = \bar{\phi}_\sigma(\vec{r})e^{i\omega_0 t} \tag{2}$$

where $\bar{\phi}_\sigma$ are eigenfunctions of:

$$\Delta\bar{\phi}_\sigma + \bar{k}_\sigma^2\bar{\phi}_\sigma = 0 \tag{3}$$

The spatial dependence of $\bar{\phi}_\sigma(\vec{r})$ is, of course, a function of the geometry of the cavity. I assume that the mirrors are spherical with appropriate curvature, so that the modes $\bar{\phi}_\sigma$ are Gaussian modes. The boundary condition $\Psi = 0$ on the wall is not very restricting for the main modes, since they decay exponentially away from the symmetry axis. Therefore, I expect that it is possible to start a meaningful perturbation analysis with the "perfect" cavity as defined above.

A more realistic model of the Fabry-Perot cavity must include:

a) the nonzero transmissivity of the mirrors;

b) scattering off the mirrors;

c) scattering off the moving walls.

The nonzero transmissivity of the mirrors can be modeled by adding an imaginary part to the eigenfrequencies $[\omega_\sigma \to \omega_\sigma + \frac{i}{\tau}]$, and the resulting losses are compensated with a weak driving field, which adds a small inhomogeneous term to the boundary conditions. Note that absorption in the mirrors has the same effect at this level, so that it need not be discussed.

I assume that the scattering off the mirrors occurs due to microroughness. In this model the actual cavity differs from the ideal cavity only through a slight perturbation of the boundary where the field Ψ vanishes. The new eigenfunctions and eigenvalues of eq. (1) governing the field Ψ inside the cavity are computed via a perturbation analysis developed in the next paragraph.

The scattering off the walls is time dependent due to vibrations produced by external acoustical noise. But note that the field inside the cavity with moving walls is again governed by eq. (1) and, again, only the boundary is perturbed as a function of time. The perturbation theory of the next paragraph is applicable for this case also.

3. The perturbation theory

In this chapter I develop a perturbation theory for the following problem: if the solutions $(\bar{\phi}_\sigma)$ of equation (1) with boundary conditions (1a) on the boundary $\partial\Sigma_0$ are known, find the solutions of eq. (1) with boundary conditions (1a) on a perturbed boundary $\partial\Sigma$ (fig. 1). A convenient way to do this is to introduce two systems of coordinates – one for the unperturbed problem and one for the perturbed problem – so that the values of the coordinates on the boundary are the same for both problems. In this way the perturbation leaves the boundary conditions unchanged, and only the form of the Laplacian operator changes. I start the unperturbed problem in cartesian coordinates (x, y, z). The perturbed coordinates (ξ, η, ς) are introduced so that the cartesian coordinates (x, y, z) of points inside the perturbed cavity are:

$$\begin{aligned} x &= \xi + U_1(\xi, \eta, \varsigma) \\ y &= \eta + U_2(\xi, \eta, \varsigma) \\ z &= \varsigma + U_3(\xi, \eta, \varsigma) \,. \end{aligned} \tag{4}$$

If the point (ξ,η,ς) belongs to the boundary, then the vector $\vec{U}(\xi,\eta,\varsigma)$ is the displacement of the boundary (mirrors or walls) with respect to the perfect cavity. Note that apart from this restriction the choice of $\vec{U}(\vec{U}\equiv\{U_1,U_2,U_3\})$ is still quite free. However, in order to make the calculations easy, I will choose a particular gauge. The metric to first order with respect to the new coordinates $(x^1=\xi, x^2=\eta, x^3=\varsigma)$ is:

$$g_{ij}=\delta_{ij}+U_{i,j}+U_{j,i} \qquad (i,j=1,2,3) \tag{5}$$

Here an index after a comma means differentiation with respect to the coordinate with the given index (for example, $U_{3,2}=\partial U_3/\partial\eta$). The Laplacian Δ in the new coordinates is (to first order also):

$$\Delta\Psi=\nabla^2\Psi-\nabla^2\vec{U}\cdot\nabla\Psi-2U_{i,j}\Psi_{,ij} \tag{6}$$

with the notation:

$$\nabla=\{\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta},\frac{\partial}{\partial\varsigma}\} \tag{7}$$

$$\nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\varsigma^2}$$

Summation over repeated indices is implied; scalar product is indicated by the dot.

The operator Δ is now obviously the sum of the old unperturbed term (∇^2) and the perturbation $\delta\Delta$, which depends on the perturbing field $\vec{U}$. The unperturbed Laplacian is certainly hermitian. It is desirable that the new Laplacian be split in such a way that both parts ∇^2 and $\delta\Delta$ are also hermitian to first order. This requirement will be met only if the divergence of the field $\vec{U}$ is a constant, as can be seen, if one remembers that the scalar product in the Hilbert space spanned by Δ is:

$$(\Phi,\Psi)=\int\Phi\Psi\sqrt{g}d^3x \quad , \tag{8}$$

where $g=1+2\nabla\cdot\vec{U}$ is the determinant of the metric tensor g_{ij}. Finally, we can write the vector field $\vec{U}$ as a sum of a gradient and a curl $(\vec{U}=\nabla\Psi+curl\vec{A})$. If we limit ourselves to perturbed surfaces $\delta\Sigma$ such that the normal to any point in $\delta\Sigma_0$ pierces only one point in $\delta\Sigma$, then we can choose $\vec{A}=0$ (fig. 1) and $\vec{U} = \nabla\Psi$. The hermiticity of the operator ∇^2 requires:

$$\nabla^2\Phi = const \tag{9}$$

and the boundary condition is that $(\partial\Phi/\partial n)\vec{n}_0$ at P_0 in $\partial\Sigma_0$ brings us to a point P in $\partial\Sigma$; here $\partial/\partial n$ means the derivative with respect to the boundary surface and $\vec{n}_0$ is the normal to the unperturbed boundary at the point P_0 (fig. 1). Note that the coordinate gauge in $\partial\Sigma$ is fixed with respect to the coordinates x, y, z and depends only on the geometry of $\partial\Sigma$.

In the system of coordinates just defined, the perturbation $\delta\Delta$ has the simple form:

$$\delta\Delta(\Psi) = -2\Phi_{,ij}\Psi_{,ij}. \tag{10}$$

It is particularly appealing to note that the matrix elements of this operator reduce to surface integrals over the boundary of the cavity, and are:

$$(\bar{\phi}_\alpha, \delta\Delta\bar{\phi}_\beta) = (\bar{\phi}_\beta, \delta\Delta\bar{\phi}_\alpha) =$$

$$= \frac{1}{4}\int(\nabla\Phi\cdot\nabla\bar{\phi}_\alpha)(\nabla\bar{\phi}_\beta\cdot d\vec{S}) -$$

$$\frac{1}{4}\int(\nabla\Phi\cdot\nabla\bar{\phi}_\beta)(\nabla\bar{\phi}_\alpha\cdot d\vec{S}) \tag{11}$$

Scattering off the mirrors is modelled as a stationary process due to perturbations in mirror surface. The main excited mode (ϕ_0) in the cavity can, therefore, be expressed as a linear combination of ideal cavity modes:

$$\phi_0 = \bar{\phi}_0 + \sum a_\lambda^{(o)}\bar{\phi}_\lambda \tag{12}$$

Assuming, for simplicity, a nondegenerate spectrum of $\bar{k}_\alpha$, the expansion coefficients $a_\lambda^{(0)}$ are computed according to standard perturbation techniques:

$$a_\lambda^{(\alpha)} = \frac{[\bar{\phi}_\lambda, \delta\Delta\bar{\phi}_\alpha]}{\bar{k}_\lambda^2 - \bar{k}_\alpha^2} \quad , \tag{13a}$$

and the corrected eigenvalues (k_α^2) are:

$$k_\alpha^2 = \bar{k}_\alpha^2 + (\phi_\alpha, \delta\Delta\phi_\alpha)\,. \tag{13b}$$

Of course, the perturbation of the mirror $\delta\Delta$ is usually not known in practice. In fact, one usually infers the coefficients a_λ from the distribution of scattered light. For example Thorne (1987) used the following probability of scattering the light into a solid angle $d\Omega$ pointing in the direction Ω subtending an angle θ ($\theta \ll 1$ and Ω is not far from the normal to the mirror) with the specularly reflected light:

$$P(\theta)d\Omega \approx \frac{\beta}{\theta^2}d\Omega \tag{14}$$

The measurements of Elson and Bennett (1979) if extrapolated to small angles are in reasonable agreement with this expression, and the coefficient β derived from these measurements is for superpolished mirrors about $1.5 \cdot 10^{-6}$.

Finally, we must include the perturbation due to moving vacuum tank walls. Let $\partial\Delta(t)$ be the time dependent perturbation of the Laplacian due to moving walls. Then, according to standard perturbation theory, the field Ψ inside the cavity can be expanded in a time dependent series of stationary eigenfunctions ϕ_α (not $\bar{\phi}_\alpha$!) as follows:

$$\Psi(\vec{r},t) = c_0[\phi_0(\vec{r}) + \sum c_\sigma(t)\phi_\sigma(\vec{r})]e^{i\omega_0 t} \ , \tag{15}$$

and the coefficients c_σ obey the equation (the superscript (0) referring to the zero order excited state quantum numbers will be omitted henceforth):

$$\frac{d^2}{dt^2}[c_\sigma e^{i\omega_0 t}] + \omega_\sigma^2[c_\sigma e^{i\omega_0 t}] = -2c^2 e^{i\omega_0 t}(\phi_\sigma, \partial\Delta(t)\phi_0) \tag{16}$$

This equation is reminiscent of a forced harmonic oscillator – the field in the main excited mode ($\Psi^{(0)} = \phi_0 e^{i\omega_0 t}$) is coupled to other modes via the force term on the right hand side of eq. (16). With this interpretation in mind it seems reasonable to model the finite lifetime of modes inside the cavity by adding a damping term to the above equation.

There is another interesting interpretation of equation (16). Let us denote:

$$\varsigma_\sigma(t) = c_\sigma(t)e^{i\omega_0 t} \tag{17}$$

and according to (15) the field Ψ is a sum of the unperturbed part Ψ_0 and the perturbation:

$$\Psi(\vec{r},t) = \Psi_0(\vec{r},t) + \psi(\vec{r},t) = \Psi_0(\vec{r},t) + \sum_\sigma \varsigma_\sigma(t)\phi_\sigma(\vec{r}) \tag{18}$$

Multiply (16) by ϕ_σ, sum over σ, take into account (3), and one obtains:

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \nabla^2 \psi = -2c^2 \sum \phi_\sigma(\vec{r})[\phi_\sigma(\vec{r}'), \partial\Delta(\vec{r}', t)\Psi_0(\vec{r}', t)]$$

Finally invoke the completeness of eigenfunctions ϕ_σ, and one realizes that ψ is the solution of the forced wave equations with the source spread over the surface of the walls as follows:

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \nabla^2 \psi = \frac{1}{2}c^2 \{ \int [(\vec{U} \cdot \nabla' \delta^3(\vec{r} - \vec{r}'))][\nabla' \Psi_0 \cdot d\vec{S}'] +$$

$$\int [\vec{U} \cdot \nabla' \Psi_0][\nabla' \delta^3(\vec{r} - \vec{r}') \cdot d\vec{S}'] \} \tag{19}$$

This result is similar to the current source model discussed by Kröger and Kretschmann (1970), which was used to describe the scattering off rough surfaces. Note, however, that our mathematical approach is closer to the calculations of Elson (1975).

4. The perturbations and the interferometer noise

Our final task is to estimate the noise measured by the photodetector in the interferometer that is produced by noise in $c_\sigma(t)$, which is in turn generated by the motion of side walls of the interferometer. In order to understand how this noise is detected by the interferometer, let us first consider a noise-free distance measurement. In this case only the fundamental mode $\Psi_{(0)}$ is excited by a laser which has a frequency $\tilde{\omega}_0$ very close to the eigenfrequency of the cavity ω_0. In order to describe the excitation of this mode, write $\Psi^{(0)}$ in the spirit of eq. (15) as $c_0(t)e^{i\omega_0 t}\phi_0(\vec{r})$. One expects that $c_0(t)$ obeys a slightly modified equation (16) as follows:

$$\frac{d^2}{dt^2}[c_o e^{i\omega_o t}] + \frac{2}{\tau}\frac{d}{dt}[c_o e^{i\omega_o t}] + \omega_o^2[c_o e^{i\omega_o t}] =$$

$$= i\frac{\alpha}{2}c^2 \int_{mirror} (\vec{n} \cdot \nabla\phi_{laser} e^{i\tilde{\omega}_o t})(\vec{n}.\nabla\phi_o)dS \tag{20}$$

I have guessed the right hand side in analogy with (11). Here $\vec{n}$ means the unit normal to the mirror, $\vec{n}.\nabla\phi_{laser}e^{i\tilde{\omega}_o t}$ is the amplitude of the field from the laser on the entrance mirror of the cavity, and α is a constant depending on the efficiency of coupling between the input beam and cavity mode.

The photodiode in the interferometer detects the interference between the laser light and the light leaking from the cavity. The phase modulation is so arranged that (in the vicinity of the resonance) the demodulated signal is proportional to the phase difference between the laser light and the cavity light, i.e. the phase measurement may be described by the following expression:

$$\Delta\phi = \arg\left[\int_{mirror} e^{i\omega_o t}\frac{\partial\Phi_{laser}}{\partial n}\frac{\partial\Psi^*}{\partial n}dS\right] \tag{21}$$

If there is no noise, then $\Psi = \Psi_o$ and if Φ_{laser} and Ψ_o are mode matched on the mirror, one obtains using (20):

$$\Delta\phi = tg^{-1}\left[\frac{\tilde{\omega}_o - \omega_o}{\tilde{\omega}_o}\tilde{\omega}_o\tau\right]$$

$$\tilde{\omega}_o\tau\left[1-\frac{\omega_o}{\tilde{\omega}_o}\right] \tag{22}$$

$$= \tilde{\omega}_o\tau\left[\frac{\delta L}{L}\right] = F\frac{\delta L}{\lambda}$$

The last interprets the detuning of the cavity from the laser as a change in length (δL), and the finesse of the cavity has been introduced in the usual way $F = 2\pi c\tau/L$.

The presence of noise in the field inside the cavity ($\psi \neq 0$) produces an additional phase shift $\Delta\phi_{noise}$ (eqs. 21 and 18):

$$\Delta\phi_{noise} = Im\frac{\int_{mirror}(\frac{\partial\psi^*}{\partial n})(\frac{\partial\Psi_o}{\partial n})dS}{\int_{mirror}|\frac{\partial\Psi_o}{\partial n}|^2 dS} \tag{23}$$

The fact that ψ is much smaller than Ψ_o has been taken into account. The phase noise ($\Delta\phi_{noise}$) is thus directly proportional to noise in the field amplitude inside the cavity.

To first order the noise in the field amplitude can be computed from eq. 19 or its integral version:

$$\psi(\vec{r}) = \int_{wall} \vec{U}(\vec{r}_s, t) \cdot \nabla\Psi_0(\vec{r}_s, t) \nabla G(\vec{r}_s, \vec{r}) \cdot d\vec{S} \qquad (19a)$$

I have assumed that $\vec{U}(\vec{r}_s, t)$ has a sinusoidal time dependence. Eq. (19a) was obtained from eq. (19) using Green's theorem, where the Green's function $G(\vec{r}, \vec{r}_s)$ for the operator in eq. (3) has been used.

The contribution due to (19) or (19a) can be thought of as follows: The light scattered off the noisy wall is frequency modulated, and if it joins the light on the photodiode it produces a phase shift which is proportional to the ratio of the amplitude in the modulated scattered beam to the amplitude in the main beam. I call this the direct contribution.

But the motion of the vacuum tank walls may also inject noise in the interferometer through the modulation of its resonant frequencies. This effect is second order in the coupling to the scattering coefficient (β); however it may, under certain circumstances, dominate the noise coupling.

The discussion so far indicates that the noise coupling coefficients can, in principle, be calculated once the precise scattering properties of mirrors and walls are given. However, a realistic calculation for any wall geometry may be quite a difficult task. Therefore, it is of some interest to study simpler limiting cases and use the insight gained as a guide in more difficult problems, as for example the baffle geometry discussed by Thorne (1987) and Drever (private communication).

In this paper I study two limiting cases. In the first example I consider a Fabry-Perot cavity enclosed in a long square prism with mirror walls. There are two good reason for choosing this example: the first is that mirror walls are expected to produce the strongest coupling between the motion of the walls and the main cavity mode, so that this will be an example from the class of worst cases. The other reason for choosing the particular prism geometry is practical - in this geometry I know how to write the complete Green's function, so that the calculation is well controlled and gives a sound physical insight into coupling mechanisms. The other example was chosen from the class of best cases. Here I choose a cylindrical wall enclosure with rough random walls. Since a rough wall scatters light in all directions and scrambles the phases, one expects the coupling to the main mode to be small, i.e. the high dispersion of light on a rough wall makes it unlikely for the scattered light to be concentrated on mirrors and the scrambled phases make the overlap integral (eq. 23) small. (One can imagine situations where the overlap integral would cancel exactly, but it is also rather clear that such a situation would be highly unstable - only a small variation of geometry might change perfect cancelation into constructive interference.) Although mirror walls are not desirable in practice and random walls (to sufficient degree of randomnes) might be difficult to make, I believe, that the two examples give a clear enough picture of the physics involved in the coupling of wall motion to the interferometer noise. These calculations also give reasonable upper and lower limits for couplings attainable.

5. The first example - mirror walls

Let the vacuum tank walls be four very long flat mirrors combined into a parallelepiped with a square cross section - the side of the square is a. The interferometer mirrors are a distance L appart on the axis of the parallelepiped which is along the z coordinate axis. The other two sides of the parallelepiped are along the x and y axis respectively (fig. 2). The Green's functions inside such an (infinite) duct can be written using the method of images as:

$$G(\vec{r},\vec{r}') = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (-1)^{p+q} \frac{e^{ik|P_x^p P_y^q \vec{r} + (\hat{m}p+\hat{n}q)a - \vec{r}'|}}{|P_x^p P_y^q \vec{r} + (\hat{m}p + \hat{n}q)a - \vec{r}'|} \qquad (24)$$

where P_x and P_y are reflection operators; $P_x\vec{r}$ mirrors the vector $\vec{r}$ in the $x = 0$ and similarly P_y mirrors in the $y = 0$ plane. Clearly $P_x^2\vec{r} = P_y^2\vec{r} = \vec{r}$. The unit vectors $\hat{m}$ and $\hat{n}$ are normals to the $y - z$ and $x - z$ planes respectively. It is easy to see that this Green's function has the proper symmetry $G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$ and vanishes on the walls at $x = \pm a/2$ and at $y = \pm a/2$.

In order to determine Ψ_0 at the wall, I first express the field (Ψ_{00}) scattered off interferometer mirrors in the absence of walls. According to

(14) it is of the form (for $r \gg \frac{1}{k}$):

$$\Psi_{00} = \sqrt{P_0 \beta} f(\frac{\vec{r}}{r}) \frac{e^{ikr}}{r} \tag{25}$$

In order to be able to complete the example, I take (14) as a model for $f(\frac{r}{r})$. Thus, denoting the angle between the vector $\frac{r}{r}$ and the normal to the mirror by θ, I write the scattering function as:

$$f(\theta) = \frac{1}{\theta} e^{i\epsilon} \tag{26}$$

where the phase ε is a random function of the direction $(\frac{r}{r})$ with coherence range in solid angle $< \Delta\Omega > \approx \frac{\lambda}{L}$ (Thorne 1989). In the above formulas I have chosen the coefficient c_0 (cf. eq. (15)) so that:

$$\frac{1}{k^2} \int_{mirror} | \frac{\partial \Psi_0}{\partial n} |^2 \, dS = P_0 \tag{27}$$

where P_0 is the circulating power in the interferometer.

One can write the scattered field inside the enclosing mirror walls if one assumes that the enclosure does not change the source on the mirror producing scattered light. In this case mirrors simply produce mirror images of the source. If the interferometer mirrors are on the axis of the prismatic enclosure, we thus obtain:

$$\Psi_{0_{scatt}}(\vec{r}) = \sqrt{P_0 \beta} \sum (-1)^{p+q} f(\theta_{pq}) \frac{e^{ikr_{pq}}}{r_{pq}} + \tag{28}$$

$$+ \; a \; similar \; term \; for \; right \; mirror$$

Here $\vec{r}_{pq} = P_x^p P_y^q \vec{r} + a(\hat{m}p + \hat{n}q)$ and θ_{pq} is the angle between $\vec{r}_{pq}$ and the z axis, i.e: $cos(\theta_{pq}) = z/r_{pq}$. The contribution from the right mirror has the same form as the one written out, except that $\vec{r}_{pq}$ should be replaced by $\hat{k}L - \vec{r}_{pq}$.

Using (19a), we can write down the expression for the perturbed field $\psi(\vec{r})$. It is quite messy, since every image of the mirror is Doppler modulated on the wall and is again imaged by the walls. For simplicity I take that only the right wall ($x = \frac{a}{2}$) of the enclosure is moving with the displacement $U(y,z)$. The result of inserting (24) and (28) into (19a) is:

$$\psi(\vec{r}) = (ka)^2 \sqrt{P_0 \beta} \sum_{q,n} \sum_{q',n'} (-1)^{q+q'} (1+4n)(1+4n' - 2\frac{x'}{a}) \times$$

$$\int dydz f(\theta_{2n,q}) \frac{e^{ik(|r_{2n',q'} - r'| + r_{2n,q})}}{r^2_{2n,q} \mid \vec{r}_{2n',q'} - \vec{r} \mid^2} U(y,z) \qquad (29)$$

Deducing the above expression I was using the fact that if $\vec{r}$ is in the plane $x = a/2$, then $\vec{r}_{2n+1,q} = \vec{r}_{2n,q}$.

The laser field is supposed to be mode matched to the TEM_{00} mode, so that on the mirror it is described by:

$$\frac{\partial \Psi_0(\vec{r})}{\partial n'} = ik\sqrt{\frac{2P_0}{\pi}} \frac{1}{w} e^{-\frac{x'^2+y'^2}{w^2}} \qquad (30)$$

Here w denotes the radius of the main beam at the exit pupil:

$$w^2 = \frac{\lambda L}{\pi\sqrt{1-g^2}} \qquad (31)$$

Here $g = 1 - \frac{L}{R}$ and R is the curvature radius of the two mirrors (they are taken to have equal curvature) of the interferometer cavity (Rudiger 1981).

Finally we may write down the projection of the perturbed field on the laser field as required by (23) using (29) and (30). The expression involves an integration over the wall surface as well as an integration over the mirror surface. It is convenient to integrate first over the mirror, since the size of the main beam (w) is the smaller linear dimension in the problem and one can neglect $\vec{r}$ in the denominator of (29) with respect to $\vec{r}_{2n',q'}$. Thus the integration over the mirror surface involves only the integral:

$$J_{2n',q'} = \int \frac{\partial \Psi_0(\vec{r})}{\partial n'} e^{ik|r_{2n',q'} - r'|} (1 + 4n' - 2\frac{x'}{a}) dx'dy' \qquad (32)$$

$$= \sqrt{4\pi P_0}\, w e^{ik|r_{2n',q'} - \hat{k}L|} e^{-(kw \frac{r_{2n',q'} - \hat{k}z}{|r_{2n',q'} - \hat{k}L|})^2} \times$$

$$\times(1+4n')[1-\frac{i}{2}\frac{w^2 k}{|\vec{r}_{2n',q'}-\hat{k}L|}]$$

The normalized projection of the perturbed field on the laser field finally becomes:

$$\frac{\int \frac{\partial\Psi_0(r')}{\partial n'}\frac{\partial\psi(r')}{\partial n'}dS'}{\int |\frac{\partial\Psi_0}{\partial n'}|^2 dS'} = (ka)^2\sqrt{2\pi\beta}w\sum_{q,n}\sum_{q',n'}(-1)^{q+q'}(1+4n)(1+4n')\times$$

$$\times\int dydzf(\theta_{2n,q})[1-\frac{i}{2}\frac{w^2 k}{|\vec{r}_{2n',q'}-\hat{k}L|}]U(y,z)\times$$

$$\frac{e^{ik(r_{2n,q}+|\vec{r}_{2n',q'}-\hat{k}L|)}e^{-\frac{(kw)^2}{4}(\frac{r_{2n',q'}-kz}{r_{2n',q'}-kL})^2}}{r^2_{2n,q}|\vec{r}_{2n',q'}-\hat{k}L|^2} \tag{33}$$

We note that the fastest varying factor in the above integral is the exponential $e^{ik(r_{2n,q}+|\vec{r}_{2n',q'}-\hat{k}L|)}$. It belongs to the product of two propagators. The first is from the center of the first interferometer mirror to the $2n,q^{th}$ image of the point $\vec{r}$ on the wall ($\vec{r}=\frac{a}{2}\hat{m}+y\hat{n}+z\hat{k}$), and the other is from the center of the second mirror to the $2n',q'^{th}$ image of the same point. Only the region of y,z where the phase of this exponential is stationary, contributes to the integral. The size of the stationary region is clearly proportional to the wavelength of light. In the limit $\lambda\to 0$ the stationary region shrinks to a point and the path $r_{2n,q},|\vec{r}_{2n',q'}-\hat{k}L|$ goes to a geometric optics trajectory from the center of one mirror to the center of the other. In fig. (2) a typical geometric optics trajectory and it's description in terms of the vectors $\vec{r}_{2n,q}$ and $\hat{k}L-\vec{r}_{2n',q'}$ is shown. It is clear, that the total path can be described as a propagation along a straight line from the left interferometer mirror to any image (say p,q^{th}) of the right interferometer mirror. Thus the stationary path can be described by a vector $\vec{R}=(pa,qa,L)$ and both vectors $\vec{r}_{2n,q}$ and any image under reflection P_x of $\hat{k}L-\vec{r}_{2n',q'}$ must be parallel to $\vec{R}$. Expanding the path length in our exponential about the stationary path, one obtains:

$$r_{2n,q}+|\vec{r}_{2n',q'}-\hat{k}L|\approx R_{p,q}+ \tag{34}$$

$$+\frac{2p^2a^2}{[(p^2+q^2)a^2+L^2]^{3/2}}\frac{p^2\xi^2+[(p^2+q^2)+(L/a)^2]\eta^2}{(4n+1)(2p-4n+1)}$$

Here the four indices n, n', q and q' are replaced by p, q and n, since the stationary path exists only if:

$$n' = \frac{p-1}{2} - n \quad ... \quad for \quad p = odd$$

$$n' = n - \frac{p}{2} \quad ... \quad for \quad p = even$$

and

$$s = Int[\frac{4n+1}{2p} q] \tag{34a}$$

$$s' = q - s \quad for \quad q = odd$$

$$s' = s - q \quad for \quad q = even$$

The coordinates ξ and η measure the distance from the stationary path in the $y - z$ plane; ξ is along the projection of the vector $\vec{R}_{p,q}$ on the $y - z$ plane, and η is perpendicular to it. The values of the y and z coordinates at the stationary point are given by: $y_0 = (-1)^{Int[\frac{4n+1}{2p}q]} Frac[\frac{4n+1}{2p}q]a$ and $z_0 = L\frac{4n+1}{2p}$. I evaluate the integral in (33) assuming that the wavelength of light is short enough for the exponential to be the only varying factor and assuming that the integration over ξ and η can be formally extended to infinity (For some paths which reflect off the edge of the wall this is obviously wrong, but the result may only be too large by at most a factor of 2). After inserting the result into (33) and ordering the terms, I obtain:

$$\frac{\int \frac{\partial \Psi_0}{\partial n'} \frac{\partial \psi}{\partial n'} dS'}{\int |\frac{\partial \Psi_0}{\partial n'}|^2 dS'} = -8i\pi\sqrt{2\pi\beta kw} \sum_{p=-\infty \; p \neq 0,-1}^{\infty} \sum_{n=0}^{p-2sgn(p)} \sum_{q=-\infty}^{\infty}$$

$$\frac{U(y_{0,p,q}, z_{0,p,q})}{L} \frac{p}{\sqrt{p^2+q^2}\sqrt{1+(p^2+q^2)a^2/L^2}} e^{-\frac{(kw)^2}{4}\frac{p^2+q^2}{p^2+q^2+L^2/a^2}} \times$$

$$\times \{1 - i\frac{w^2 kp}{L(2p-4n-1)[1+(p^2+q^2)a^2/L^2]^{1/2}}\} \times$$

$$\times \begin{pmatrix} (\frac{1+2p-4n}{1-2p+4n})^2 & for \; p..odd \\ \frac{1+2p-4n}{1-2p+4n} & for \; p..even \end{pmatrix} e^{ikR_{p,q}} e^{is} \tag{35}$$

Note that contributions from different terms add incoherently, since different directions $\vec{R}_{p,q}$ and $\vec{R}_{p',q'}$ are sufficiently separated in angle to make the phases $\varepsilon(\theta_{pq})$ and $\varepsilon(\theta_{p'q'})$ uncorrelated.

The above result can be written in a somewhat more transparent form if one takes into account that for a gravity wave interferometer $L >> a$. I also use the confocal beam size for w ($w^2 = \frac{\lambda L}{\pi}$). Inserting this into (35) and using (22), I obtain the $\delta L/L$ noise in the form:

$$\frac{\delta L}{L} \approx -16i\pi\sqrt{2\pi\beta}\sum\sum\sum \frac{U(y_{0,p,q}, z_{0,p,q})}{w}\frac{\lambda}{LF}\frac{p}{\sqrt{p^2+q^2}} \tag{36}$$

$$e^{-(a/w)^2(p^2+q^2)}\left[1-\frac{ip}{2p-4n-1}\right]\begin{pmatrix} \left(\frac{1+2p-4n}{1-2p+4n}\right)^2 & for\ p\ odd \\ \frac{1+2p-4n}{1-2p+4n} & for\ p\ even \end{pmatrix} e^{ikR_{p,q}}e^{i\varepsilon}$$

In most cases the above expression can be even further simplified. If $a/w >> 1$, then the sum in (36) converges so rapidly that only the first term contributes and we get a simple form for the noise coupling:

$$\frac{\delta L}{L} \approx 8\pi\sqrt{2\pi\beta}\frac{U}{w}\frac{\lambda}{LF} \times 16e^{-(\frac{a}{w})^2} \tag{36a}$$

Here U is the displacement at the center of the moving wall.

This first order contribution to noise coupling of the walls can be thought as consisting of two factors. The first factor $(8\pi\sqrt{2\pi\beta}\frac{U}{w}\frac{\lambda}{LF})$ takes into account the main dimensional characteristics of the interferometer and is independent of the wall geometry, while the second factor depends strongly on the wall geometry and on the coherence length of scattered light. The exponential term in the result is so small due to an efficient cancellation in the interference of the modulated beam coming at an angle $\theta_{p,q}$ with the main beam. This destructive interference works so well because beams with different (p,q) are not correlated and cannot interfere constructively to produce a wave with more or less constant phase on the exit pupil. However, each beam (p,q) has a coherent phase across the exit pupil, so that it's cancellation is very well described by the above exponential. I should like to remark in passing that the above exponential cancellation is characteristic of the particular wall geometry of this example and most of all on the small coherence angle of scattered light. It is possible to think of geometries where the wall would focus one interferometer mirror on the other. In such a case the cancellation could be much weaker, however it would depend much more strongly on the relative position of the interferometer mirror with respect to the wall.

It will be shown in the next section that this first order contribution is not the most important contribution, therefore, we now turn to the modulation of the main beam. (I have shown in an internal report (Čadež,1989) that this is true even in a focusing geometry.)

ii) The contribution of phase modulation to the main beam

In order to calculate the effect of the motion of the walls on the phase modulation of the main beam, we return to eq. (16) for $\sigma = 0$. Since the right hand side is resonant to this mode, we must add a damping term as in (20), so that we get:

$$\frac{d^2}{dt^2}[c_0 e^{i\omega_0 t}] + \frac{2}{\tau}\frac{d}{dt}[c_0 e^{i\omega_0 t}] + \omega_0^2 [c_0 e^{i\omega_0 t}] = -2c^2 e^{i\omega_0 t}(\phi_0, \partial\Delta(t)\phi_0) \quad (37)$$

If the displacement $\vec{U}$ is normal to the wall, the source on the right hand side becomes:

$$Rhs = \frac{c^2 e^{i\omega_0 t}\int_{wall} U(\vec{r},\Omega)(\partial\phi_0/\partial n)^2 dS}{\int \mid \phi_0 \mid^2 dV} \quad (38)$$

where $\partial/\partial n$ is the derivative with respect to the normal to the surface dS. This source has a very simple interpretation: $(\frac{\partial\phi_0}{\partial n})^2 dS$ is the radiation pressure force of the field ϕ_0 on the surface element dS, so that $\int(\frac{\partial\Phi_0}{\partial n})^2 \vec{U}(\vec{r}).d\vec{S}$ is the work done on the radiation field inside the cavity by the moving wall. Noting that the motion of the cavity mirror by δL_m also does work on the mode ϕ_0 in the amount $dE_0 = \delta L_m \int \mid \frac{\partial\Psi_0}{\partial n} \mid^2 dS$, and using eqs. (22), (37) and (38), one finds the displacement noise:

$$\delta L = \frac{\int(\frac{\partial\phi_0}{\partial n})^2 \vec{U}.d\vec{S}}{\frac{dE_0}{\delta L_m}} \quad (39)$$

In order to calculate ϕ_0 at the walls, one must find a complete set of modes inside the cavity consisting of the two main (interferometer) mirrors and the mirror walls. If the main mirrors were perfect, the modes inside the complete cavity could be classified into (fig. 3):

i) gaussian modes resonating between the two main mirrors (*fundamental modes*) which decay exponentially from the optical axis,

ii) gaussian modes propagating from one main mirror to the wall, where they are reflected any number of times before being reflected off the other main mirror, to retrace the mirror image of their path back to the first mirror (*circulating modes*) and

iii) *running modes* describing the propagation of light from $+\infty$ ($-\infty$) past the right (left) mirror to be reflected off the left (right) mirror back to $+\infty$ ($-\infty$).

The microroughness of the main mirror couples the fundamental mode ($\bar{\phi}_{00}$) to circulating modes and to running modes. The main excited mode in the complete cavity (ϕ_0) is, therefore, a linear combination of the fundamental $\bar{\phi}_{00}$ mode and of all the circulating and running modes (eq. (12)). The coefficients in this expansion are expressed by eq. (13a) and the relevant matrix elements can be computed using (11).

Since there is no field coming in from $+\infty$ or $-\infty$, running modes can only couple to the main mode in such a way as to drain the energy out of the main mode. Therefore, these modes do not couple the motion of the walls to energy in the main mode and should not be included in ϕ_0 in (39).

The only relevant modes having nonvanishing amplitudes at the wall are the circulating modes. I characterize them by three quantum numbers p, q, s; p and q count the number of reflections the beam suffers off the $x-z$ and $y-z$ planes respectively on it's path from one interferometer mirror to the other. The third number s is the number of longitudinal nodes in the mode.

Let us describe the mode (2,1,s) as an example of circulating modes (fig.4). It starts at $\vec{r} = (0,0,0)$ and propagates along the vector $\vec{R}_{2,1}$ until it hits the surface $x = a/2$. There it is reflected and propagates along the vector $\vec{R}^{(2)}_{2,1}$ connecting the points $\vec{P}_2 = (a,0,0)$ and $\vec{P}_2{}' = (-a,a,L)$ until it impinges on the wall in the plane $y = a/2$. Next it follows the path along the vector $\vec{R}^{(3)}_{2,1}$ connecting the points $\vec{P}_3 = (a,a,0)$ and $\vec{P}_3{}' = (-a,0,L)$ until reaching the wall at $x = -a/2$ where it is reflected along the vector $\vec{R}^{(4)}_{2,1}$ connecting the points $\vec{P}_4 = (-2a,a,0)$ and $\vec{P}_4{}' = (0,0,L)$ on its final stretch to the center of the second mirror. From there on the mode path retraces its mirror image back to the first mirror. One can see from this picture that the field of a mode is a linear superposition of gaussian T_{00} modes propagating between the appropriate images of the two interferometer mirrors ($\vec{P}_n$ and $\vec{P}_n{}'$). Symmetry of paths guarantees that such a superposition of wave functions vanishes on the walls and on the mirrors, where two gaussian modes always meet at equally opposite angles (One may argue that gaussian $T_{l,m}$ modes should also be considered as candidates for wave functions of the pieces of circulating modes - indeed they do solve the wave equation. However, going through the argument of the completeness of wave functions, one finds, that these modes must be linear combinations of modes $\bar{\phi}_{pq}$, built with gaussian T_{00} modes.). With this picture in mind it is easy to calculate the normal derivative of the modes $\bar{\phi}_{pq}$, at a position $\vec{\rho}$ on the mirror to obtain:

$$\frac{\partial \bar{\phi}_{p,q}}{\partial n} = \frac{k}{\sqrt{2\pi L_{eff}}\, w_{p,q}} e^{-\frac{\rho^2}{w_{p,q}^2}} \left[e^{ik \frac{\vec{R}_{p,q}}{R_{p,q}}.\vec{\rho}} + e^{ik \frac{\vec{R}_{-p,-p}}{R_{-p,-q}}.\vec{\rho}}\right] \frac{\vec{R}_{p,q}}{R_{p,q}}.\vec{n} \tag{40}$$

Here $L_{eff} = | \vec{R}_{p,q} |$. This expression is approximate in the sense that I have neglected the astigmatism seen by light impinging on the main mirror at an angle with respect to the optical axis.

Circulating modes are coupled to the main excited interferometer mode through the surface roughness of main mirrors, i.e. as indicated by the perturbation theory in the second paragraph. The expansion of the mode ϕ_0 into the fundamental mode(s) $\bar{\phi}_0$, circulating modes and running modes is described by eq. (13a) and the relevant matrix elements are given in eq. (11). In our case the only relevant matrix elements are those between the mode $\bar{\phi}_0$ and circulating modes. Inserting (40) and (30) into (11), I obtain:

$$(\bar{\phi}_{p,q}, \delta\Delta\bar{\phi}_{00}) = \frac{1}{2\pi} \int_{mirror1} \frac{\partial \Phi}{\partial n} \frac{k^2}{w_{pq} w_{00} \sqrt{L_{eff} L}} e^{-\rho^2\left(\frac{1}{w_{pq}^2} + \frac{1}{w_{00}^2}\right)} \times$$

$$\times \left[e^{ik \frac{\vec{R}_{pq}}{R_{pq}}.\vec{\rho}} + e^{-ik \frac{\vec{R}_{pq}}{R_{pq}}.\vec{\rho}}\right] \frac{\vec{R}_{pq}}{R_{pq}}.dS +$$

$$+\ \textit{similar term for other mirror} \tag{42}$$

This matrix element can be expressed in terms of the scattering function $f(\theta)$, if one remembers that scattering is generated by microroughness of the mirror $(\frac{\partial\Phi}{\partial n})$. Using (25) and expressing Ψ_0 with (19a), where the integration is taken over the rough mirror, one obtains:

$$f(\hat{\nu}) = \frac{2k^2}{(2\pi)^{3/2}\beta^{1/2}w_{00}} \int \frac{\partial\Phi}{\partial n} e^{-\frac{\rho^2}{w_{00}^2} - ik\hat{\nu}.\vec{\rho}}\, \hat{\nu}.d\vec{S} \qquad (43)$$

If the main cavity is confocal, then w_{pq} is always larger then w_{00}, and one doesn't make much of an error if one neglects $\frac{1}{w_{pq}^2}$ in the exponent in (42). Then (43) can be inserted into (42) to obtain the matrix element in the form:

$$(\bar{\phi}_{pq}, \delta\Delta\bar{\phi}_{00}) = \sqrt{\frac{2\pi\beta}{L_{eff}L}}\frac{2}{w_{pq}}[f(\frac{\vec{R}_{pq}}{R_{pq}}) + f^*(\frac{\vec{R}_{pq}}{R_{pq}})] +$$

$$+ similar\ term\ for\ the\ other\ mirror \qquad (44)$$

(Star denotes the complex conjugate.) Inserting (44) into (13a), we obtain the mode coefficients:

$$a^0_{pqs} = \sqrt{\frac{8\pi\beta}{L_{eff}L}}\frac{1}{w_{pq}}\frac{\Re[f(\frac{\vec{R}_{pq}}{R_{pq}})]}{k^2_{pqs} - k^2} \qquad (45)$$

The separation between neighboring k_{pqs}'s is $\frac{\pi}{L_{eff}}$,and $\Re[f]$ denotes the real part of f. Also, if the mirror at the wall is lossy, k_{pqs}'s have imaginary components $\frac{i}{c\tau_{pq}}$, where τ_{pq} is the decay time of the (pqs) mode. Let T be the absorption coefficient of the wall, then it can easily be seen that $\tau_{pq} = \frac{L_{eff}}{T(|p|+|q|)c}$. Moreover, for decay times shorter then the circulation time $(\tau_{circ} = 2L_{eff}/c)$ our theory is no longer adequate, since the assumption of a loss less wall obviously breaks down. However, the introducing argument gives a clue. The moving wall does no work on the absorbed fraction of the wave, which suggests that only the truly circulating fraction of the field should be included in the numerator of (39). The wave function $\bar{\psi}_{pqs}$ bounces $p+q$ times off the wall before reaching the other mirror. Since the amplitude of the wave function drops after each reflection by a factor $(1-T)$, a return trip from one mirror to the other and back diminishes the wave function amplitude by

$$\gamma_{pq} = (1-T)^{2(|p|+|q|)} \qquad (46)$$

$$\approx e^{-2T(|p|+|q|)}$$

$$= e^{-\tau_{circ}/\tau_{pq}}$$

Finally we have all the elements to calculate the noise contributions due to excitation of different modes. Expressing ϕ_0 at the wall with circulating modes ($\phi_0 = \bar{\phi}_{00} + \sum a^0_{pqs}\bar{\phi}_{pqs}$), we obtain the numerator of (39) in the form:

$$\sum_{p,q,s}\sum_{p',q',s'} a_{pqs}a_{p'q's'}\gamma^2_{pq}\int_{wall}\frac{\partial\bar{\phi}_{pqs}}{\partial n}\frac{\partial\bar{\phi}_{p'q's'}}{\partial n}\vec{U}.d\vec{S} \approx$$

$$\approx \sum_{p,q,s,s'}\gamma^2_{pq}a_{pqs}a_{p,q,s'}\int_{wall}\frac{\partial\bar{\phi}_{p,q,\frac{s+s'}{2}}}{\partial n}\frac{\partial\bar{\phi}_{p,q,\frac{s+s'}{2}}}{\partial n}\vec{U}.d\vec{S} \qquad (47)$$

It is clear that modes with different p and q do not overlap, since they are spatially separated. However, it is easy to see that modes $\bar{\phi}_{pqs}$ and $\bar{\phi}_{pqs'}$ overlap almost completely as long as $| s - s' | \ll \sqrt{\frac{L}{\lambda}\frac{L/a}{\sqrt{p^2+q^2}}}$. Both observations were used to write the above equality.

In order to complete the calculation, one must choose a form of $\vec{U}$. I assume the worst case, where $\vec{U}$ is independent of position, normal to the wall $x = \frac{a}{2}$, and vanishes on the other three walls. The $\vec{U}$ in (47) can be taken in front of the integral, so that the integral can be calculated by simply remembering that $(\frac{\partial\bar{\phi}}{\partial n})^2$ is proportional to the radiation pressure in the direction orthogonal to n. This pressure is exerted $| p | /2$ (or $| p{\pm}1 | /2$) times as the mode hits the moving wall,so we get:

$$U\int[\frac{\partial\phi_{pq}}{\partial n}]^2 dS = U\frac{|p|}{2}\frac{k^2\cos(\theta)}{2L_{eff}} \qquad (48)$$

I have dropped the index p, knowing that only those p's which correspond to wave vectors close to k are of any importance. Putting (45) and (48) into (47) and this into (39) I get:

$$\delta L = 64\pi\beta U a\sum_{pq}\frac{|p|\sqrt{p^2+q^2}}{L^3_{eff}w^2_{pq}}\gamma^2_{pq}[\Re f(\frac{\vec{R}_{pq}}{R_{pq}})]^2\times$$

$$\times\sum_{s,s'}\frac{1}{k^2_{pqs}-k^2}\frac{1}{k^{*\,2}_{pqs'}-k^2} \qquad (49)$$

The sum over s (and s') gives:

$$\sum_{s} \frac{1}{k_{pqs}^2 - k^2} = -\frac{iL_{eff}}{2k}\{1 - \frac{2}{\pi}tan^{-1}[\frac{T(|p| + |q|)}{\pi k L_{eff}}]\} \approx -\frac{iL_{eff}}{2k} \quad (50)$$

Using (26) for $f(\theta)$ and approximating the sum over m and n with an integral, I obtain finally a simple result:

$$\delta L = \pi\beta U\frac{\lambda}{a}\frac{1}{T^2}ln(\sqrt{2}+1)$$

$$= 2.8\beta U\frac{\lambda}{a}\frac{1}{1-\alpha} \quad (49a)$$

Here $\alpha = 1 - T^2$ is the albedo.

6. The second example - random walls

Previous calculations with mirror walls indicated, that the amount of energy stored in the parallel mirror-wall-mirror cavity is the most important factor determining the coupling of the main interferometer mode to the motion of the walls. One has two choices for reducing this energy:
i) choosing lower reflectivity walls reduces the finesse of the parallel cavity and thus the coupling.
ii) stronger coupling of reflected light at the wall to running modes also drastically reduces the effective energy stored in the parallel cavity.

One way of implementing the second mechanism, is to introduce rough walls, i.e. walls which scatter incident light in as wide a solid angle as possible.. Thus, I assume, that the walls are rough enough that the scattering off their surface can be described by Lambert's law (M. Born, E. Wolf 1980; M.V. Klein, T.E. Furtak 1986), i.e. scattered light is uniformly distributed over the solid angle irrespective of the angular distribution of incident light. (Lambert's law is valid for rough surfaces if the incidence angle of light is smaller then $cos^{-1}(\frac{\lambda}{H})$, where λ is the wavelength of light an H is the surface roughness scale.)

In the rough wall approximation the field amplitude (Ψ) is not the most natural quantity to describe the distribution of scattered light, since the phase of scattered light is completely scrambled. It is more appropriate to use the surface brightness distribution (B) and the incident flux j. In order to introduce these quantities in terms of the field amplitude, it is convenient to describe the field in a neighborhood of a point near a random surface as a superposition of plane waves as follows:

$$\Psi(\vec{r}) = \frac{1}{4\pi}\int \varsigma(\hat{n})e^{ik\hat{n}.\vec{r}}d^2\hat{n} \tag{51}$$

And the inverse is:

$$\varsigma(\vec{n}) = \frac{1}{4\pi}lim_{s\to 0}\int \Psi(\vec{r}+\hat{\nu}s)e^{-ik\hat{n}.(\vec{r}+\hat{\nu}s)}d^2\vec{\nu} \tag{52}$$

I write the correlation function between wave amplitudes (ς) as:

$$< \varsigma(\hat{n})\varsigma^*(\hat{n}') > = K(\hat{n})\delta^2(\hat{n}-\hat{n}') \tag{53}$$

Here $< ... >$ denotes the average over an ensemble of statistically equivalent samples. The incident flux on a surface perpendicular to $\hat{n}_0$ becomes:

$$j = \frac{1}{2k^2c}(\frac{\partial\psi}{\partial t}\hat{n}_0.\nabla\psi^* + \frac{\partial\psi^*}{\partial t}\hat{n}_0.\nabla\psi) = -\frac{1}{(4\pi)^2}\int_{ingoing} d^2\hat{n}K(\hat{n})\hat{n}_0.\hat{n} \tag{54}$$

And the brightness distribution is given by:

$$B(\hat{n}) = \frac{1}{(4\pi)^2}\int < \varsigma(\hat{n})\varsigma^*(\hat{n}')\hat{n}'.\hat{n}_0 > d^2\hat{n}' = \frac{1}{(4\pi)^2}K(\hat{n})\hat{n}_0.\hat{n} \tag{55}$$

For a Lambert scatterer the outgoing part of $K(\hat{n})$ is a constant independent of the direction of incoming light. If the scatterer has albedo α, then

$$K_{out}(\hat{n}) = 16\pi\alpha j \tag{55a}$$

Suppose that the wall of a Fabry-Perot cavity is a circular pipe with radius R and the two interferometer mirrors on the axis of the pipe. The incident flux on the wall comes from scattered light off the interferometer mirrors plus scattered light off other portions of the walls. The problem of the brightness distribution inside a circular duct is an old one (P. Moon 1940). Solving the appropriate integral equation with the source function given by (14), I observed that the brightness $B_0(\vec{r})$ is roughly given by

$$B_0(\vec{r}) = \frac{\alpha}{1-\alpha} j_{inc}(\vec{r}) \tag{56}$$

Suppose now that an incident flux scatters off a rough wall vibrating with the frequency Ω. The previously monochromatic brightness is Doppler modulated, so that two side bands $B^{\pm}(\hat{n})$ emerge with frequencies $\omega_0 + \Omega$ and $\omega_0 - \Omega$ (ω_0 is the frequency of incoming light). The equations (19) and (19a) are not directly useful for determining the brightness distribution, since the roughness function of the wall is not explicitly given. It is more convenient to consider the local expansion into plane waves. One can break (51) into two parts: an integration over incoming waves and another one over outgoing waves. The random wall scatters incoming waves into outgoing waves, which I describe with a linear operator as follows:

$$\varsigma_{out}(\hat{n}) = \int \varsigma_{in}(\hat{n}')R(\hat{n}',\hat{n})d^2\hat{n}', \tag{57}$$

where ς_{out} denotes the amplitudes of outgoing waves and ς_{in} goes with ingoing waves. It is easy to show that, if $R(\hat{n}',\hat{n})$ is to describe a Lambert scatterer, it's correlation function must be of the form:

$$< R(\hat{n}',\hat{n})R(\hat{n}',\hat{n}") > = \frac{\alpha}{\pi}\delta^2(\hat{n}-\hat{n}") \tag{58}$$

In order to see what happens to an incoming wave when scattered off a moving wall, go into a coordinate system ($\vec{r}'$) which is moving with the wall ($\vec{r}' = \vec{r} - \vec{U}cos(\Omega t)$). One must insure that the wave function vanishes at the wall ($\vec{r}' = \vec{r}_s$), i.e. in the moving system. Going back in the inertial coordinate system ($\vec{r}$), one finds that this boundary condition determines the coefficients of reflected plane waves as follows:

$$\varsigma_{out}(\hat{n}) = \int \varsigma_{in}(\hat{n}')[1 - ik(\hat{n}-\hat{n}').\vec{U}cos(\Omega t)]R(\hat{n}',\hat{n})d^2\hat{n}'$$

The side band amplitudes are, therefore:

$$\varsigma^{\pm}_{out} = -\frac{i}{2}k\int \varsigma_{in}(\hat{n}')(\hat{n}-\hat{n}').\vec{U}R(\hat{n}',\hat{n})d^2\hat{n}' \qquad (59)$$

Inserting (59) in (55) the source of the brightness distribution in side bands can be computed. Assuming that the random functions $\varsigma_{in}(\hat{n})$ and $R(\hat{n}',\hat{n})$ are stochastically independent (i.e.$<\varsigma(\hat{n})\varsigma^*(\hat{n}')R(\hat{n}'',\hat{n}^{iii})R(\hat{n}^{iv},\hat{n}^{v})> =$ $=<\varsigma(\hat{n})\varsigma(\hat{n}')><R(\hat{n}'',\hat{n}^{iii})R(\hat{n}^{iv},\hat{n}^{v})>$) one obtains:

$$B^{\pm}(\hat{n}) = (\frac{1}{4\pi})^2\hat{n}_0.\hat{n}(\frac{k}{2})^2\int_{incoming} d^2\hat{n}''K(\hat{n}'')[\vec{U}.(\hat{n}''-\hat{n})]^2 \qquad (60)$$

The side bands produced by the motion of the wall propagate toward the exit pupil of the interferometer either directly, or through scattering off the walls, i.e. the actual brightness of a side band at the position $\vec{r}_s$ at the wall ($B^{\pm}_a(\hat{n},\vec{r}_s)$) is the sum of the source ($B^{\pm}(\hat{n},\vec{r}_s)$) plus the contribution from all other parts of the wall illuminating the region at $\vec{r}_s$. Thus B_a obeys the integral equation:

$$B^{\pm}_a(\vec{r}_s,\hat{n}) = B^{\pm}(\vec{r}_s,\hat{n})+\frac{\alpha}{\pi}\hat{n}_0.\hat{n}\int B^{\pm}_a(\vec{r}',\frac{\vec{r}_s-\vec{r}'}{|\vec{r}_s-\vec{r}'|})\frac{dS'}{|\vec{r}_s-\vec{r}'|^2} \qquad (61)$$

The Lambert scattering law (55a) has been taken into account.

The brightness of light impinging on the exit pupil from the direction $\hat{n}$ is the brightness of the wall seen from this direction, i.e. $B^{\pm}_{ex}(\vec{r}_{ex},\hat{n}) = B^{\pm}_a(\vec{r}_{ex}-\hat{n}s,\hat{n})$, where s is such that $\vec{r}_{ex}-\hat{n}s$ is a position on the wall.

In order to calculate the first order noise coupling, one must evaluate the overlap integral in the numerator of eq. 23. In the average it vanishes, since the average value of any amplitude $\varsigma(\hat{n})$ vanishes. However, the variance of this expression is not zero - using the expansions (51) and (53), one obtains:

$$<|\int_{ex} dS\frac{\partial\Psi_0}{\partial n}\frac{\partial\psi}{\partial n}|^2> = <\int_{ex} dS\int_{ex} dS'\frac{\partial\Psi_0}{\partial n}\frac{\partial\psi}{\partial n}\frac{\partial\Psi_0^*}{\partial n'}\frac{\partial\psi^*}{\partial n}> =$$

$$= (\frac{k}{4\pi})^2\int_{ex} dS\int_{ex} dS'\frac{\partial\Psi_0}{\partial n}\frac{\partial\Psi_0^*}{\partial n'}\int d^2\hat{n}(\hat{n}.\hat{n}_{ex})^2K(\hat{n})e^{ik\hat{n}.(\vec{r}-\vec{r}')} \qquad (62)$$

Inserting (30) for $\frac{\partial \Psi_0}{\partial n}$, one can integrate first over dS and dS' to obtain:

$$<|\int dS \frac{\partial \Psi_0}{\partial n}\frac{\partial \psi}{\partial n}|^2> =$$

$$= 2\pi P_0 k^4 w^2 \int d^2\hat{n}(\hat{n}.\hat{n}_{ez})e^{-\frac{(kw)^2}{2}(1-(\hat{n}.\hat{n}_{ez})^2)} B_a^{\pm}(\hat{n}) \qquad (63)$$

The exponential factor in the integrand strongly prefers the forward direction, so that we may take an approximate $B_a^{\pm}$ in the forward direction in front of the integral to obtain:

$$w^2 \int d^2\hat{n}(\hat{n}.\hat{n}_{ez})e^{-2k^2w^2(1-(\hat{n}.\hat{n}_{ez})^2)} B_a^{\pm}(\hat{n}) \approx \frac{\sqrt{2\pi}}{4} < B_a^{\pm}(\hat{n}_{ez}) > \lambda w$$

Using this in (63) and inserting it in (23), we obtain the phase noise in the form:

$$< \Delta\Phi^2 >^{\frac{1}{2}} = \{\sqrt{\pi^3/2}\frac{\lambda w < B_a^{\pm}(\hat{n}_{ez}) >}{P_0}\}^{1/2} \qquad (64)$$

The meaning of (64) is transparent: the phase noise is proportional to the square root of the ratio of the side band power incoming on an area $\sqrt{2}\pi^{3/2}\lambda w$ of the exit pupil, to the circulating power in the interferometer. The noise receiving area $(\sqrt{2}\pi^{3/2}\lambda w)$ is so much smaller then the mode area (πw^2), since the noise signal comes with small random speckles with size $d_{speckle} \approx \lambda$. There are approximately $(\frac{\pi w^2}{\lambda^2})$ speckles in the mode and the square root of this number contributes to noise. (Remember that $< B_a^{\pm}(\hat{n}_{ez}) >$ is the average brightness in the forward direction and the contribution from wider angles is damped even more for essentially the same reasons as in (36)).

The calculation of $B_a^{\pm}(\hat{n})$ according to (61) could be quite a laborious exercise and, therefore, I will not go into detail here. However, eq. (56) suggests that for small albedo the incoming brightness of the carrier on the wall($K(\hat{n},\vec{r}_s)$) is given mostly by the direct contribution from the interferometer mirror, and the brightness of the side bands at the mirror is due mainly to the side band source at the wall ($B^{\pm}$) (i.e. the integral in (61) is small so that $B_a^{\pm} \approx B^{\pm}$). The carrier brightness is in this case (cf. (14)):

$$K(\hat{n},\vec{r}_s) = (4\pi)^2\beta\frac{P_0}{|\vec{r}_s-\vec{r}_m|^2}\frac{1}{\theta^2}\delta^2(\hat{n}-\frac{\vec{r}_s-\vec{r}_m}{|\vec{r}_s-\vec{r}_m|}) \quad , \qquad (65)$$

where θ is the angle between the direction at which the light is incoming at the wall and the normal to the interferometer mirror (see fig. 5). If the interferometer mirror is on the axis of the pipe with radius R, then $sin\theta = \frac{R}{|r_s - r_n|}$ (fig. 5) and, since all angles are small, we may replace θ by $sin\theta$ and the brightness becomes:

$$K(\hat{n}, \vec{r}_s) = (4\pi)^2 \beta \frac{P_0}{R^2} \delta^2 (\hat{n} - \frac{\vec{r}_s - \vec{r}_n}{| \vec{r}_s - \vec{r}_n |}) \tag{66}$$

This incoming light gives birth to side bands whose brightness when expressed by (60) becomes:

$$B^{\pm}(\hat{n}, \vec{r}_s) = \hat{n}_0 . \hat{n} (\frac{k}{2})^2 \alpha\beta \frac{P_0}{R^2} [\vec{U} . (\frac{\vec{r}_s - \vec{r}_m}{| \vec{r}_s - \vec{r}_m |} - \hat{n})]^2 \tag{67}$$

The brightness in the direction $\hat{n}$ at the exit mirror (denoted by ex, see fig. 5) is the brightness of the wall seen from the exit mirror in the direction $\hat{n}_{ex}$, i.e. $\hat{n}_{ex} = \frac{r_{ex} - r_s}{|r_{ex} - r_s|}$. Since in our geometry the brightness distribution is cylindrically symmetric, it depends only on the angle θ'. Using (67) and reading the quantities off the diagram in fig. 5, it is easy to see that:

$$B^{\pm}(\theta') = U^2 (\frac{k}{2})^2 \alpha\beta \frac{P_0}{R^2} sin^3 \theta' \times$$

$$\times \left[1 + \frac{1}{\sqrt{(L/R)^2 sin^2 \theta' - (2L/R) sin\theta' cos\theta' + 1}} \right]^2 \tag{68}$$

When this side band brightness is inserted in (63), we obtain the average square of the matrix element, which inserted in (23) and (22) gives the phase noise and distance noise respectively. In the limit $L >> R$ (which is the only interesting limit), I obtain for the distance noise:

$$\frac{\delta L}{L} = \frac{\pi\lambda}{FL} \sqrt{\frac{\alpha\beta}{8}} \frac{U}{\sqrt{LR}} \{ e^{-2(\frac{R}{w})^2} + \int_{\sqrt{2}R/w}^{\infty} e^{-\xi^2} d\xi \}^{1/2} \tag{69}$$

Comparing this result to the noise coupling in the mirror wall case given by eq. (36a), one can recognize the similarities and differences in processes responsible for noise coupling. The ratio of the two expressions is

$$\frac{\delta L_{mirror}}{\delta L_{random}} \approx 512\pi \sqrt{\frac{R}{\lambda}} \times e^{-\frac{a^2 - 2R^2}{w^2}} \tag{70}$$

One can immediately recognize the same exponential factors in both expressions which are due to the fact that light scattered off the wall always interferes with the main beam at an angle which brings about the exponential cancelation of the effect. There is a difference in the intensity of the coupling which can be attributed solely to the difference in the probability that the photon reaches the exit pupil after scattering at the wall. In the random case the probability for a photon to reach the exit pupil after scattering at the wall at z (see fig. 5) is proportional to the solid angle at which the photon at this position sees the exit pupil: $p \approx \pi w^2/(z^2 + R^2)$. The average of this probability is $< p >= (1/L)\int_0^L p.dz \approx \lambda/R$ (I am always using the confocal value for the size of the beam w.). In the mirror wall case, a photon heading in the right direction certainly hits the exit pupil, so that the square root factor in (70) is precisely the square root of the ratio of the respective probabilities. Taking typical parameters for a large gravity wave interferometer: $L = 4km$, $R = .5m$, $\lambda = .5\mu m$, one thus expects that a random wall is about 10^6 times less coupled to the main interferometer then a mirror wall of about the same size.

ii) The contribution of phase modulation to the main beam

The calculation of phase modulation should go along the same steps as in the mirror case. However, there is a difference; in the mirror case modes are relatively well defined and their occupation , in particular the occupation of circulating modes, can be (at least approximatelly) calculated from first principles. In the random case modes are not so well defined, but (due to the randomizing effect of the walls), one can use classical radiative transfer equations to get a very good estimate for the total brightness distribution $(K(\vec{r}, \hat{n}))$ at the walls. I use this distribution as a basis, and calculate the probability $(\gamma^2(\vec{r}, \hat{n}))$ that a particular component $K(\vec{r}, \hat{n})$ belongs to a circulating mode.

Consider a small surface element dS_0 in fig. (6). The light emitted from it belongs to the circulating component only if it can be scattered back to the original position and direction to interfere with itself. There are many ways by which this may happen and we can classify them by the number of scatterings off the walls on the return trip (just as in the mirror case). It is clear that for the case of small albedo only those paths requiring just one additional scattering off the wall are of consequence, since more complicated paths are considerably less probable.All possible such rays propagate within the shaded region in fig. (6a,b). It will also be clear that the largest contribution comes from those paths which do not wander too far from one of the mirrors. Therefore, for long interferometers ($L \gg R$) the contributions corresponding to paths in fig. (6b) are much smaller then those corresponding to fig. (6a) and will thus be neglected.

The field emanating from a sufficiently small illuminated surface element dS_0 can be written as a spherical wave:

$$\Psi_{ds} = \sqrt{j(\vec{r}_0)dS_0}\frac{e^{ik|r-r_0|}}{|\vec{r}-\vec{r}_0|} \tag{71}$$

Here $j(\vec{r}_0)$ is the flux density impinging on dS_0 at $\vec{r}_0$. This spherical wave propagates toward ΔS either directly or by reflection off the left mirror. At ΔS the two components interfere and are scattered back in the direction of dS_0. Just after scattering off ΔS the field is a supperposition of the two spherical waves modulated in phase by the random surface roughness (I. Yamaguchi, 1977):

$$\Psi(\vec{r}) = \sqrt{\alpha j_0 dS_0}\left\{\frac{e^{ik|r-r_0|}}{|\vec{r}-\vec{r}_0|} + \frac{e^{ik|r-r_0^*|}}{|\vec{r}-\vec{r}_0^*|}\right\}e^{i\phi(r)} \tag{72}$$

The correlation function of the random phase $e^{i\phi(r)}$ follows from (57) and (58):

$$< e^{i\phi(r)}e^{-i\phi(r')} > = \frac{1}{4\pi k^2}\delta^2(\vec{r}-\vec{r}') \tag{73}$$

From ΔS the field $\Psi(\vec{r})$ propagates to a point $\vec{r}_0'$ close to $\vec{r}_0$ by two paths - directly or through a reflection off the left mirror. We can use Green's theorem to calculate it's amplitude:

$$\Psi'(\vec{r}_0') = -\frac{ik}{4\pi}\sqrt{\alpha j_0 dS_0}\times$$

$$\times\int_{\Delta S} dS e^{i\phi(r)}[\frac{\hat{n}_0.\hat{n}_1}{s_1}e^{iks_1} + \frac{\hat{n}_0.\hat{n}_1'}{s_1'}e^{iks_1'}][\frac{e^{iks_2'}}{s_2'} - \frac{e^{iks_2}}{s_2}] \tag{74}$$

For brevity I have introduced $\hat{n}_0$ as the unit normal to the wall at $\vec{r}$, and the following notation:

$$s_1 = |\vec{r}_0 - \vec{r}| \qquad s_2 = |\vec{r}_0' - \vec{r}|$$

$$s'_1 = |\vec{r}_0^{\,*} - \vec{r}\,| \qquad s'_2 = |\vec{r}_0^{\,\prime *} - \vec{r}\,|$$

and

$$\hat{n}_1 = \frac{\vec{r}_0 - \vec{r}}{s_1} \qquad \hat{n}_2 = \frac{\vec{r}_0^{\,\prime} - \vec{r}}{s_2}$$

$$\hat{n}'_1 = \frac{\vec{r}_0^{\,*} - \vec{r}}{s'_1} \qquad \hat{n}'_2 = \frac{\vec{r}_0^{\,\prime *} - \vec{r}}{s'_2}$$

Finally, I calculate the flux density corresponding to the field Ψ' (eq. (54)). The expression is quite messy. It simplifies somewhat, when one takes into account (73), but it still contains 16 terms, which are products of combinations of exponentials in (74). The first four terms give the usual diffuse light flux that is calculated in geometric optics. Twelve terms are oscillating rapidly in the region of interest giving no net contribution and the last two terms give:

$$\vec{j}_c(\vec{r}_0^{\,\prime}) = \frac{1}{(4\pi)^3}(\alpha j_0\, dS_0)\Re\Big(\int_{\Delta S} dS \frac{(\hat{n}_0.\hat{n}_1)(\hat{n}_0.\hat{n}'_1)}{s_1\, s'_1\, s_2\, s'_2} \times$$

$$\times\{\hat{n}_2 e^{ik(s_1 - s_2 - s'_1 + s'_2)} + \hat{n}'_2 e^{-ik(s_1 - s_2 - s'_1 + s'_2)}\}\Big) \tag{75}$$

When the size of the system is much, much larger then the wavelength of light (which is the case in practical applications), the only important varying factors are the exponentials, and we get:

$$\int_{\Delta S} dS e^{ik(s_1 - s_2 - s'_1 + s'_2)} \approx \lambda^2 \delta_0^2(\hat{n}_1 + \hat{n}'_1 + \hat{n}_2 + \hat{n}'_2) = \lambda^2 \frac{(s_1 s'_1)^2}{(s_1 + s'_1)^2}\delta_0^2(\vec{r}_0 - \vec{r}_0^{\,\prime}) \tag{76}$$

Here $\delta_0^2(\vec{a}) = \delta^2(\vec{a} - \hat{n}_0(\vec{a}.\hat{n}_0))$ a Dirac distribution of the projection of the vector $\vec{a}$ on the plane perpendicular to $\hat{n}_0$. Integrating $\vec{j}_c(\vec{r}_0^{\,\prime})$ over dS'_0, we obtain the energy received by the surface element dS'_0. The delta function insures that this contribution is nonvanishing only if dS_0 and dS'_0 coincide. Note that only the contribution due to (75) remains finite as dS_0 goes to zero. Thus, we may interpret $\int j_c(\vec{r}_0^{\,\prime})dS'_0/dS_0$ as the current in the circulating mode. (It is worth remarking that this term makes the holographic interferometry of diffuse surfaces posssible (I. Yamaguchi, 1977).)

The probability that $K(\hat{n})$ belongs to a circulating mode is clearly the ratio of the circulating current to the current in all modes. According to the above discussion I get:

$$\gamma^2(\vec{r},\hat{n}) = \frac{\alpha}{(8\pi)^2}\left(\frac{\lambda}{s_1 + s_1'}\right)^2 (\hat{n}_0 . \hat{n})(\hat{n}_0 . \hat{n}') \tag{77}$$

$$\gamma^2(\vec{r},\hat{n}) = 0 \quad ... \quad if \;\; \hat{n}' \; does \; not \; exist$$

Here $\hat{n}'$ is conjugate to $\hat{n}$ in the sense that if a ray starting at dS_0 along $\hat{n}$ hits the surface ΔS, then a ray starting at dS_0 along $\hat{n}'$ will hit the same point on ΔS. One of the directions must be of course reflected on the way to ΔS. The conjugate direction $\hat{n}'$ exists only for $\hat{n}$'s pointing toward the mirror or toward the portion of the wall illuminated by the reflection off the mirror.

We may finally calculate the circulating component of the stress tensor $|\frac{\partial\Psi_0}{\partial n}|^2$. I express Ψ_0 in terms of ς (51). When the ensemble average is taken, ς's are reduced to $K(\vec{r},\hat{n})$ (53), which must be multiplied by $\gamma^2(\vec{r},\hat{n})$ to project out the outgoing circulating part. The total brightness distribution $K(\vec{r},\hat{n})$ is calculated from (66), Lambert's law (55a) and taking note of (56). In this way I obtain the following expression:

$$(\frac{\partial\Psi_0}{\partial n})^2 = \left(\frac{k}{4\pi}\right)^2 \int d^2\hat{n} K(\hat{n})\gamma^2(\hat{n})(\hat{n}_0 . \hat{n})^2 =$$

$$= \frac{\alpha\beta}{16\pi(1-\alpha)}\frac{P_0}{R^2} cos\theta \int d^2\hat{n}\frac{(\hat{n}' . \hat{n}_0)(\hat{n} . \hat{n}_0)^3}{(s_1 + s_1')^2} \tag{78}$$

The $d^2\hat{n}$ integration runs only over those solid angles where $\hat{n}$ has a conjugate direction. I suppose that the mirros are small compared to the diameter of the enclosure to simplify the solid angle integration. The displacement noise is finally calculated by multiplying (78) with the displacement of the wall $(\vec{U}(\vec{r}) = \hat{n}_0 U)$ and integrating over the wall. When this is inserted in (39), I get after a straightforward calculation and a tediuos integration:

$$\delta L = U\frac{\alpha\beta}{256\pi^3(1-\alpha)}\left(\frac{\lambda}{R}\right)^2 \frac{S_m}{R^2}\left(ln\frac{L}{3.87R} + O(\frac{R}{L})\right) \tag{79}$$

Comparing (79) with the equivalent expression (49a) for mirror walls, one can see, that the main difference is in the factor λ/R, which is the ratio of probabilities that a photon scattered off the wall reaches the exit pupil. This same factor was already encountered in the comparison of direct contributions (see eq. 70). However, comparing direct contributions, the probability factor comes with the square root and in the ratio of main beam modulations the factor is first order. The difference is a clear consequence of the fact that direct contributions couple through the field amplitude and the main beam phase modulation couples through the energy flux.

Conclusion

This article discusses the physics involved in coupling Fabry-Perot cavities to their enclosures due to mirror imperfections. In the first four paragraphs the problem is formulated and a perturbation theory is developed, which allows one to calculate the coupling of the cavity to the wall. The perturbation formulae are explicitly written to first order only, but it is straightforward to extend them to higher orders if necessary. The results are Green's type formulae (19) or (19a) describing the field produced by changing boundary conditions. It is important to note that the Green's function propagating the field from the wall to the exit pupil is the Green's function belonging to the whole cavity enclosure. Multiple scatterings off the walls are thus generally included in first order equations. An additional expansion of the first order term into a series corresponding to an increasing number of wall scatterings might not be converging sufficiently fast unless the albedo of the walls is quite low. This is strikingly illustrated by the discussion of the main beam modulation in the second part of the fifth paragraph. The general theory in the first four paragraphs only touches on the question of the decreased coupling due to losses at the walls. The details are spelled out in examples in the following two paragraphs.

In the 5^{th} and 6^{th} paragraph two limiting examples are worked out in some detail. The results of these examples are expressions (36a) and (49a) if the enclosure is a mirror, and (69) and (79) if the enclosure is a perfectly random scatterer. Both cases predict the stronger coupling through the essentially second order effect called here the phase modulation of the main beam. The reason for the prevalence of the second order effect is the canceling interference between coherent beams coming from the enclosure and the main beam. Since scattered light at the wall has a sufficiently short coherence length, it is very unlikely to produce a coherent coupling in a more or less standard geometry. It must be stressed, that these results were calculated for a particular scattering function $f(\frac{r}{r})$ chosen in eq. (26). Probably the most important effect of such a sharply peaked scattering function is its short coherence range. Therefore, one might expect that the exponential cancelation factors in the direct contribution could be smaller if the scattering function is more wide open.

It is interesting to calculate the limits on noise coupling predicted by eqs. (36a) and (49a) for mirror walls, and (69) and (79) for random walls, for a planned Large Interferometer Gravitational wave Observatory (LIGO). The length of the planned interferometer is $L = 4km$, the radius of the pipe R is 0.5m, the wavelength of light $\lambda = 500nm$, the finesse of the cavity F is 100, for the coupling coefficient β I take 10^{-6} and I suppose that the albedo (α) is .5.

For mirror walls the strongest coupling predicted by (49a) is:

$$\frac{\delta L}{L} \approx 2.8\beta\frac{2\lambda}{R}\frac{1}{1-\alpha}\frac{U}{L} = 2.8 \times 10^{-21} \times \frac{U}{1\mu m}$$

Reading this formula one should remember that it was derived for a square enclosure with the interferometer axis on the symmetry axis of the enclosure. The equivalent result for a circular cylinder enclosure should differ from the above by a numerical factor, which is probably of order 1. The derivation of the above result also indicated that many modes contribute to the coupling, so that U is an average displacement of the whole pipe.

For a random wall the strongest coupling is predicted by (79) with the result:

$$\frac{\delta L}{L} = \frac{\alpha\beta}{128\pi^3(1-\alpha)}\left(\frac{\lambda}{R}\right)^2\frac{S_m}{R^2}\frac{U}{L}ln\frac{L}{3.87R} = 7.6 \times 10^{-33} \times \frac{U}{1\mu m}$$

The very large factor between the two predictions is explained by the low probability in the random case that a scattered photon belongs to a circulating mode. The comparison of results for mirror walls and for random walls suggests that quenching of circulating modes is the most effective mechanism to decrease coupling to the walls. Baffles in the pipe are considered as the best technical solution to do that (Thorne 1987,89).

Finally I should like to remark, that the calculations indicated in paragraphs 5 and 6 are rough lower and upper limits. More detailed calculations are now being performed at MIT and Caltech in order to estimate the expected noise coupling in the geometry of the large gravity wave interferometer. The above calculation indicates that there is plenty of room between the best and the worst result, so that these effects should not hamper the ultimate sensitivity of large gravity wave interferometers.

7. Acknowledgements

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Figure captions

Fig. 1

The unperturbed boundary ($\delta\Sigma_0$) and its coordinate system (square grid), and the perturbed boundary ($\delta\Sigma$) with the coordinate system $\xi, (\eta), \zeta$.

Fig. 2

The mirror wall enclosure, with interferometer mirros at the symmetry axis of the enclosure a distance L appart. Images of interferometer mirrors are also indicated. An optical path from one mirror to the other is a straight line from one mirror to the appropriate image of the other. The coordinate system is centered on the left mirror.

Fig. 3

Schematic representation of fundamental, circulating and running modes.

Fig. 4

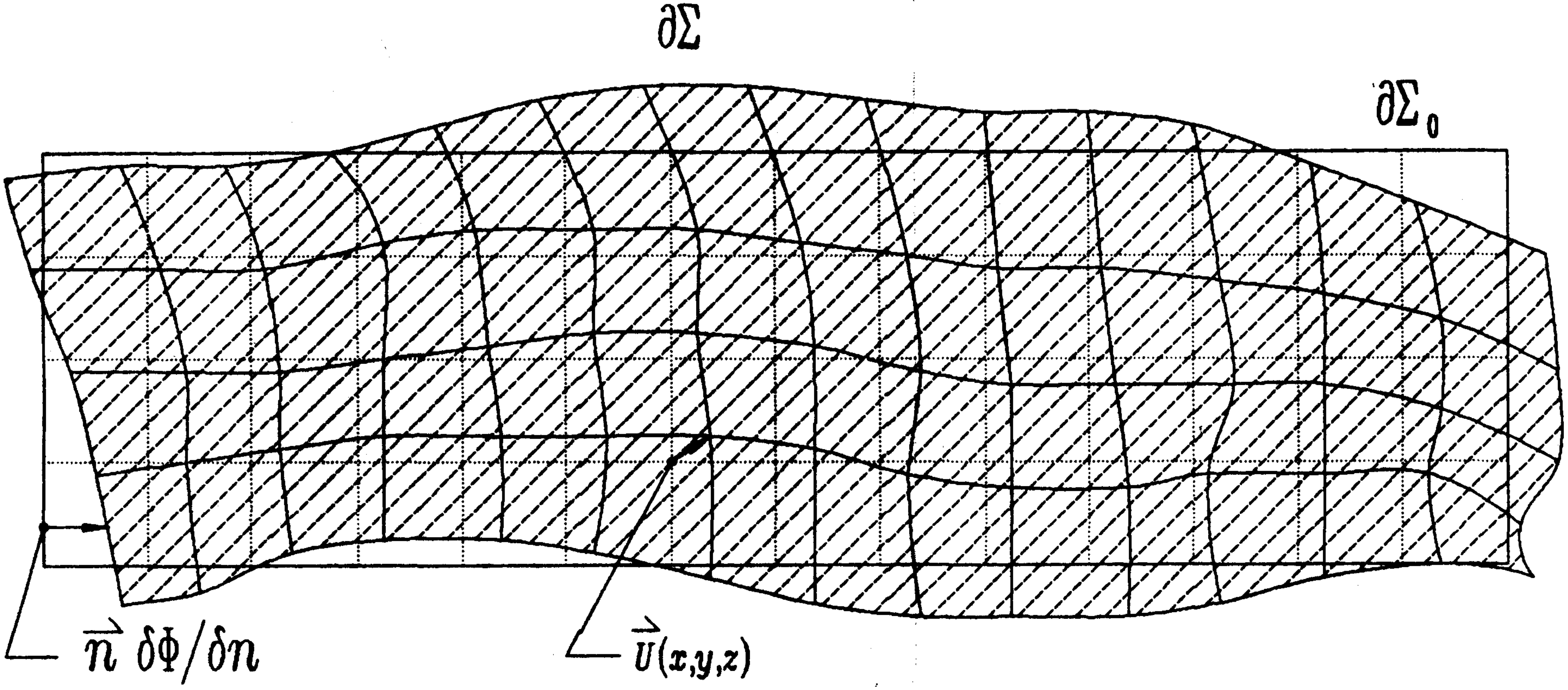
The elements of the (2,1,s) mode as a representative description of circulating modes (See text.).

Fig. 5

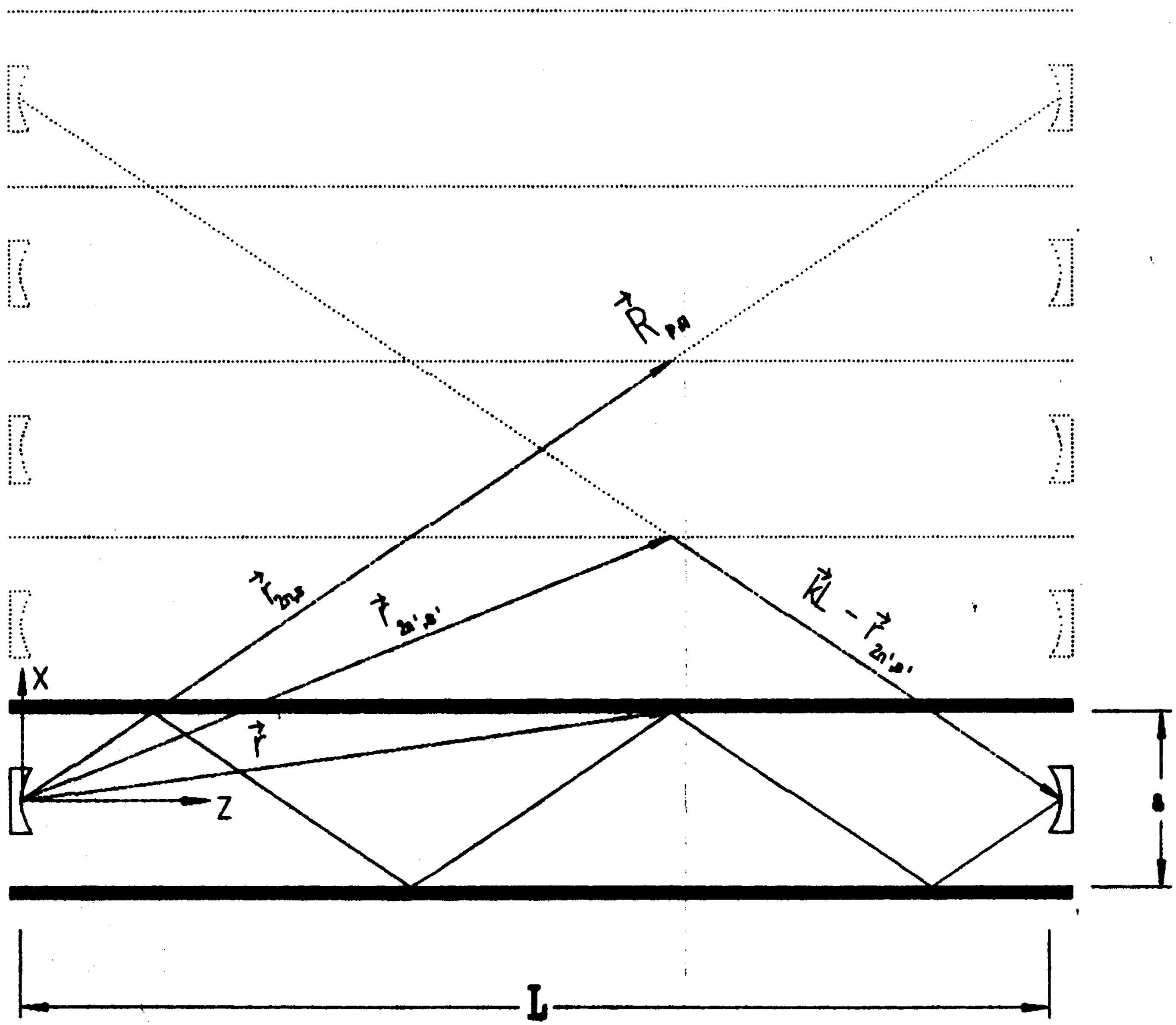
The geometry of a rough circular wall and the notations used in the text.

Fig. 6

Light circulating paths in the random wall geometry. Note the definition of conjugate vectors $\hat{n}$ and $\hat{n}'$.

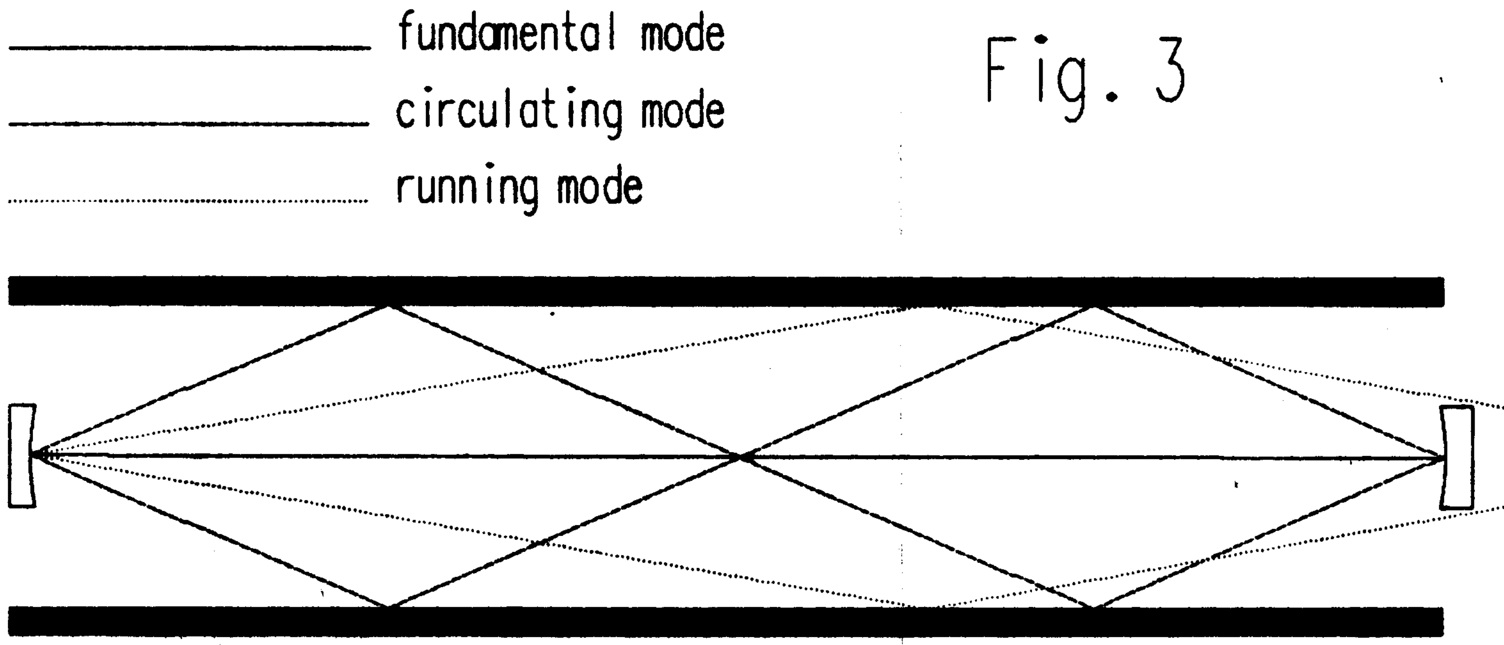


∂Σ
∂Σ₀
$\vec{n}\ \delta\Phi/\delta n$
$\vec{U}(x,y,z)$



$\vec{R}_{pq}$
$\vec{r}_{2n,s}$
$\vec{r}_{2n',s'}$
$\vec{k}L - \vec{r}_{2n',s'}$
X
$\vec{r}$
Z
a
L

Fig. 3



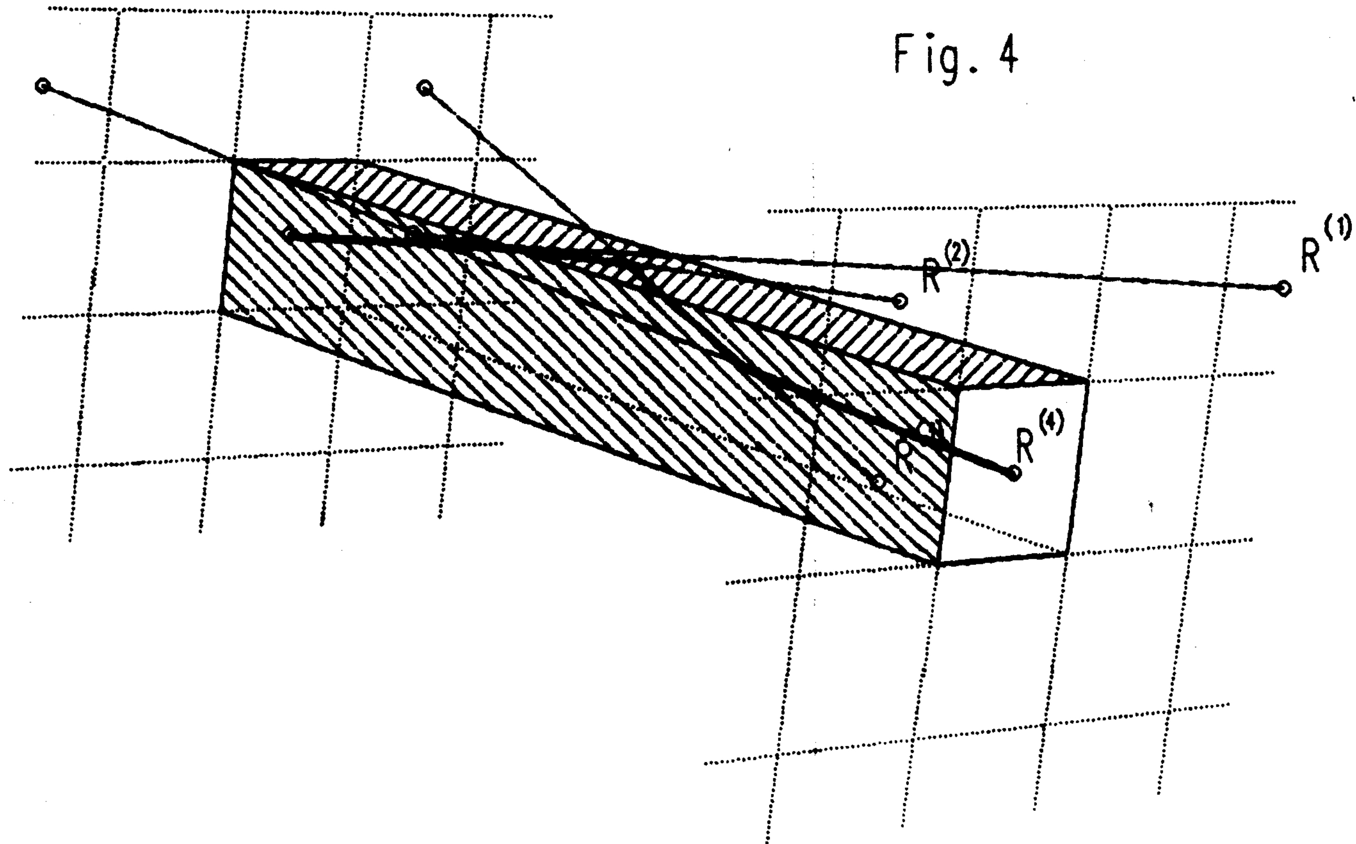
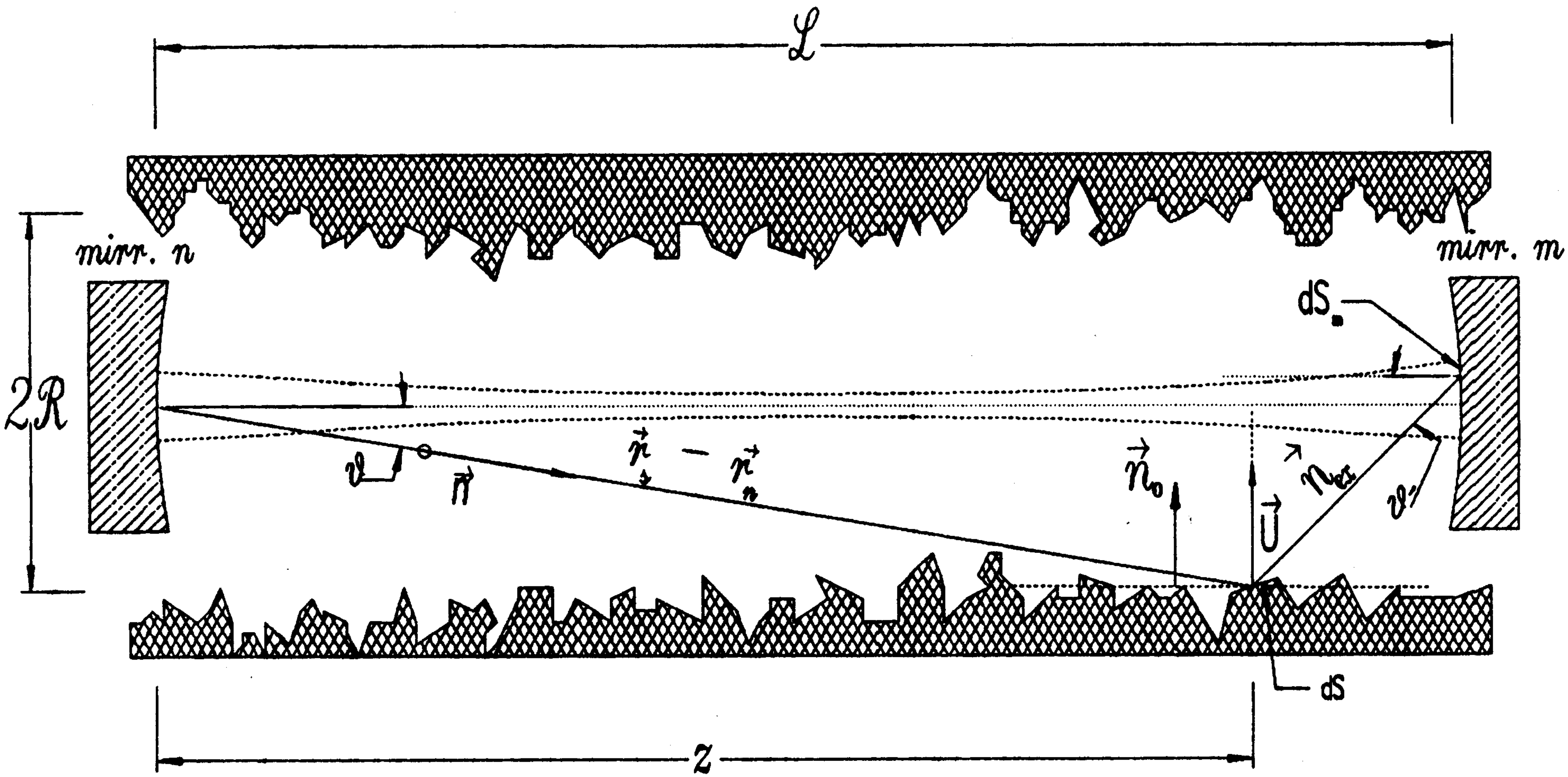
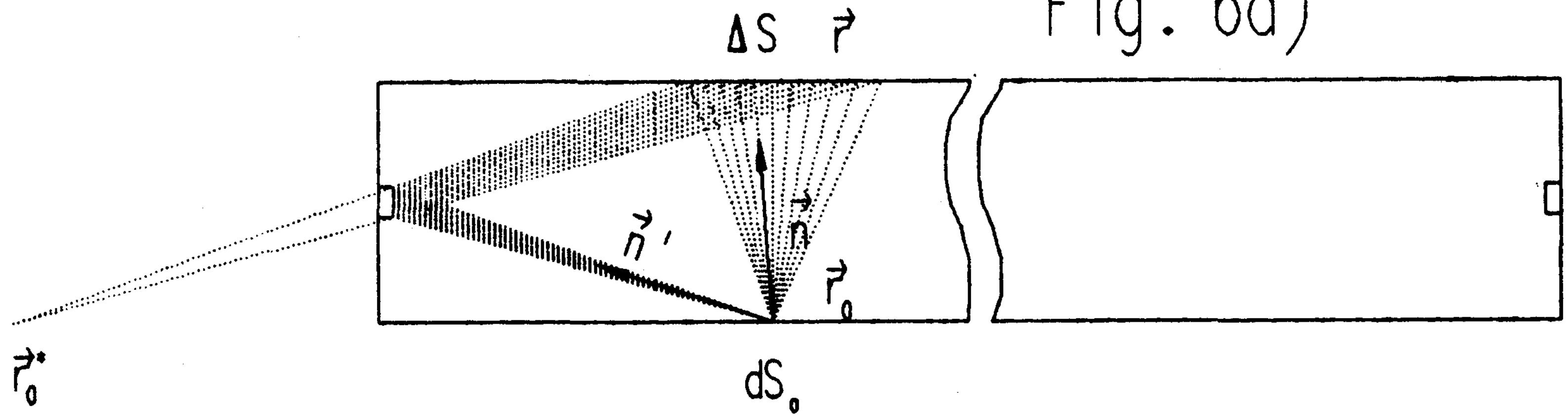


Fig. 4



L
mirr. n
mirr. m
2R
dS
z

Fig. 6a)



6b)