New Folder Name Analysis of Interferometer

Analysis of an Externally Modulated Recycled Interferometer

David Shoemaker and Rainer Weiss

Introduction

The analysis presented here was made in 1990 and led to the modintsm.f program as a means of dealing with the complexity of the algebra. If one would like real results, it is still best to use this program. A second program called gravnoiseplot2.f (gnp2.f modified by dhs to be more user friendly) is a compendium of our present estimates of the interferometer noise from many sources. The shot noise contribution in that program is calculated for perfectly locked but finite contrast interferometers based on the analysis given in this note.

The analysis in this note is organized as follows:

1. The fields in the interferometer

The section defines the symbols and uses the operators defined in the appendix to manipulate the carrier and the phase modulated sideband fields. The carrier and two sideband fields at ω_1 (first order in the Bessel function expansion of the field) are propagated through the interferometer schematized in figure 1. These sidebands, resonant in the recycling cavity, are used to lock the interferometer components: the position of the recycling mirror, the two arm cavity resonances and, from a small path length unbalance between the main beam splitter and the front cavity mirrors, the path length in the Mach - Zehnder interferometer. The Michelson interferometer, comprised of the main beam splitter and the front two cavity mirrors, is maintained in lock by the phase modulated sidebands of a single sideband (frequency shifted) beam designed not to be resonant with the arm cavities but in resonance with the recycling cavity. The detailed analysis of the various discriminants generated by the sidebands of the main carrier and the sidebands on the frequency shifted single sideband, in a basis using the symmetric and antisymmetric motions of the arm cavity mirrors, has been made with modintsm.f and is not included explicitly in the analysis. The calculational procedures to get these discriminants is identical to that used in this note.

The principal results of this section are the field at the antisymmetric port of the main beam splitter (this beam carries the differential phase information from the arm cavities) and the field of the reference beam derived from the symmetric port of the main beam splitter by a pick off between the main splitter and the recycling mirror. The reference beam is modulated by an external phase modulator at modulation frequency ω_3 with a modulation amplitude $\Gamma \approx 1.8$, the maximum of $J_1(\Gamma)$. The modulated reference beam and the antisymmetric output beam are brought to interference in the Mach - Zehnder interferometer with a photodetector at both the symmetric and antisymmetric outputs. The final output of the interferometer is the antisymmetric combination of the two photodetector currents.

2. The shot noise analysis

The shot noise is calculated explicitly for the case of the two arm cavities on resonance, the recycling cavity on resonance and a symmetric main beam splitter. In this analysis no allowance is made for propagation of the shot noise from servo loops that maintain the mirror and beam splitter positions. The imperfections of the interferometer are buried in the finite contrast at the antisymmetric port of the main beam splitter. Photodetector and preamplifier noise are included in the analysis. The shot noise for an idealized interferometer with perfect contrast is given in closed form. For other cases the shot noise is minimized by finding the optimum reflectivity of the reference pick off (analogous to the optimization of the modulation depth in a more conventional interferometer) when transmission of the recycling mirror is optimized. The optimization of the reference pick off parameters, in this model of the interferometer, is independent of the recycling mirror parameters. Again, numerical results are most easily determined by using the gravnoiseplot2.f program. The program calculates the shot noise for several variants of the interferometer design. The text includes a table to compare the phase noise limited gravitational wave sensitivity of these interferometers.

3. The matrix of signals developed to lock the interferometer using the common and differential mode basis.

The various discriminants used to hold the interferometer on lock are calculated with modintsm.f. The beam identifications used in the script for the program are indicated in figure 3. The resultant discriminant matrix assumes an input beam *intensity* at the entrance to the system of unity.

rw_ext_schem.draw

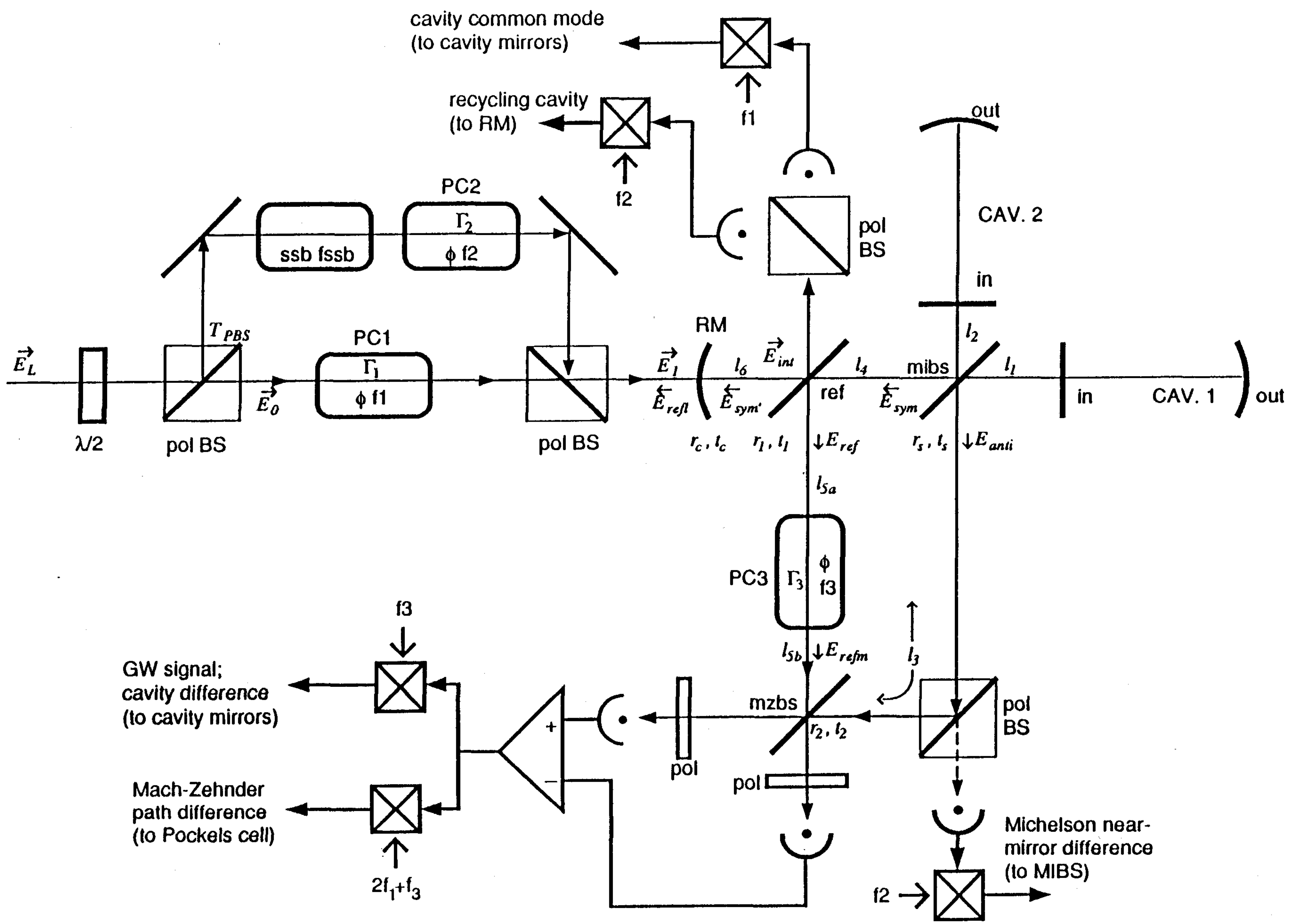

Figure 1:
Schematic of an externally modulated broad-band recycled interferometer
using Fabry-Perot cavities as optical storage elements

The fields in the interferometer

Main interferometer matrix transfer functions

Define several quantities in the main interferometer for ease of calculation and understanding. The unbalance in the main interferometer has symmetric and antisymmetric parts: the Michelson lengths, the amplitude of the main cavity reflection coefficients, and the phase deviation of the main cavities from resonance can be split into symmetric and antisymmetric parts,

$$l_2 = l_0 + \Delta \qquad A_2(\omega_0) = A_0(\omega_0) + \Delta A(\omega_0) \qquad \phi_2(\omega_0) = \phi_s(\omega_0) + \phi_a(\omega_0)$$

$$l_1 = l_0 - \Delta \qquad A_1(\omega_0) = A_0(\omega_0) - \Delta A(\omega_0) \qquad \phi_1(\omega_0) = \phi_s(\omega_0) - \phi_a(\omega_0)$$

In subsequent calculations it is useful to express the main cavity unbalance in terms of the fringe contrast of the optical carrier at the antisymmetric port. The contrast is defined as

$$c = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

where $I_{\max}$ is the intensity of the bright (constructive) fringe at the antisymmetric port and $I_{\min}$ is the intensity in the dark fringe. These quantities are measured on the optical carrier and taken to the limit of the RF modulation approaching 0. In the external modulation system the contrast with the modulation on or off should be close to the same value.

The contrast and the main cavity unbalance are related as

$$c = \frac{((1 - (\frac{\Delta A(\omega_0)}{A_0(\omega_0)})^2)}{(1 + (\frac{\Delta A(\omega_0)}{A_0(\omega_0)})^2))} \qquad \Delta A(\omega_0) = \sqrt{\frac{(1-c)}{(1+c)}} A_0(\omega_0)$$

In the limit of the perfect locked interferometer with losses only in the main cavity mirrors $\Delta = 0$, $c = 1$, and $A_0(\omega_0) \approx 1 - \frac{2(Am_1 + Am_2)}{Tm_1}$. Am_1 and Am_2 are the intensity loss due to absorption and scattering in the main cavity mirrors and Tm_1 is the intensity transmission of the main cavity input mirror (valid approxiation if $Tm_1/(Am_1 + Am_2) > 10$). The cavity reflection amplitude reexpressed in terms of the main cavity energy storage and transit times is $A_0(\omega_0) \approx 1 - (Am_1 + Am_2)\tau_{st}/\tau_{trans}$.

The properties of the main beam splitter can also be given in terms of symmetric and antisymmetric parts,

$$r_s^2 = R_s = S_s + S_a \qquad t_s^2 = T_s = S_s - S_a$$

so that

$$R_s + T_s = 2S_s \qquad R_s - T_s = 2S_a$$

In the perfect interferometer using a completely symmetric beam splitter with negligible loss $S_s = 1/2$ and $S_a = 0$.

The ratio of the output field at the antisymmetric port to the input field at the main beam splitter is

$$T_{anti} = r_s t_s \left(P(2l_2) Rc_2 - P(2l_1) Rc_1 \right)$$

where Rc_j designates the complex reflection coefficient of arm cavity j.

The antisymmetric transfer function matrix is

$$T_{\rm anti} = r_s t_s [2k_0 l_0] *$$

$$\begin{pmatrix} [2k_1 l_0] 2i\sin(2(k_0+k_1)\Delta) \\ -2[\phi_s(\omega_0)]A_0(\omega_0)\left(i\sin(2k_0\Delta+\phi_a(\omega_0)) + \sqrt{\frac{(1-c)}{(1+c)}}\cos(2k_0\Delta+\phi_a(\omega_0))\right) \\ [-2k_1 l_0] 2i\sin(2(k_0-k_1)\Delta) \end{pmatrix}$$

The perfect interferometer when locked has no optical field leaving the antisymmetric port at either the carrier or the sidebands generated by PC1.

The ratio of the output field at the symmetric port to the input field at the main beam splitter is

$$T_{\rm sym} = \left(r_s^2 P(2l_2)Rc_2 + t_s^2 P(2l_1)Rc_1\right) = S_s\left(P(2l_2)Rc_2 + P(2l_1)Rc_1\right) + \left(\frac{S_a}{r_s t_s}\right)T_{\rm anti}$$

The symmetric transfer function matrix is

$$T_{\rm sym} = [2k_0 l_0] *$$

$$\begin{pmatrix} 2[2k_1 l_0]\left(S_s\cos(2(k_0+k_1)\Delta) + iS_a\sin(2(k_0+k_1)\Delta)\right) \\ -2[\phi_s(\omega_0)]A_0(\omega_0)\left(S_s\left(\cos(2k_0\Delta+\phi_a(\omega_0)) + i\sqrt{\frac{(1-c)}{(1+c)}}\sin(2k_0\Delta+\phi_a(\omega_0))\right)\right. \\ \left. +S_a\left(i\sin(2k_0\Delta+\phi_a(\omega_0)) + \sqrt{\frac{(1-c)}{(1+c)}}\cos(2k_0\Delta+\phi_a(\omega_0))\right)\right) \\ 2[-2k_1 l_0]\left(S_s\cos(2(k_0-k_1)\Delta) + iS_a\sin(2(k_0-k_1)\Delta)\right) \end{pmatrix}$$

The perfect interferometer (c=1) when locked returns $A_0(\omega_0)$ of the optical carrier and unattenuated sidebands to the recycling mirror.

The fields outside of the main interferometer

The phase reference is chosen at the recycling mirror.

The field incident on the recycling mirror from the laser is

$$E_1 = M(\Gamma_1, \omega_1)E_0$$

where E_0 is the laser field incident on the input beam phase modulator PC1.

The optical feedback provided by the recycling mirror is given by

$$E_{\rm int} = r_c E_{\rm sym'} + t_c E_1$$

where $E_{\rm int}$ is the total internal recycled field leaving the recycling mirror toward the main interferometer and $E_{\rm sym'}$ is the field returned to the recycling mirror from the main interferometer after passing through the reference pickoff splitter.

The field reflected by the recycling mirror is

$$E_{\mathrm{refl}} = t_c E_{\mathrm{sym}'} - r_c E_1$$

The recycling forward transfer function is the ratio of the field returned from the interferometer at the recycling mirror, $E_{\mathrm{sym}'}$, to the internal field, E_{int}, given by

$$T_{\mathrm{recy}} = t_1^2 P(2(l_4 + l_6)) T_{\mathrm{sym}}$$

The matrix formulation of the recycling forward transfer function is

$$T_{\mathrm{recy}} =$$

$$\begin{pmatrix} 2t_1^2[2(k_0+k_1)(l_0+l_4+l_6)]\Big(S_s\cos(2(k_0+k_1)\Delta) + iS_a\sin(2(k_0+k_1)\Delta)\Big) & \\ -2t_1^2[2k_0(l_0+l_4+l_6)][\phi_s(\omega_0)]A_0(\omega_0)\left(S_s\left(\cos(2k_0\Delta+\phi_a(\omega_0)) + i\sqrt{\frac{(1-c)}{(1+c)}}\sin(2k_0\Delta+\phi_a(\omega_0))\right) \right. & \\ \left. +S_a\left(i\sin(2k_0\Delta+\phi_a(\omega_0)) + \sqrt{\frac{(1-c)}{(1+c)}}\cos(2k_0\Delta+\phi_a(\omega_0))\right)\right) & \\ 2t_1^2[2(k_0-k_1)(l_0+l_4+l_6)]\Big(S_s\cos(2(k_0-k_1)\Delta) + iS_a\sin(2(k_0-k_1)\Delta)) \end{pmatrix}$$

To optimize the recycling cavity for the carrier, the path length, $(l_4 + l_6 + l_0)$, is chosen to make $[2k_0(l_4 + l_6 + l_0)] = -1$. To simultaneously optimize the recycling cavity for the sidebands at ω_1, the modulation frequency and lengths are chosen to make $[2k_1(l_4 + l_6 + l_0)] = -1$, then $[2(k_0 \pm k_1)(l_4 + l_6 + l_0)] = 1$ which satisfies the condition that the arm cavities have a phase inversion on reflection at the carrier resonance and (approximately) no phase shift off resonance.

When the perfect interferometer is locked the recycling forward transfer function matrix element for the optical carrier becomes $t_1^2 A_0(\omega_0)$. The ideal value approaches 1 for an interferometer with small losses in the main cavities, high contrast and therefore small power removed by the reference pickoff. The optimum value in a real interferometer will be less than 1 and is primarily determined by the main cavity loss and the imperfect contrast. The contrast determines the optimum value for the reference pickoff reflection coefficient, r_1.

The recycling feedback closed loop gain matrix, the ratio E_{int}/E_1, is determined from the above definitions as

$$G_{\mathrm{recy}} = \frac{t_c}{(1 - r_c T_{\mathrm{recy}})}$$

The perfect interferometer would have a recycling mirror with a reflection coefficient $r_c \approx t_1^2 A_0(\omega_0)$. The recyling field gain for the carrier becomes $G_{\mathrm{recy}} \approx 1/\sqrt{1 - t_1^2 A_0^2(\omega_0)} \approx 1/\sqrt{\text{total power loss}}$ when the interferometer is locked. The imperfect contrast of a real interferometer changes this substantively.

With the above definitions, the explicit formulation of the vector fields in the interferometer and at the various output ports in terms of the input laser field with phase reference at the recycling mirror are:

The internal field,

$$E_{int} = G_{recy} M(\Gamma_1, \omega_1) E_0$$

The antisymmetric output field at the main splitter

$$E_{anti} = t_1 G_{recy} T_{anti} P(l_4 + l_6) M(\Gamma_1, \omega_1) E_0$$

The reference field at the pickoff,

$$E_{ref} = -r_1 t_1 G_{recy} P(2l_4 + l_6) T_{sym} M(\Gamma_1, \omega_1) E_0$$

The reflected field at the recycling mirror,

$$E_{refl} = (G_{recy} t_c T_{recy} - r_c) M(\Gamma_1, \omega_1) E_0$$

The field returning from the interferometer to the recycling mirror,

$$E_{sym'} = G_{recy} T_{recy} M(\Gamma_1, \omega_1) E_0$$

The fields at the interferometer output

The main output of the interferometer is developed in the Mach - Zehnder interferometer comprised of path lengths l_3, l_4 and l_5. The reference and antisymmetric output beams are recombined on splitter mzbs and photodetectors are placed at both the symmetric and antisymmetric port of mzbs. The reference beam has sidebands at ω_3 impressed on it by electroptic modulator PC3. The reference field incident on the secondary splitter is

$$E_{refm} = P(l_{5a}) M(\Gamma_3, \omega_3) P(l_{5b}) E_{ref}$$

The notation l_{5a} and l_{5b} indicates 5th length divided into a path before and after the modulator.

The principal antisymmetric output beam at the secondary splitter

$$E_{antimz} = P(l_3) E_{anti}$$

The fields incident on the output photodetectors are

$$E_{PDA1} = t_2 E_{refm} - r_2 E_{antimz} \qquad E_{PDA2} = r_2 E_{refm} + t_2 E_{antimz}$$

The optical power on the two output photodetectors when $r_2 = t_2 = 1/\sqrt{2}$ is

$$P_{PDA1} = 1/2(E_{refm} E^*_{refm} - E_{refm} E^*_{antimz} - E^*_{refm} E_{antimz} + E_{antimz} E^*_{antimz})$$
$$P_{PDA2} = 1/2(E_{refm} E^*_{refm} + E_{refm} E^*_{antimz} + E^*_{refm} E_{antimz} + E_{antimz} E^*_{antimz})$$

The detected photocurrents are proportional to the incident power. The shot noise fluctuations are uncorrelated in the two detectors but the excess amplitude noise in the light is correlated and of equal sign in the two detectors. The main interferometer output is of opposite sign in the two detectors. The main output is the difference in the photocurrents of the two detectors proportional to the difference in power

$$P_{PDA2} - P_{PDA1} = E_{refm} E^*_{antimz} + E^*_{refm} E_{antimz}$$

Here it would be appropriate to calculate explicit expressions for the power on the photodetectors but the algebra for the complete case is a mess. The program modintsm.f comes to the rescue if one wants numerical results. In the next section explicit expressions are derived for the special case of the three cavities being on resonance.

Phase noise in the interferometer

Assumptions

In order to reduce the algebra several assumptions have been made:

1) The interferometer components are held in position to optimize the differential output signal. The noise from the servo systems used to hold the interferometer in this locked state is not included in the calculation. The modintsm.f analysis of the discriminants shows this to be a reasonable approximation.

2) The main and Mach - Zehnder beam splitters are symmetric, $S_a = 0$.

3) The modulation index, $\Gamma_1 < 0.1$, so that the unbalance in the ω_1 sidebands at the main antisymmetric port is neglected relative to the unbalance of the carrier. Also, the ω_1 sidebands are neglected in the reference beam. This latter assumption, though fair for the phase noise calculation, is not a good approximation in calculating the discriminant to hold the Mach - Zehnder interferometer in lock. Furthermore, the frequency shifted single sideband and its sidebands at ω_2 are neglected in the analysis since these are in an orthognal linear polarization to the main beam and not part of the signal on the photodetectors at the Mach - Zehnder output. The existence of these fields is, however, acknowledged by including the power removed from the main beam by the input polarizing beam splitter which is assigned an intensity transmission for the main beam designated by $T_{\rm pbs} = t_{\rm pbs}^2$.

4) The Michelson path difference $\Delta = 0$. This assumption is slightly more dubious than the others in this list but is compensated by the following argument. The only reason for a non zero Δ is to provide a discriminant to hold the path length difference $l_5 - l_3$ of the Mach - Zehnder interferometer in lock if the main interferometer has a contrast of 1. If the main interferometer has imperfect contrast, the carrier unbalance can be used to derive this discriminant with $\Delta = 0$. In either case the interferometer main output signal is only second order sensitive to the Mach - Zehnder path length difference when properly set.

Should one want the phase noise without any assumptions, this can be calculated directly with the modintsm.f program which determines both the discriminant at the modulation frequency and the total "DC" intensity at the photodetector placed at any port of the interferometer.

The approximate fields

With the above assumptions and the results given in the prior section for the fields external to the main interferometer, the two relevant output fields are

$$E_{\rm anti} = a\Big(i\sin(\phi) + b\Big)$$

$$E_{\rm ref} = -r_1 a \qquad \text{(pick off)}$$

where $\phi = \phi_a$ and the functions a and b are defined as

$$a = \frac{t_1 t_c A_0(\omega_0) J_0(\Gamma_1) t_{\rm pbs} E_{\rm L}}{1 - r_c t_1^2 A_0(\omega_0)}$$

$E_{\rm L}$ is the field leaving the mode cleaner and incident on the $\lambda/2$ wave plate at the interferometer input,

$$b = \left(\frac{1-c}{1+c}\right)^{1/2}$$

These fields appear at the input port of the beam splitter of the Mach - Zehnder, mzbs, as

$$E_{\mathrm{refm}} = -r_1 a\Big(J_0(\Gamma_3) - 2iJ_1(\Gamma_3)\sin(\omega_3 t) + H\Big)[\psi_\mathrm{r}]$$

where H is a sum of the higher order Bessel function terms, eventually to be eliminated by the Bessel function sum rules, and ψ_r is the optical phase (assumed only for the carrier) accumulated in the reference arm of the Mach - Zehnder intereferometer, and

$$E_{\mathrm{antimz}} = a\Big(i\sin(\phi) \; + \; b\Big)[\psi_\mathrm{a}]$$

where ψ_a is the optical phase accumulated in the signal arm of the Mach - Zehnder intereferometer.

The approximate intensities at the Mach - Zehnder photodetectors

When $\phi \rightarrow 0$, the self conjugate terms in the intensity at either photodetector becomes

$$1/2\Big(E_{\mathrm{refm}}E^*_{\mathrm{refm}} + E_{\mathrm{antizm}}E^*_{\mathrm{antizm}}\Big) = 1/2\Big(r_1^2 + b^2\Big)a^2$$

The cross term becomes

$$1/2\Big(E_{\mathrm{refm}}E^*_{\mathrm{antizm}} + E^*_{\mathrm{refm}}E_{\mathrm{antizm}}\Big) =$$

$$r_1 a^2\Big(J_0(\Gamma_3)\;\phi\;\sin(\psi) + 2J_1(\Gamma_3)\;\phi\;\cos(\psi)\sin(\omega_3 t) + H\;\phi\;\sin(\psi)$$

$$-b\;J_0(\Gamma_3)\cos(\psi) + 2J_1(\Gamma_3)\;b\;\sin(\psi)\sin(\omega_3 t) - b\;H\cos(\psi)\Big)$$

where $\psi = \psi_\mathrm{a} - \psi_\mathrm{r}$ is the relative optical phase difference for the optical carrier.

The optical phase difference is chosen $\psi = 0,\ \pi$ so that all the terms in $\sin(\psi)$ vanish. The choice of 0 or π depends on the type of contrast defect. If the beams at the reference port and the antisymmetric port of the interferometer are in the same spatial mode (this depends on the type of mechanism causing the contrast defect), it is possible to reduce the net "DC" intensity on one of the photodetectors by making the proper choice of ψ and this could lead to a better phase noise limit than calculated here. In the remainder of this analysis $\psi = 0$ and the "DC" term in the intensity cross term is neglected (incoherent modes).

Here it is worth noting that external cavities in the arms of the Mach - Zehnder interferometer may, in future versions of such interferometers, have some useful properties. A mode cleaning cavity in the signal arm, besides reducing the noise from tube and component scattering, could be used to improve the contrast by passing only the main spatial mode of the interferometer which is dominant in the reference beam. Another concept that has been analysed and incorporated into the gravnoiseplot2.f program is to reduce the unnecessary carrier component in the reference beam by driving the modulator PC3 to have $\Gamma_3 = 2.4$ where $J_0(\Gamma_3) = 0$, the carrier is suppressed, and placing an optical filter cavity tuned to pass only the $\pm\omega_3$ sidebands between PC3 and the Mach - Zehnder beam splitter. This is expected to improve the shot noise amplitude limit by $\sqrt{2}$.

The "DC" and "AC" intensity terms on one detector become

$$P(0) = 1/2\Big(r_1^2 + b^2\Big)a^2$$

$$P(\omega_3) = 2r_1 a^2 J_1(\Gamma_3)\;\phi\;\sin(\omega_3 t)$$

The approximate interferometer phase noise

The shot noise due to $P(0)$ on the two detectors is uncorrelated and when the output of the two detectors is differenced, the shot noise powers add. The "AC" signal powers have opposite sign on the two detectors and the differenced output is double that of a single detector. A coherent (sometimes called excess) amplitude noise in the light is reduced by the differencing.

The net signal expressed as a photodetector current is

$$I_{\rm sig}(\omega_3) = \Big(e\eta/h\nu\Big)\Big(4r_1 a^2 J_1(\Gamma_3)\ \phi\ \sin(\omega_3 t)\Big)$$

e is the charge on the electron, h Planck's constant and ν the optical frequency.

When calculating the phase noise, it is useful to express ϕ as a spectral amplitude $\phi(f)$. After phase detection in the mixer, a phase noise amplitude $\phi(f)$ in the interferometer would produce a current noise amplitude $i(f)$

$$i_{\rm sig}(f) = \Big(e\eta/h\nu\Big)\Big(2r_1 a^2 J_1(\Gamma_3)\ \phi(f)\ \delta(f)\Big)$$

since $< \sin^2 >= 1/2$.

The shot noise current power spectrum summed for the two detectors before demodulation is

$$i^2_{\rm shot}(\omega_3 \pm f) = \Big(2e^2\eta/h\nu\Big)a^2(r_1^2 + b^2)$$

The photodetector and preamplifier noise for the differenced detectors expressed in terms of an input current noise is

$$i^2_{\rm elec}(\omega_3 \pm f) = \Big(2e^2_{\rm amp}(\omega_3 \pm f)/Z^2(\omega_3 + f) + 4kT\ Re(Z(\omega_3 \pm f))/Z^2(\omega_3 + f) + i^2_{\rm amp}(\omega_3 \pm f) + 2eI_{\rm dark}\Big)$$

$e_{\rm amp}(f)$ and $i_{\rm amp}(f)$ are the voltage and current noise of the amplifier at the modulation frequency, $Z(f)$ is the photodetector load impedance and $Re(Z(f))$ is its real part, k is Boltzmann's constant and T is the load impedance's absolute temperature, finally, $I_{\rm dark}$ is the photodetector dark current.

The relevant quantity, to determine the equivalent phase noise power in the interferometer associated with the shot noise and the electronics noise power, is the rms current noise after demodulation. The shot noise and electronics noise power in both sidebands at $\omega_3 \pm f$ are added in the demodulation, the equivalent rms noise power in the detection bandwidth that ultimately defines $\phi^2(f)$ is therefore doubled. This is accounted for by equating the rms noise power of the signal integrated over the bandwidth ΔB with the rms shot noise and electronics noise power spectrum over a bandwidth $2\Delta B$. The equivalent phase noise power spectrum in the interferometer becomes

$$\phi^2(f) = \Big(\frac{h\nu}{4\eta r_1^2 a^2 J_1^2(\Gamma_3)}\Big)\Big((r_1^2 + b^2) + (\frac{h\nu}{e^2\eta})\frac{i^2_{\rm elect}(f)}{a^2}\Big)$$

The idealized phase noise power for perfect contrast, $b = 0, c = 1$, no electronics noise, $i^2_{\rm elec} = 0$, $r_c = A_0(\omega_0)$ and $t_1^2 \approx 1$ becomes

$$\phi^2_{\rm ideal}(f) = \frac{h\nu(1 - A_0^2(\omega_0))}{4\eta A_0^2(\omega_0) J_0^2(\Gamma_1) J_1^2(\Gamma_3) t^2_{\rm pbs} E_L^2}$$

For some what less idealized interferometers, providing that the electronics noise can still be neglected, it is possible to minimize the phase noise power by choosing the recycling mirror reflection coefficient so that

$$r_c = t_1^2 A_0(\omega_0).$$

Providing the loss in the recycling mirror can be neglected, the phase noise power becomes

$$\phi^2(f) = \left(\frac{h\nu}{4\eta}\right)\left(\frac{(1 - t_1^4 A_0^2(\omega_0))(r_1^2 + b^2)}{r_1^2 t_1^2 A_0^2(\omega_0) J_1^2(\Gamma_3) J_0^2(\Gamma_1) t_{\text{pbs}}^2 E_L^2}\right)$$

The reference pickoff intensity reflectivity, $R = 1 - t_1^2$, which minimizes the above expression, is a solution of the cumbersome fourth order equation

$$R^4 - 2R^3 + R^2\frac{(1 + A(b^2 + 1))}{A} + 2R\frac{b^2(1 - A)}{A} - b^2\frac{1 - A}{A}$$

here $A = A_0^2(\omega_0)$. Unfortunately, all but the quartic term must be considered for our range of parameters. Figure 2a shows the ratio of the more realistic phase noise power to the idealized phase noise power as function of the contrast defect $1 - c$. Figure 2b plots the solutions of the quartic equation for R as a function of the contrast defect. The calculation assumes that the electronics noise remains negligible. This assumption should be checked as one gets closer to fixing a design.

The interferometer strain noise

The antisymmetric phase change, $\phi_a = \phi$, used in deriving the interferometer fields in section 1 and 2 of this note is the phase change in one cavity. The relation of the gravitational strain to the phase change is

$$h(f) = \frac{\delta h}{\delta \phi}(f)\phi(f)$$

where the interferometer transfer function, providing the arm cavity input mirror intensity transmission is much larger than the sum of the mirror intensity losses, is

$$\frac{\delta h}{\delta \phi}(f) = \left(\frac{\tau_{\text{optical}}}{4\pi\tau_{\text{store}}}\right)\sqrt{(1 + (f/f_0)^2)}$$

with the corner frequency

$$f_0 = \frac{1}{4\pi\tau_{\text{store}}}$$

In terms of the arm cavity mirror parameters, the cavity reflection amplitude on resonance, $A_0(\omega_0)$, and the arm cavity storage time, τ_{store}, are defined as

$$A_0^2(\omega_0) = 1 - 4(A_1 + A_2 + T_2)/T_1$$

where A_i are the mirror intensity loss due to both scattering and absorption and T_i are the mirror intensity transmissions,

$$\tau_{\text{store}} = \frac{2L}{c(T_1 + T_2 + A_1 + A_2)}$$

Figure 2c shows the strain amplitude spectral density for a variety of contrasts calculated by using the above formalism with the following parameters: $A_1 = A_2 = 1.0 \times 10^{-4}$, $T_1 = 3.0 \times 10^{-2}$, $T_2 = 0$, $\eta E_L^2 = 2$ watts, $T_{pbs} = .81$, $\Gamma_1 = 0.1$, $\Gamma_3 = 1.8$, $L = 4.0 \times 10^5$ cm, $\lambda = 5.145 \times 10^{-5}$ cm. The derived parameters are $A_0^2(\omega_0) = 9.73 \times 10^{-1}$, $\tau_{\text{store}} = 8.84 \times 10^{-4}$ sec, $f_0 = 90.0$ sec.

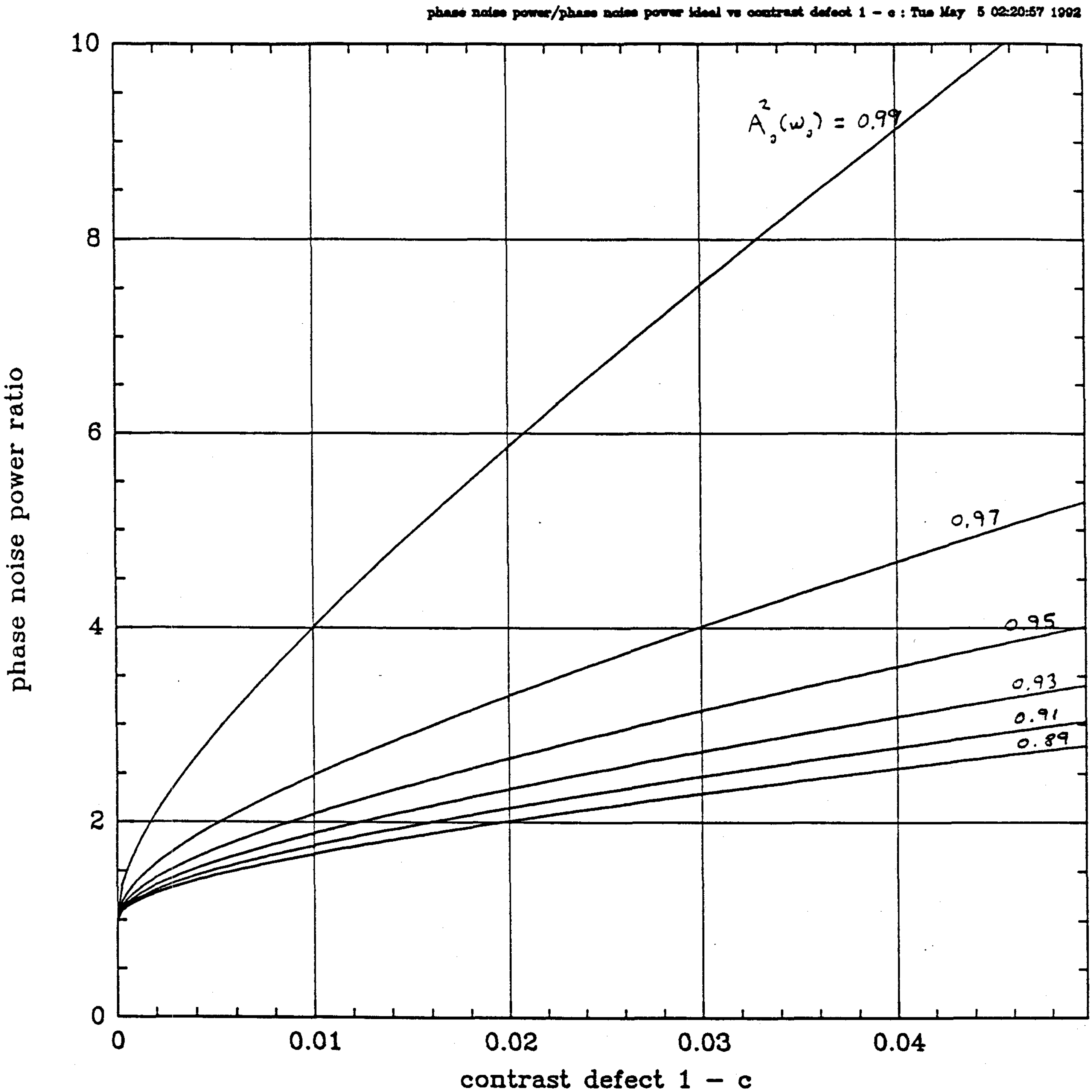
phase noise power/phase noise power ideal vs contrast defect 1 − c : Tue May 5 02:20:57 1992
phase noise power ratio
contrast defect 1 − c
0
2
4
6
8
10
0.01
0.02
0.03
0.04
$A_0^2(\omega_0) = 0.99$
0.97
0.95
0.93
0.91
0.89

FIGURE 2a

pickoff reflectivity vs contrast defect 1 - c : Tue May 5 02:12:42 1992

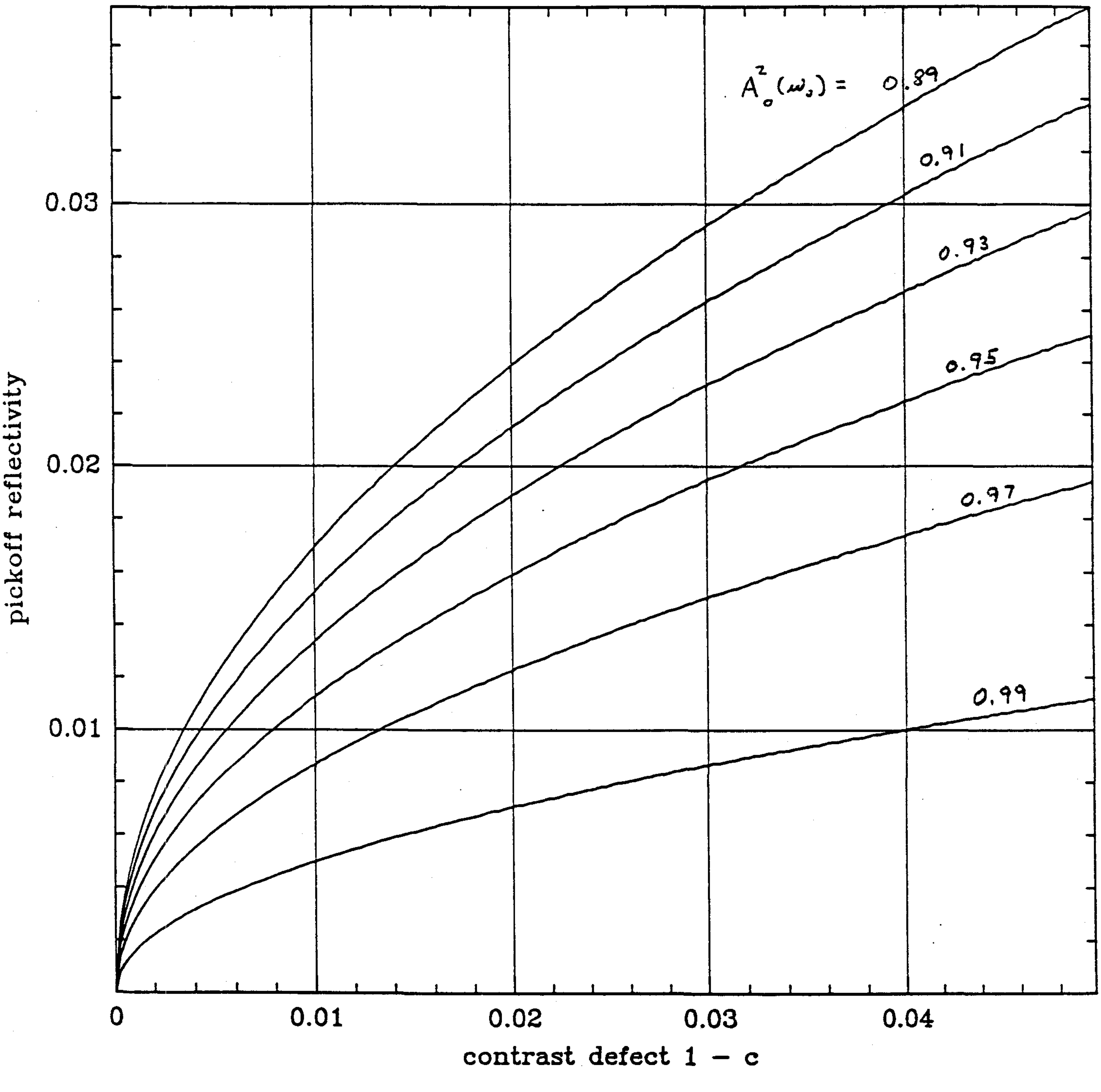

FIGURE 2b

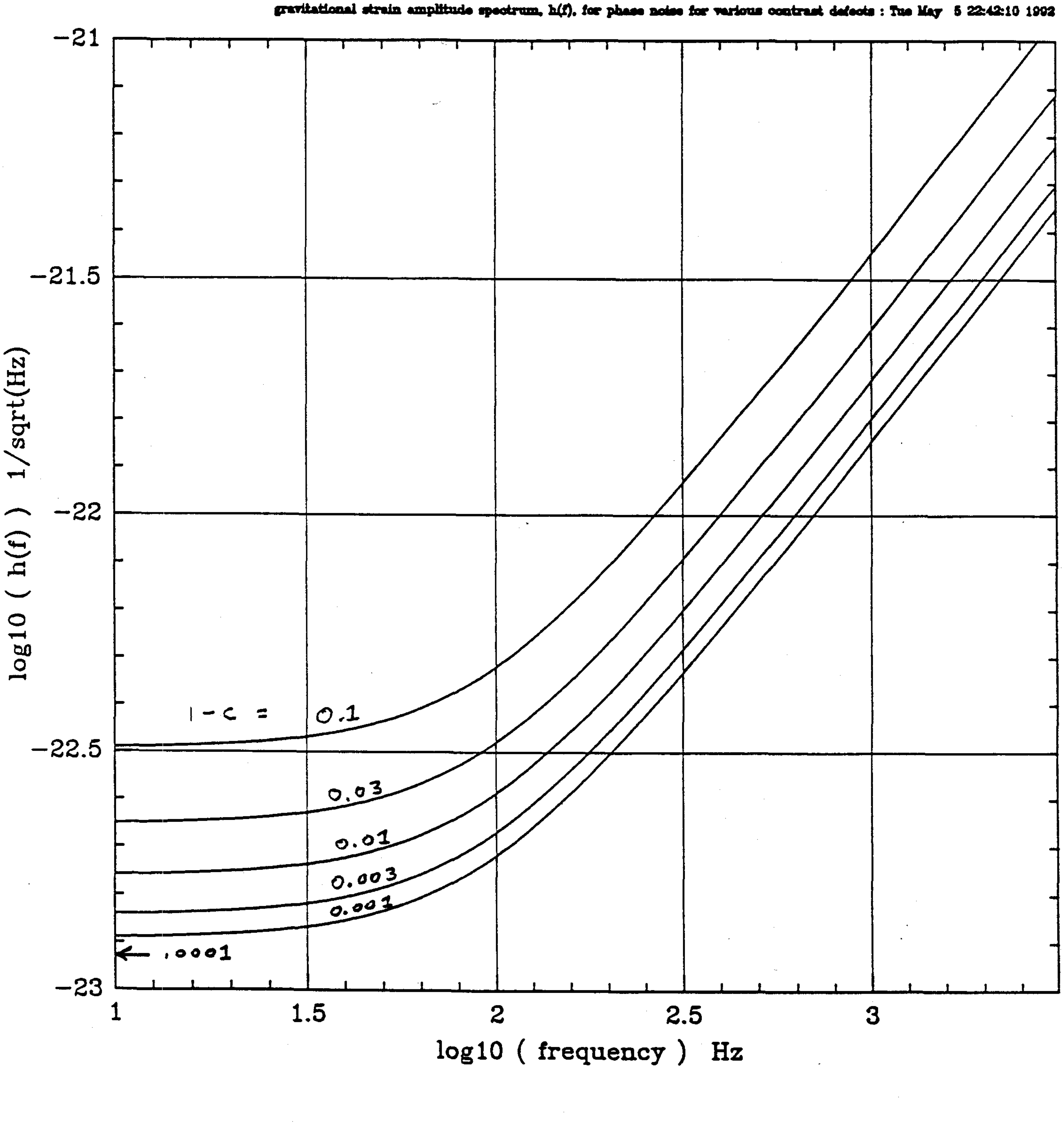
gravitational strain amplitude spectrum, h(f), for phase noise for various contrast defects : Tue May 5 22:42:10 1992
-21
-21.5
-22
-22.5
-23
log10 (h(f)) 1/sqrt(Hz)
1
1.5
2
2.5
3
log10 (frequency) Hz
1 - c = 0.1
0.03
0.01
0.003
0.001
.0001

FIGURE 2c

Interferometer Parameters

The table below gives a comparison of various externally modulated interferometers using the numerical output of gravnoiseplot2.f. All the interferometers have a contrast defect, $1 - c = 1 \times 10^{-3}$, and the parameters (arm cavity mirrors, optical power, etc) specified in the text.

?

Interferometer 1: Is the configuration analysed in the text with the reference pickoff beam lost on both sides of the reference splitter.

Interferometer 2: Has a return mirror on the unused side of the reference pickoff which is interferometrically controlled to return the unused reference beam into the interferometer with the proper phase. There are Brewster angle pickoffs between the arm cavities and the main beam splitter with a two sided reflectivity of 2×10^{-4}.

Interferometer 3: The same configuration as interferometer 2 but the electro-optic modulator PC3 in the Mach - Zehnder interferometer is driven with $\Gamma_3 = 2.4$ to suppress the carrier in the reference beam An optical cavity with total transmission 0.8 for the $\pm\omega_3$ sidebands is placed between PC3 and the Mach - Zehnder beam splitter.

	h(100Hz) $1/\sqrt{\text{Hz}}$	G^2_{recy}	R_1	T_c	arm cav watts	recyc cav watts	elect/shot
interf.1	1.88×10^{-23}	29.1	2.0×10^{-3}	3.4×10^{-2}	3.06×10^{3}	47.1	.040
interf.2	1.36×10^{-23}	33.1	1.0×10^{-3}	3.3×10^{-2}	3.48×10^{3}	53.6	.037
interf.3	1.10×10^{-23}	33.1	1.0×10^{-3}	3.3×10^{-2}	3.48×10^{3}	53.6	.049

Note: The limiting value (no contrast defect) of interferometer 1 with the chosen mirror parameters, optical efficiency and laser power is

$$h(100) = 1.62 \times 10^{-23} \quad 1/\sqrt{\text{Hz}}$$

recyextmod_modint.draw

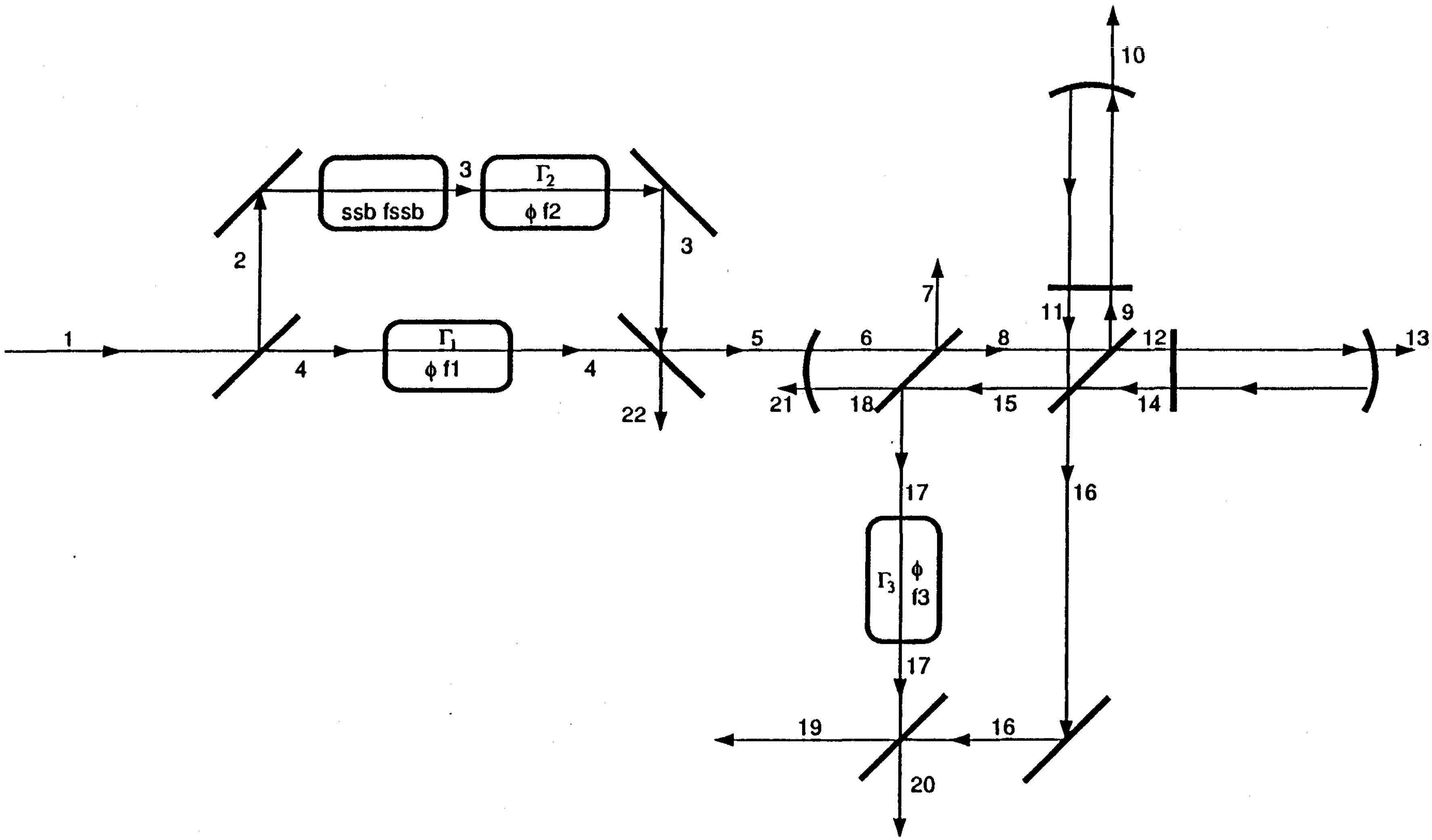

Figure 3:
Model of the interferometer
numbers indicate beam index in the MODINTSM script

	common mode cavity	diff mode cavity	common mode Michelson	diff mode Michelson	MZ arm
ref splitter f_1 (173°)	**972** **0.136**	0.0 0.136	6.17 0.136	0.0 0.136	0.0 0.136
MZ output f_3 (180°)	0.0 0.1024*	**4440** **0.1024***	0.0 0.1024*	38.68 0.1024*	0.0 0.1024*
ref. splitter SSB, f_2 (15°)	0.0 0.00388	0.0 0.00388	**0.0598** **0.00388**	0.0 0.00388	0.0 0.00388
Mich output SSB, f_2 (−35°)	0.0 0.000268	0.0 0.000268	0.0 0.000268	**0.3115** **0.000268** (2.873×10^{-4} **offset)**	0.0 0.000268
MZ output $2f_1 \pm f_3$ (180°)	0.0 0.1024*	0.0 0.1024*	0.0 0.1024	0.0 0.1024	**0.0021** **0.1024**

The entries in the table are $(\Delta I/I)/\lambda$ for the upper entry (slope of discriminant), and $\Delta I/I_{in}$ for the lower entry (DC photocurrent). * indicates the sum of both output photodetectors $I_{\text{pd sym}} + I_{\text{pd anti}}$

The conditions for the matrix are the following:

At input to rec. mirror
$I/I_{laser} = 0.81$
$I_{ssb}/I_{laser} = 0.01$

Modulation:
$\Gamma_1 = 0.1$
$\Gamma_2 = 0.8$
$\Gamma_3 = 1.8$

$\Delta l = 1.4$ cm
$f_1 = 31.197$ MHz
$f_2 = 87.352$ MHz
$f_3 = 5.38$ MHz
$f_{ssb} = 6.239$ MHz
$f_{fsr} = 12.48$ MHz

$T_{\text{arm m1}} = 3 \times 10^{-2}$
$A_{\text{arm m1}} = 1 \times 10^{-4}$
$T_{\text{arm m2}} = 3 \times 10^{-2}$
$A_{\text{arm m2}} = 1 \times 10^{-4}$
$T_{\text{recycl}} = 5 \times 10^{-2}$
$R_{\text{refpick}} = 1 \times 10^{-2}$
$A_{\text{recycl}} = 1 \times 10^{-4}$
$A_{\text{refpick}} = 1 \times 10^{-4}$
$A_{\text{beamsp}} = 1 \times 10^{-4}$

Michelson:
$\frac{l_1+l_2}{2} = 1201.1$ cm

Arm cavities:
$\frac{L_1+L_2}{2} = 4 \times 10^5$ cm

$l_{MZ\ sig} = 1120.03$ cm

$l_{ref} - l_{bs} = 500.0$ cm

$l_{ref\ mz} = 620.03$ cm

APPENDIX

Definition of the operators

Notation: $[x] = e^{ix}$

In order to make the algebra more compact, it is useful to write all the fields of one polarization in terms of a set of basis components of the optical carrier and the sidebands. The propagating fields are vectors with components defined as

$$E = [\omega_0 t] \begin{vmatrix} [(\omega_1 + \omega_3)t] \\ [(\omega_1 + \omega_2)t] \\ [(\omega_1 - \omega_3)t] \\ [(\omega_1 - \omega_2)t] \\ [2\omega_3 t] \\ [\omega_3 t] \\ [2\omega_2 t] \\ [\omega_2 t] \\ [2\omega_1 t] \\ [\omega_1 t] \\ 1 \\ [-\omega_1 t] \\ [-2\omega_1 t] \\ [-\omega_2 t] \\ [-2\omega_2 t] \\ [-\omega_3 t] \\ [-2\omega_3 t] \\ [-(\omega_1 - \omega_2)t] \\ [-(\omega_1 - \omega_3)t] \\ [-(\omega_1 + \omega_2)t] \\ [-(\omega_1 + \omega_3)t] \end{vmatrix}$$

Each field vector, E, is associated with a complex conjugate dual vector, E^* used in determining intensities ($I = E * E^*$).

$$E^* = [-\omega_0 t] \Big| \; [-(\omega_1 + \omega_3)t], \quad \dots \quad \dots \quad [-\omega_1 t], \quad 1, \quad [\omega_1 t], \quad \dots \quad [(\omega_1 + \omega_3)t] \quad \Big|$$

The individual elements of the interferometer operate on the field vectors,

$$E_{\text{out}} = \Big|\text{OPERATOR}\Big| E_{\text{in}}$$

The operators are :

Main cavity reflection operator

(assuming sidebands are outside resonance)

$$Rc_i = \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & & 1 & 0 & 0 & \\ & & 0 & -A_i(\omega_0)[\phi_i(\omega_0)] & 0 & \\ & & 0 & 0 & 1 & \\ & & & & & 1 \\ & & & & & & \ddots \end{pmatrix}$$

Propagation operator

$$P(l) = [k_0 l] \begin{pmatrix} [(k_2+k_3)l] & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & [k_1 l] & & & & \\ & & & & 1 & & & \\ & & & & & [-k_1 l] & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \\ & & & & & & & & [-(k_2+k_3)l] \end{pmatrix}$$

Propagation difference operator

$$l_2 = l_0 + \Delta \qquad l_1 = l_0 - \Delta$$

$$P(l_2) - P(l_1) = [k_0 l_0] \begin{pmatrix} \ddots & & & & \\ & [k_1 l_0] 2i\sin((k_0+k_1)\Delta) & & & \\ & & 2i\sin(k_0\Delta) & & \\ & & & [-k_1 l_0] 2i\sin((k_0-k_1)\Delta) & \\ & & & & \ddots \end{pmatrix}$$

Phase modulation operators

Input beam phase modulator

(to second order in modulation index)

J_n: Matrix symmetric, n = even; antisymmetric, n = odd

$$M(\Gamma_1, \omega_1) = \begin{pmatrix} \ddots & & & & & & & \\ & J_0 & & & & & & \\ & & J_0 & -J_1 & J_2 & 0 & 0 & \\ & & J_1 & J_0 & -J_1 & J_2 & 0 & \\ & & J_2 & J_1 & J_0 & -J_1 & J_2 & \\ & & 0 & J_2 & J_1 & J_0 & -J_1 & \\ & & 0 & 0 & J_2 & J_1 & J_0 & \\ & & & & & & & J_0 \\ & & & & & & & & \ddots \end{pmatrix}$$

$M(\Gamma_3, \omega_3) =$

(2 + 3)	(1 + 3)	(1 − 2)	(1 − 3)	2 * 3	2	3 * 3	3	3 * 1	1	0	−1	−3 * 1	−3	−3 * 3	−2	−2 * 3	−(1 − 3)	−(1 − 2)	−(1 + 3)	−(2 + 3)		
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
J_0										0											⋮	(2 + 3)
	J_0		J_2						$-J_1$	0											⋮	(1 + 3)
		J_0								0											⋮	(1 − 2)
	J_2		J_0						J_1	0											⋮	(1 − 3)
				J_0						0											⋮	2 * 3
					J_0					0											⋮	2
						J_0	$-J_1$			J_2											⋮	3 * 3
						J_1	J_0			$-J_1$			J_2								⋮	3
								J_0		0											⋮	3 * 1
	J_1		$-J_1$						J_0	0											⋮	1
0	0	0	0	0	0	J_2	J_1	0	0	J_0	0	0	$-J_1$	J_2	0	0	0	0	0	0	⋮	0
										0	J_0						J_1		$-J_1$		⋮	−1
										0		J_0									⋮	−3 * 1
							J_2			J_1			J_0	$-J_1$							⋮	−3
										J_2			J_1	J_0							⋮	−3 * 3
										0					J_0						⋮	−2
										0						J_0					⋮	−2 * 3
										0	$-J_1$						J_0		J_2		⋮	−(1 − 3)
										0								J_0			⋮	−(1 − 2)
										0	J_1						J_2		J_0		⋮	−(1 + 3)
										0										J_0	⋮	−(2 + 3)