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Near-Thermal Excitation of Violin Modes in the Test-Mass Suspension Wires of the 40–m Prototype Interferometer

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Abstract

The rms longitudinal motion of a test mass due to thermal excitation of the violin resonances of its suspension wires is estimated using a perturbation technique. Comparison with recent data indicate that the motion of test masses in the 40–meter interferometer at these resonances is very close to the minimum level consistent with such a system in thermal equilibrium with a reservoir at 300 K.

I. Introduction

A high mechanical-quality (Q) test mass suspension is required in a gravitational-wave laser interferometer to minimize test mass motions due to seismic and thermal noise. Inescapable features of high-Q pendulum suspensions are the violin resonances of the suspension wires, typically occurring at frequencies of several hundred Hz. Because of the high Q of these resonances, they may be filtered out of the interferometer output with little loss of observing bandwidth, and thus do not seriously affect the sensitivity of the interferometer.

From a different point of view, the violin resonances can be used as a diagnostic for mechanical forces affecting the motion of the test masses. The current work was motivated by a desire to measure the result of stochastic mechanical forces exerted on the test masses through the pendulum suspension, in effect establishing an effective temperature for the suspension. Such an effective temperature would provide a collective measure of all stochastic forces (whether seismic, electromagnetic, thermal, etc.) which act on the test masses through the suspension wires. The resonant nature of the violin modes allows a clean separation between actual displacement noise for the masses and "sensing" noise which causes the masses to appear to move.

Generally a comparison of thermal noise calculation and experimental data uses a detailed lineshape for the resonant system and then relies on the Fluctuation-Dissipation Theorem to fit this lineshape to a given temperature. A more robust (less model dependent) procedure is followed here. Instead of resolving the resonance lines we obtain the root-mean-square (rms) displacement x_{rms} from more coarsely-binned interferometer data, and compare this to the displacement calculated from the Equipartition Theorem.

II. Thermal Motion of a Violin String in One Dimension

The Equipartition Theorem states that the energy of a system in thermal equilibrium is $\frac{1}{2}$ kT per degree of freedom, where k is Boltzman's constant and T is the temperature (absolute). For a harmonic oscillator executing one-dimensional motion, the average kinetic energy is equal to the average potential energy. Therefore we can write the average kinetic energy as

$$<K_{wire}>= \frac{1}{2} \int_0^L \rho_0 < \dot{\psi}^2(y) > dy = \frac{1}{2} kT \tag{1}$$

where ρ_0 is the density per unit length of the violin string, L is its length (see Figure 1), and $\dot{\psi}(y)$ is the velocity of the point y of the string. Assuming ψ = 0 everywhere at t = 0, and applying the boundary conditions $\psi(0) = \psi(L) = 0$ for all time, we can write

$$\psi_n(y,t) = A_n \sin\left(\frac{n\,\pi y}{L}\right) \sin \omega_n t \tag{2}$$

where A_n is the amplitude of vibration, and ω_n is the resonance angular frequency, for the n^{th} order mode of the string. The time-averaged squared velocity is then

$$< \dot{\psi}_n^2(y) >= \frac{1}{2}\, \omega_n^2 A_n^2 \sin^2\left(\frac{n\,\pi y}{L}\right) \tag{3}$$

which gives an average kinetic energy for the n^{th} mode

$$< K_{wire} >= \frac{\rho_0 L \omega_n^2 A_n^2}{8} \tag{4}$$

after integration.[1] Substituting $L = \frac{n\lambda_n}{2}$ into equation (4) we obtain

$$< K_{wire} > = \frac{n \rho_0 \lambda_n\, \omega_n^2\, A_n^2}{16} \tag{5}$$

which, using equation (1) gives

$$A_n^2 = \frac{8kT}{n \rho_0 \lambda_n} \frac{1}{\omega_n^2} = 2A_{rms}^2 \tag{6}$$

[1] It is interesting to note the similarity of equation (4) to the expression for the average kinetic energy of a point mass oscillator, given by equation (4) with the substitution $\frac{1}{2}\rho_0 L = m$, where m is the mass of the oscillator.

III. Motion of a Test Mass Due to Thermally Excited Violin Mode of the Suspension Wire

We solve for test mass motion due to vibrations of the suspension wire by a perturbation technique, where we assume that the test mass recoils from the wire without seriously affecting the motion of the wire. We therefore use the zeroeth order solution for wire motion in the case of an infinite mass test mass (equation 6) to calculate the recoil motion of the actual test mass with $M >> \rho_0 L$. This is a good approximation when the kinetic energy carried by the test mass is small compared to the kinetic energy carried by the moving wire. For mass supported by a single wire in one dimension, we have at the mass-wire interface (see Figure 2):

$$M\frac{d^2 z}{dt^2} = T_0 \sin\theta \tag{7}$$

where M is the suspended mass, z is the coordinate of the center-of-mass of the test mass, T_0 is the tension in the wire, and θ is the angle at which the wire leaves the test mass. Since both the mass and the wire move at the same frequency we have

$$-M\,\omega_n^2 z = T_0 \left.\frac{d\psi_n}{dy}\right|_{y=L} \tag{8}$$

which has the solution

$$z = (-1)^n \frac{2\pi A_n T_0}{\lambda_n M\,\omega_n^2} \sin\omega_n t = (-1)^n z_n \sin\omega_n t \tag{9}$$

Substituting $\lambda_n \nu_n = (T_0/\rho_0)^{\frac{1}{2}}$, where $2\pi\nu_n = \omega_n$, we obtain

$$\frac{z_n}{A_n} = \frac{(\rho_0 T_0)^{\frac{1}{2}}}{M\omega_n} = \frac{Z_{wire}}{Z_{mass}} \tag{10}$$

Where the last term on the right is the ratio of mechanical impedances for the wire and the test mass. Using equation (6) and setting $T_0 = Mg$ gives

$$z_{rms} = \left(\frac{4\,kT}{Mn} \cdot \frac{g}{\lambda_n}\right)^{\frac{1}{2}} \cdot \frac{1}{\omega_n^2} \tag{11}$$

Using $\lambda_n = \frac{2L}{n}$, and recognizing $\sqrt{g/L}$ as the pendulum's angular frequency ω_p we finally obtain for the test mass motion due to the thermally excited wire:

$$z_{rms} = \left(\frac{2\,kT}{M}\right)^{\frac{1}{2}} \cdot \frac{\omega_p}{\omega_n^2} \tag{12}$$

IV. Check of Validity of Perturbation Treatment

In this section we compare the average kinetic energies of the test mass and wire. The average kinetic energy of the text mass is

$$< K_{mass} >= \frac{1}{4} M \omega_n^2 z_n^2 \tag{13}$$

Combining equations (6 and 12) we have

$$\frac{< K_{mass} >}{< K_{wire} >} = 2\left(\frac{\omega_p}{\omega_n}\right)^2 \tag{14}$$

Since typical pendulum frequencies are about 1 Hz and typical wire resonance frequencies are several hundred Hz we see that the assumption that the kinetic energy is carried principally by the wire is verified.

V. Application to a Two Wire-Loop Suspension

The cylindrically-shaped test masses in a laser interferometer gravitational-wave detector are typically suspended by two wire loops for a total of four suspension wires (see Figure 3). Each wire is free to oscillate in two orthogonal polarizations (one with motion z along the interferometer arm and one perpendicular to the arm). Due to small differences in loading of the four wires, each wire has a distinctive resonance frequency. The analysis of Sections II and III can be generalized to the four wire case, where the reaction mass is still M, but we now have $T_0 \simeq \frac{1}{4} Mg$ for the tension in each wire. We assume that the violin modes polarized perpendicular to the interferometer arm do not contribute to the mass motion measured by the interferometer.[2] We then expect four distinct sets of violin modes for each test mass, each of which contributes

$$z_{rms} = \left(\frac{kT}{2M}\right)^{\frac{1}{2}} \cdot \frac{\omega_p}{\omega_n^2} \tag{15}$$

to the interferometer output signal. Using parameters typical of the current 40–m prototype interferometer ($M = 1.6$ kg, $\omega_p = 2\pi$ s^{-1}, $\omega_1 = 2\pi \cdot 300$ s^{-1}) we can rewrite equation (15) as

$$z_{rms} = 6.4 \times 10^{-17}\,\mathrm{m} \cdot \left(\frac{1.6\,\mathrm{kg}}{M}\right)^{\frac{1}{2}} \cdot \left(\frac{300\,\mathrm{Hz}}{f_1}\right)^2 \tag{16}$$

where f_1 is the fundamental resonance frequency.

[2] This is true only to the extent that rotations of the test mass do not appear in the interferometer output.

VI. Comparison with Test Mass Motion at Violin-Mode Frequencies in the 40–m Prototype

The noise equivalent displacement of the 40–m interferometer as measured on October 3, 1991 is given in Figure 4. Groups of wire resonances around 320 Hz (originating in the left-arm end mass suspension wires) and around 600 Hz (originating in the right-arm end mass suspension wires) are marked. The horizontal levels marked at each set of wire resonances give the minimum level of test mass motion at the violin resonances that is consistent with the interferometer operating in equilibrium with a heat bath at 300 K. Given the 1.8 Hz channel bandwidth we expect

$$\tilde{z}\,(320\,\mathrm{Hz}) = 4.2 \times 10^{-17}\,\mathrm{m}/\sqrt{\mathrm{Hz}} \tag{17}$$

$$\tilde{z}\,(600\,\mathrm{Hz}) = 1.2 \times 10^{-17}\,\mathrm{m}/\sqrt{\mathrm{Hz}} \tag{18}$$

for each wire in the respective group of resonances.[3]

We can see that the ambient excitation of these resonances is about a factor of two greater than the predicted level due to thermal noise. A subsequent investigation by S. Kawamura has identified the excess noise above thermal noise near 320 Hz as due to electronic noise in the test mass damping system. With this information we may conclude that other external sources of displacement noise, for example seismic motion, have been sufficiently attenuated at these frequencies.

[3] Accidental overlaps of two wire resonances in a single bin can raise these values an additional factor of $\sqrt{2}$.

Appendix:

Thermal Noise Calculated from the Fluctuation — Dissipation Theorem

We have calculated thermal noise in the violin modes assuming that all of the thermal energy of a given mode is concentrated at the resonant frequency of that mode, i.e., that the lineshape of the resonance is a delta function centered at the resonant frequency. Here we show that this approximation is reasonable for the current 40–meter prototype, since the FWHM of the resonances (typically ~0.01 Hz) is much less than the bandwidth at which measurements are taken (typically ~1 Hz)

To make a more detailed analysis, we assume the noise can be modelled as that of a simple harmonic oscillator with a damping force proportional to its velocity, driven by a white force derived from the Fluctuation-Dissipation Theorem. The lineshape of the amplitude of the violin modes is then as follows:

$$\tilde{A}_n(\omega) = \frac{1}{\pi}\left(\frac{32k\,T\gamma}{M}\right)^{\frac{1}{2}} \frac{\omega_n}{\omega_p}\left[\frac{1}{(\omega_n^2 - \omega^2)^2 + (\gamma\omega)^2}\right]^{\frac{1}{2}} \tag{19}$$

Here k is Boltzman's constant, T is the temperature, γ is the energy damping rate constant, M is the mass of the test mass, ω_n is the violin mode resonant frequency, and ω_p is the pendulum resonant frequency. From this we can calculate

$$A_{rms} = \sqrt{\int_{\text{Bandwidth}} \left[\tilde{A}(f)\right]^2 df} \tag{20}$$

The comparison which we wish to make is between the energy near the resonance and the total energy of the system. The numbers which we will use are those for the primary arm end mass (f_0 = 320 Hz, γ = 0.13 s^{-1}, $\Delta f \equiv$ FWHM = 0.02 Hz), but the comparison is basically independent of these particular values.

Bandwidth Centered at the Resonance	Percentage of Total Energy (kT)
Δf	48
$2\Delta f$	71
$3\Delta f$	79
$4\Delta f$	84
$10\Delta f$	94

Since 10 Δf is still much less than our present typical measurement bandwidth, the approximation that the bulk of the energy is contained in the resonance peak is a good one.

As a closing statement we should note that the velocity damping model may not be applicable to the violin modes of our system, and other damping models with different lineshapes have been proposed.[1] Although these models differ substantially away from resonance, they give the same shape near resonance and therefore give similar results for the amount of energy concentrated at the resonant frequency.

1 P. R. Saulson. *Phys. Rev. D*, **42**, 2437 (1990).

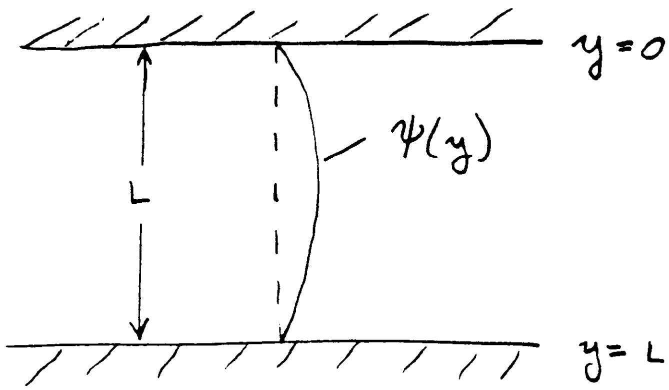

Figure 1. A violin string with infinite mass terminations.

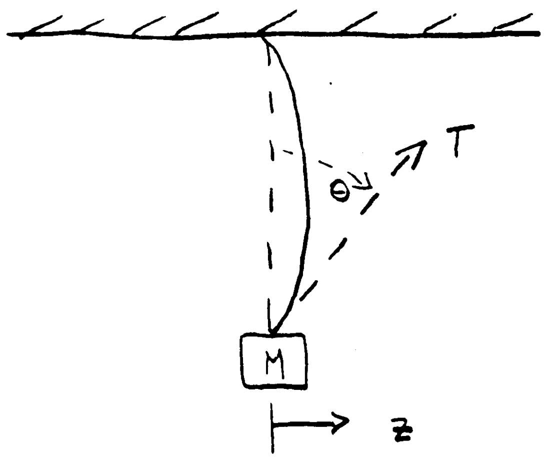

Figure 2. Violin mode of a simple pendulum.

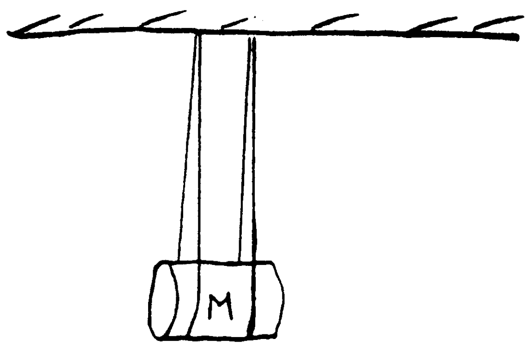

Figure 3. A double-loop test mass suspension.

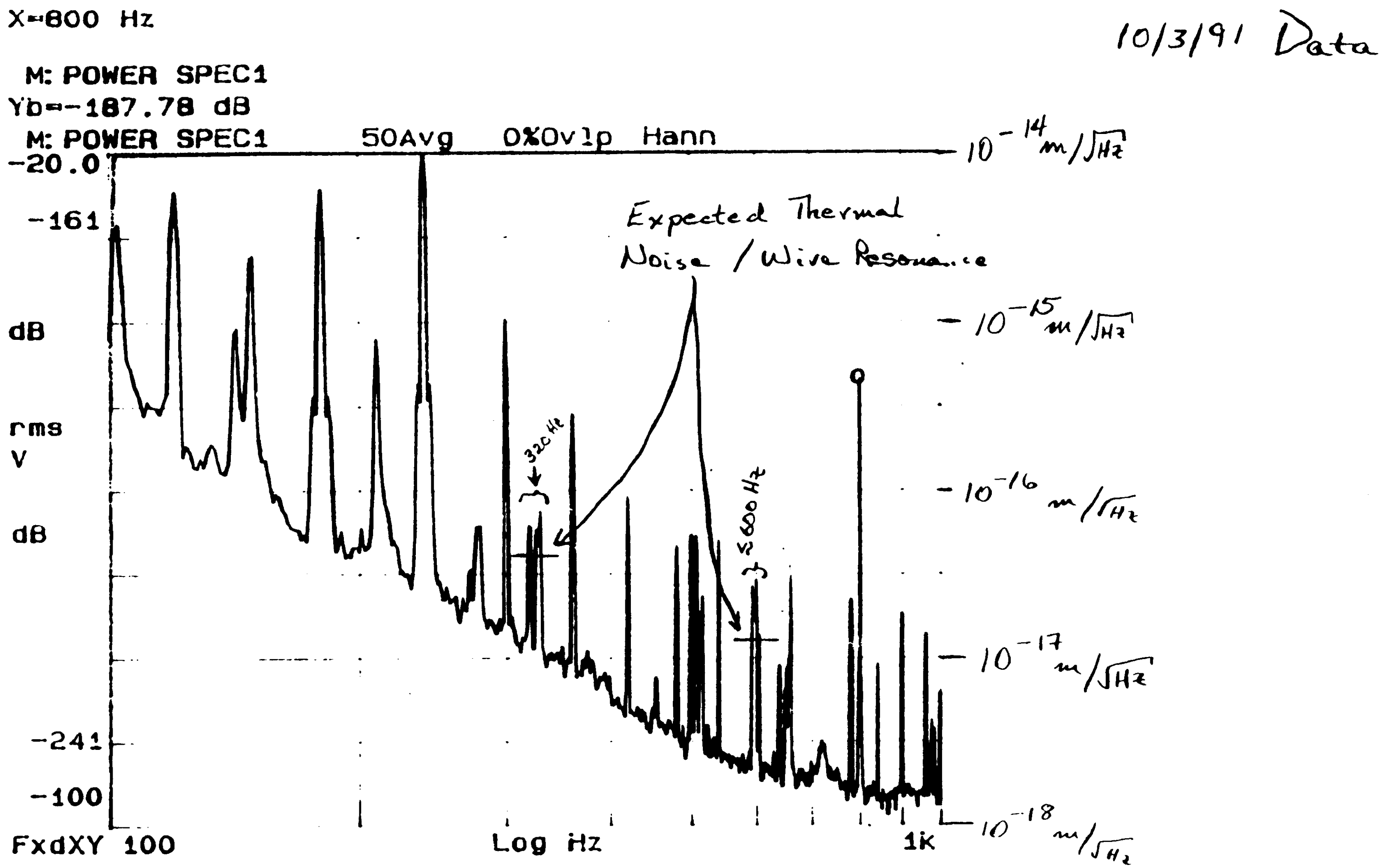

Figure 4. Comparison of observed violin mode excitation with prediction based on thermal excitation.