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Stress-dependent damping in Cu-Be torsion and flexure suspensions at stresses up to 1,1 GPa

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Abstract.— Measurements are presented of the damping in Cu-Be torsion and flexure suspensions at stresses up to 1,1 GPa observed in freely oscillating pendulums having periods between 1,5 s and 70 s. Surprisingly, the modulus defect of Cu-Be appears to increase, from $4,3 \times 10^{-5}$ to only about 6×10^{-5}, from zero stress up to 1,1 GPa, some 95 % of the yield stress. Additional damping, originating at the clamping interface between the suspension and its support, has been measured and shown to have a similar frequency dependence to that of the intrinsic damping in the Cu-Be. The measurements of this additional damping thus support the hypothesis that stick/slip damping can be described in terms of a self-organized critical process. A new expression is derived for the damping in flexure strip suspensions that, for the first time, correctly takes into account subtle distinctions between elastic and gravitational potential energies. As a test of our understanding of the damping in pendulums, a simple pendulum was made, suspended by a Cu-Be flexure strip and having a 1 kg bob with a period of 1 s, which was found to have a Q of 4×10^6, in good agreement with the predicted value of $(4,2 \pm 2) \times 10^6$.

1. Introduction

In an earlier paper [1] we reported observations of frequency-dependent damping of a long-period pendulum supported on a copper-2 % beryllium (Cu-Be) flexure strip. In that study we presented measurements of anelastic after-effect and logarithmic decrement of free oscillations of the pendulum which indicated that the modulus defect of Cu-Be was independent of frequency over a broad range of frequencies. We suggested that a distribution of dislocation relaxation processes of equal strength, with

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relaxation times between a few seconds and a few thousand seconds and with a population that varied inversely with the relaxation time would explain the observations. We also noted, as had been previously pointed out [2] that this damping would give rise to a spectrum of perturbing noise torques in a weak force detector whose spectral density would vary as 1/f. Further, the microscopic theory we developed predicted that the modulus defect would increase as the stress increased in the material. The theory was unable, however, to predict at what level of stress the modulus defect would begin to rise at a significant rate, although it did predict that as the stress, σ, approached the yield stress, $\hat{\sigma}$, we should observe

$$\frac{\Delta E}{E} \propto \frac{\sigma/\hat{\sigma}}{1 - \sigma/\hat{\sigma}} \tag{1}$$

For precipitation hardened Cu-Be, the yield stress is about 1 GPa, which corresponds to a strain of about 0,6 %. Our 1992 paper, op. cit., was principally concerned with the frequency dependence of the modulus defect and no systematic study was made of its stress dependence, all measurements being made at about 5 % of yield stress. We did report, however, a proportional increase in damping when the flexure was thinned by some 20 % to remove surface damage. We took this as an indication of the presence of a stress-dependent increase in the modulus defect consistent with Equation (1).

We now report the results of a more systematic study of the stress-dependent damping in the same Cu-Be alloy measured in both flexure and torsion. At an early stage in the work it became apparent that accurate measurements of stress-dependent damping in a flexure are difficult due to the presence of unknown contributions to the measured damping coming from the mounting of the flexure and the mechanism for supporting the loads and due also to the difficulty of manufacturing a strip whose geometry and dimensions are, in certain critical features, not sufficiently well known. To demonstrate that the results represented the intrinsic damping of the Cu-Be, measurements were made with flexure and torsion specimens in a number of different configurations. In this way certain significant contributions to the damping coming from various extraneous sources were identified and eliminated. Among the specimen configurations used was one similar to that used in our earlier work [1]. Using this it was confirmed that the earlier results are consistent with those presented here.

Three quite different configurations of pendulum were used : a vertical pendulum supported close to its centre of mass by various designs of flexure pivot (this pendulum was similar to that used in our earlier work but was rather smaller), a simple pendulum with a 1 s period and a torsion pendulum supported by torsion wires or torsion strips. The flexures and torsion wires and strips were all made from cold-worked age-hardened Cu-Be alloy.

A critical feature of each flexure or torsion suspension is the manner in which it is attached to the pendulum and to the base upon which it rests or from which it hangs. Ideally, the suspension is rigidly fixed to a rigid base so that damping due to stick/slip at the junctions is absent. We show that such a stick/slip process not only leads to a measured damping that can be nearly two orders of magnitude larger than the intrinsic damping of the Cu-Be, but that it also produces frequency-independent damping easily confused with intrinsic damping.

Damping due to residual gas in the experimental chamber is relatively easy to eliminate. At the frequencies of oscillation used, namely below about 1 Hz, residual pressures of 10^{-5} torr (10^{-3} Pa) or below are sufficient to reduce the gas damping to negligible levels compared with the intrinsic damping of the material.

2. The relation between the logarithmic decrement and modulus defect for flexure and torsion strips

In all the experiments reported here, the measured quantity is the logarithmic decrement η of a freely swinging pendulum in vacuum, defined by

$$\eta = \frac{1}{\pi} \ln \frac{A_n}{A_{n+1}} \qquad (2)$$

where A_n and A_{n+1} are the amplitudes of the n^{th} and $(n+1)^{th}$ oscillations respectively. Note that here the Q of the system is given simply by $1/\eta$.

We showed in our earlier paper [1] how the modulus defect ΔE, which is the difference between the stressed and unstressed modulus of elasticity, can be written

$$\frac{\Delta E}{E} = \frac{\delta e}{E} \int_{\tau_o}^{\tau\infty} f(t) \frac{\omega t}{1 + \omega^2 t^2} dt \qquad (3)$$

assuming $f(t) = 1/\ln(\tau_\infty / \tau_o) \times 1/t$ this becomes

$$\frac{\Delta E}{E} = \frac{\delta e}{E \ln(\tau_\infty/\tau_o)} (\tan^{-1} \omega \tau_\infty - \tan^{-1} \omega \tau_o) \qquad (4)$$

where δe is the relaxation strength of each of the individual relaxation elements of our proposed hierarchical structure and τ_o and τ_∞ are the extremes of their relaxation times. We suggested that $\tau_o \leq 10$ s and $\tau_\infty \geq 5000$ s. Thus $\Delta E/E$ is independent of frequency for $1/\tau_\infty < f < 1/\tau_o$. The same formalism was, we have discovered, originally intorduced by Van der Ziel [3] in 1950 to account for 1/f electrical noise in semiconductors.

The logarithmic decrement of a freely oscillating pendulum of mass m and moment of inertia I was then written

$$\eta = \frac{f(W)}{I \omega^2} \frac{\delta e}{E} \frac{1}{\ln(\tau_\infty/\tau_o)} (\tan^{-1} \omega \tau_\infty - \tan^{-1} \omega \tau_o) \qquad (5)$$

where W = mg and f(W) is a geometrical factor that is a function of the dimensions of the strip and the load, W. The thermoelastic effect contributes a frequency-dependent component to $\Delta E/E$, which is negligible in our experiments, as we show in Section 5.4 below.

For a freely oscillating pendulum we can write

$$\eta = \frac{f(W)}{I \omega^2} \frac{\Delta E}{E} \qquad (6)$$

For a flexure strip of length L a relatively simple expression for f(W) is derived in Appendix I and is :

$$f(W) = (WEH)^{1/2} \left(\coth \alpha L + \frac{\alpha L}{\sinh^2 \alpha L} + \frac{2\alpha^2 L^2 \coth \alpha L}{\sinh^2 \alpha L} \right) \qquad (7)$$

where $\alpha = (W/EH)^{1/2}$ in which H is the second moment of area of the strip given by $bt^3/12$ for a strip of width b and thickness t. Equation (7) agrees very well, for the geometries discussed below, with numerical calculations based on a more general analysis due to Haag [4].

The shape of a flexure strip for the case of the load having its centre of mass at the lower end of the flexure and deflected through an angle θ is shown schematically, later on, in Figure 7. As the load increases, the bending becomes increasingly confined to the lower end of the flexure and the bending thus becomes more pronounced for a given deflection. The bending radius R of the strip close to its lower end is given by

$$R = \frac{E\,H}{c_f\,\theta} \tag{8}$$

where c_f is the sum of the elastic and gravitational restoring torques given by

$$c_f = (WEH)^{1/2} \coth \alpha L \tag{9}$$

It is clear, therefore, that for a flexure under load the restoring torque, and hence the stiffness of the flexure, soon ceases to depend on L as either W or L increases. As αL increases, $\coth \alpha L \rightarrow 1$ and $C_f \rightarrow (WEH)^{1/2}$, independent of L. It was this argument that in the first place led us to design a flexure strip that was short and could easily be cut from the solid [5].

This behaviour of a flexure strip does, however, make it difficult to obtain accurate values of the modulus defect as a function of load from measurements of logarithmic decrement of a pendulum swinging on a flexure strip. The difficulty arises from the need to have quite thin and narrow flexure strips if high stresses are to be obtained without having excessive loads on the pendulum. Such thin and narrow strips, say 100 µm thick and 10 mm wide, are difficult to make while still maintaining an accurate shape close to the ends so that the calculated value of f(W) based on measurements of b and t correctly represents reality. Even with a flexure of this size, to reach a stress of 1 GPa would require a load of 100 kg.

We have, therefore, limited the range of stresses explored using a flexure strip to a maximum of about 230 MPa, some 20 % of the yield stress. Higher stresses were obtained using a strip in torsion.

For a torsion strip pendulum the logarithmic decrement is related very simply to the modulus defect because there is no gravitational component to the restoring force and the stress is distributed uniformly along the length of the strip. We can write

$$\eta = \frac{c_o}{I\omega^2}\,\frac{\Delta E}{E} \tag{10}$$

where c_o is the torsional stiffness of the strip and I is the moment of inertia of the suspended mass. But, since $c_o = I\omega^2$ we have

$$\eta = \frac{\Delta E}{E} \quad (11)$$

so that η is a direct measure of $\Delta E/E$ provided that no other damping mechanisms are present. There is, of course, no thermoelastic contribution in torsion.

In what follows we describe experiments with both flexure strips and torsion strips that demonstrate the presence of damping coming from the clamping mechanisms that hold the strip. In some cases the magnitude of this additional damping exceeds by nearly two orders of magnitude the intrinsic damping of the material.

Damping due to residual gas in the experimental chamber is small. Taking the expression given by Saulsen [6] for the effective Q of a system damped only by residual gas pressure, we can write

$$Q \approx \frac{D\rho\,\omega_o}{p}\sqrt{\frac{kT}{M}} \approx \frac{15\,\rho\,\omega_o}{p} \quad (12)$$

where D is a characteristic length of the pendulum which we take here to be 0,05 m, ρ is the mean density of the material of the pendulum which is in the range from about 3×10^3 kg m^{-3} to 8×10^3 kg m^{-3}, ω_o is the frequency of oscillation, p is the pressure and M is the molecular mass of the residual gas in the system. We find in most cases (i.e. ρ and ω_o minimum and a pressure of 7×10^{-4} Pa) a Q of about 4×10^8. The maximum Q's observed were about 20 000 for the flexure and torsion pendulums and 5×10^6 for the simple pendulum. Thus damping from residual gas is negligible in all cases.

3. The pendulums

3.1 Vertical pendulum

The vertical pendulum used with the flexure strip is shown in Figures 1 and 2. It was designed to have a high rigidity and as low a mass as possible consistent with being able to support loads of up to 10 kg. The pendulum was made from duralumin and had a mass of 1,24 kg.

The period of the pendulum was adjusted by placing brass balls up to 10 mm in diameter as shown in the Figures to raise or lower the centre of mass with respect to the centre of rotation of the flexure while keeping the moment of inertia of the whole assembly nearly unchanged. Adjusted in this way, Equation 6 tells us that for $\Delta E/E$ independent of frequency the measured logarithmic decrement η should be proportional to ω^{-2}. This we observed in our earlier work and confirmed again here. If, on the other hand, we had changed the period by changing the moment of inertia I, then because $I\omega^2$ would have been a constant under these circumstances no frequency dependence of η would have been observed. This is, of course, the case with the torsion pendulum.

With brass balls placed as shown in Figure 2 to give a period of about 30 s, the total mass was 1,47 kg. Additional loads could be added to give total masses of 2,63 kg, 5,47 kg and 9,52 kg by placing brass cylinders on either side of the pendulum in such a way that the moment of inertia is changed as little as possible on loading. The moment of inertia of the unloaded pendulum was measured to be 0,0214 kg m^2. Fully loaded with 9,52 kg, it increased only to 0,0275 kg m^2.

The amplitude of oscillation of the pendulum was measured using a laser beam reflected from a mirror attached to the pendulum. The reflected beam was detected by a photo-cell attached to a servo-controlled pen recorder. Amplitudes of oscillation were in the range from 1 mrad to 10 mrad and could be read with a precision of about 0,05 mrad, no significant non-linearity was present.

The pendulum was mounted in a vacuum chamber having two optical windows and provision for external manipulation of the pendulum so that it could be set swinging with an appropriate amplitude. Pressures below 5×10^{-6} torr (7×10^{-4} Pa) could easily be obtained.

3.2 The torsion pendulum

The torsion pendulum is shown in Figures 3(a) and 3(b) and is much simpler than the vertical pendulum. Additional loads up to 8,5 kg, to give a stress in the strip above 1 GPa, were added simply by clamping them to the vertical rod extending below the pendulum cross beam. Note that in the torsion pendulum the additional loads have a significant effect on the period, increasing it from about 28 s to 72 s. However, in accordance with Equation 6 the modulus defect, $\Delta E/E$, remains equal to the measured values of η irrespective of the period.

The oscillations of the pendulum were observed, as before, using a laser beam reflected from the mirror on the beam. The support was made rigid against torsional stresses. It was remarkable that during all of the measurements with the torsion strips, no zero adjustment to the top support of the pendulum had to be made. On loading the strips and in subsequent observation of the decay of oscillations over some days the zero drift for a given strip was below the resolution of the optical system, namely, about 0,05 mrad even at the highest loads, see Appendix II for some remarks on the significance of this natural zero stability.

4. The Cu-Be flexure and torsion elements

4.1 Flexure elements

Figure 4 shows the various designs of flexure element. The first, Figure 4(a), is one of the now standard elements used in the BIPM flexure-strip balance and is similar to that used in [1] but with slightly modified dimensions and shape of the block and strip.

The pendulum was designed to rest on a block of the design shown in Figure 4(a), experience having shown that a kinematic mounting of this sort is satisfactory. All other shapes of flexure element were thus made to be held in blocks of this shape. Observed differences in damping could thus be considered as coming either from the flexure itself or from the clamping interfaces in the block.

Composite flexure blocks were made using rolled sheet of Cu-Be, either 80 μm thick or 1,4 mm thick, and steel clamping blocks as shown in Figures 4(b) and (c). Initial experiments were carried out using the 80 μm sheet spark cut to have flexure elements of various lengths from 3 mm to 10 mm and widths from 2 mm to 10 mm. The flexure elements made from the 1,4 mm thick sheet were ground and hand polished to the dimensions shown in the Figure.

The faces of the steel clamping blocks were cut away so that the flexure elements were tightly gripped close to the end of the flexure. This was done in an attempt to minimize the stick/slip damping that is expected to occur at this interface. It was, of course, to guard against such damping that the original Cu-Be flexure block was made with the flexure cut from the solid, see Figure 4(a).

4.2 Torsion elements

Two types of torsion element were used, the first was a simple drawn wire of diameter 0,3 mm and the second, see Figure 5, included three slightly different forms of strip 1 mm wide and about 80 μm thick.

The wires were soldered at each end to a small cylindrical block that was clamped to the support or torsion beam.

The torsion strips were made either by spark cutting the strip from a sheet of 80 μm Cu-Be Figure 5(a) and (c) or by grinding the strip from 1,4 mm thick Cu-Be sheet, Figure 5(b). In the case of the strips cut from the thin sheet the ends were either plane or broadened to a width of 10 mm. The ends of the plane strip were either soldered or crimped into copper cylinders. The ends of the broadened strip were clamped as shown in the Figure. The ends of the strip cut from the 1,4 mm sheet were clamped in a similar way.

5. Results

5.1 Torsion wires

The first measurements of load dependency in torsion of $\Delta E/E$ were those made using 0,3 mm diameter torsion wires. These extended up to stresses of 190 MPa, about 20 % of the yield stress. The measured values of $\Delta E/E$ were between $5,3 \times 10^{-5}$ and $5,8 \times 10^{-5}$ and showed no significant stress dependence. It was, however, necessary to anneal the wires to remove a set resulting from winding on a spool. It was difficult to control satisfactorily the annealing process and being thus unable properly to characterize the wires the measurements were not pursued. The results were however taken to indicate an upper limit to the intrinsic modulus defect of the Cu-Be at these stresses.

5.2 Flexure strips

Measurements using flexure strips were made first using the strip cut from the solid Cu-Be block (Figure 4(a)) as in our earlier paper [1], to provide a link with the results reported there and also to provide a comparison between flexures cut from a solid block and those subsequently made from rolled sheet. It was not known, for example, whether grain orientation and cold work would make a significant difference to the damping at high loads. In fact, most of the differences between the measured values of logarithmic decrement observed between the different configurations can be explained by the presence of damping either in the structure of the pendulum or at the kinematic mounting points of the additional loads.

Results obtained using the solid-block flexure gave a value for the modulus defect at the lowest load ($\sigma \approx$ 10 MPa) of 6,6 x 10^{-5}. This was very close to that obtained using the torsion wires but lower than the corrected value of 2,3 x 10^{-4}, from our earlier work on the solid block, which we now calculate using the new expression for f(W) given in Equation 7. As the load was increased, however, the modulus defect apparently increased reaching a value of 3,2 x 10^{-4} at a load of 95 N (σ = 66 MPa). No amplitude dependency was observed and the reproducibility of the values after changing loads was about 10 %. Measurements were made at a variety of periods within the range 16 s to 25 s. At a given load a strict $1/\omega^2$ proportionality of the measured values of η was found to within the measurement uncertainty, which for a single set of measurements at a given load was about 5 %.

Results obtained with the composite flexures made with the 80 µm rolled sheet were very much higher, and were strongly amplitude dependent, see Figure 6. Large variations in measured η were observed depending on the way in which the steel plates were clamped to the flexure sheet. Attempts at glueing, soldering and simple mechanical clamping all gave different results, the first two methods sometimes giving values of η an order of magnitude higher than those shown in Figure 6.

In all of these measurements, however, the same $1/\omega^2$ dependence was always maintained. To test this over a wider frequency range, the logarithmic decrement of the 80 µm clamped flexure was measured at T = 1,45 s and at T = 45 s and the $1/\omega^2$ relation was found to hold within 5 %.

To explain these results the hypothesis was adopted that the additional damping observed in the 80 µm composite flexure originated in the stick/slip that must be present at the surface of the 80 µm thick sheet as it enters the steel clamping plates. A calculation of the magnitude of the shear forces existing at these interfaces supports this hypothesis. Figure 7 shows how these forces arise. We have already seen that a flexure under a load W has a bending radius R at its lower end inversely proportional to

$W^{1/2}$ and given by Equation 8. The bending moment at the end of the flexure is balanced by a shear stress applied across its thickness by the clamping plates. The shear strain across a thin strip bent through a radius R is t/R so that the shear stress σ_s at the surface required to produce this curvature is $\sigma_s = Gt/R$, where G is the shear modulus.

For the case of an 80 μm thick strip supporting a load of 95 N and bent through an angle of 10 mrad, the shear stress σ_s at the surface interface between the flexure and the clamp is about 5×10^7 N m^{-2}. Assuming a coefficient of friction of unity, the pressure required to suppress slipping is, equally, 5×10^7 N m^{-2} or 50 N mm^{-2}. While such an average pressure can be applied by a 3 mm diameter steel screw, it is most unlikely that the same pressure can be maintained right up to the edge of a clamp. It is at the edge that stick/slipping will take place. Such a process would be strongly amplitude dependent since σ_s is inversely proportional to θ. This is in fact what is observed, see Figure 6.

Recently, Cagnoli et al. [7] have shown that the $1/\omega^2$ dependence of η, which implies a frequency independent modulus defect, observed in this and in our earlier work can be explained on the basis of a stick/slip mechanism in the dynamics of dislocation movement. The arguments they use to reach this conclusion invoke the ideas of self-organized criticality. They assume that in the presence of a high stress, the dislocation network arranges itself at or close to a critical state. A small perturbation stress will then lead to a self-adjusting unlocking cascade of events over a wide range of length and time scales. Such a sequence of events is not dissimilar to the locking and unlocking of a highly stressed interface such as exists at the clamping of a strip as described here which exhibits exactly the same frequency independence as does the dislocation damping.

If stick/slip at the flexure-clamp interface is indeed at the origin of the additional damping, it should disappear or be much reduced if the flexure is already much thicker at the point where it enters the clamp, as in Figure 4(c). This is what is observed. Figure 6 also shows the values of η as a function of load for such a flexure. No significant amplitude dependence was observed in these measurements which further supports the hypothesis that the high damping seen in the 80 μm sheet flexures resulted from stick/slip at the clamp interface. The damping is, however, still apparently load dependent. The values of $\Delta E/E$ calculated from the measurements shown in Figure 6, using Equation 6, increase significantly with pendulum load and are consistent neither with those obtained with the torsion wire nor with those from the torsion strip to be given later. The inset to Figure 6 shows the values of η extrapolated to zero amplitude plotted as a function of stress. Those from the Figure 4(c) design of flexure are very similar to the data obtained from the solid flexure Figure 4(a).

We suspected that some or all of the apparent load or stress dependence could be due to damping in the structure of the pendulum itself or in the kinematic mountings of the additional loads. To check this we measured the damping as a function of stress in the flexure at constant pendulum load by progressively reducing the width of the flexure strip. For each width of the strip, namely 35, 24, 10 and 5 mm, measurements of η were made at each of the penduum loads of 14,7; 25,3; 54,7 and 95,2 N. It was immediately clear that the addition of an extra load to the pendulum led to an increase in damping that was much higher than when the same increase in stress was brought about by reducing the width of the strip without adding any additional load to the pendulum. At a load of 14 N (the unloaded pendulum) the damping obtained on reducing the width increases only slowly with stress, whereas the addition of any load immediately leads to a much increased damping. We were able to eliminate the largest part of this increase, however, by screwing the extra loads tightly to their kinematic mountings on the pendulum. Previously the extra loads had simply rested by their own weight on three phosphor bronze balls in 'V' grooves. Almost all the remaining part of the increased damping was eliminated by adding strengthening pieces to the pendulum. The damping subsequently measured at the higher loads using the 5 mm wide flexure is shown in Figure 8. It is much lower than before and in good agreement with that measured using the unloaded pendulum.

Having removed the largest source of additional damping there remained the possibility of there still being a small residual component of damping coming from the structure of the pendulum itself. On the hypothesis that this existed, the relation between the measured values of η and $\Delta E/E$ can be written in the form

$$\frac{\eta I\omega^2}{f(W)} = \frac{\Delta E}{E} + \frac{A}{f(W)} \qquad (13)$$

Table 1 shows the measured values of η and the calculated values of $\Delta E/E$ obtained by fitting the data to an equation of this form. The value of A, $4{,}8 \times 10^{-7}$ Nm, is taken to represent the damping component of the structure of the pendulum. The values given in Table 1 are consistent with the model upon which equation (13) is based and lead to a value of $4{,}7 \times 10^{-5}$ for $\Delta E/E$ in the flexure strip. The data obtained before the additional loads were screwed to the pendulum can be treated in the same way except that a much larger value of A is necessary. If this is done a similar result is obtained but at the higher loads the spread of the resulting values of $\Delta E/E$ is somewhat larger since the damping produced by the slipping kinematic mounts is, presumably, variable. To check that the kinematic mounting of the flexure block was not a significant source of damping, the diameter of the balls was changed in order to change the pressure at the contacts. No significant difference in $\Delta E/E$ was observed.

The final results of the measurements of $\Delta E/E$ using the flexure strip are shown in Figure 8 and indicate that there is no significant stress dependence, at least up to a stress of 220 MPa about 25 % of the yield stress. These results are consistent with those obtained using both the torsion wire and torsion strip.

5.3 Torsion strip

Having been confronted by the problem of clamping of the flexure strip and by damping originating in the pendulum and in the way the loads were applied, great care was taken in the design and construction of the torsion pendulum and torsion strip.

Preliminary measurements were made using a plain strip cut from 80 μm thick sheet, as shown in Figure 5(a) whose ends were either soldered or crimped into copper end pieces. The measured values of η and hence $\Delta E/E$ were 2,5 x 10^{-4} with soldered ends and 6 x 10^{-5} with crimped ends. Measurements were made at loads up to 9,5 kg to give a stress of 119 MPa. Supporting the top of the strip on a crossed knife-edge gimbal led to no change in the measured value.

To reach much higher stresses two other designs of strip were used. The first was a strip cut from the same 1,4 mm thick plate that was, in the end, successfully used for the flexure strip. Its shape is shown in Figure 5(b).

The results of the measurements made with this strip are shown in Figure 9. The modulus defect, $\Delta E/E$, surprisingly increases very little right up to a stress of 850 MPa, about 85 % of the yield stress. This strip failed (about 10 mm below the upper support) as a load of 85 N was slowly being applied, equivalent to a stress of 950 MPa.

A second torsion strip cut from the 80 μm thick sheet, Figure 5(c) was then tested. Its results are also shown in Figure 8. This strip failed about 10 minutes after a load of 1156 MPa had been applied. At stresses above about 250 MPa the results are similar to those obtained with the strip cut from the thick plate, but at lower stresses the modulus defect appears higher. We interpret this to indicate that some stick/slip process occurs at low load which disappears as the load increases. In this case no significant amplitude dependency was observed.

5.4 Simple Pendulum

As a test of our understanding of the damping in pendulums, once we had characterised the the behaviour of our Cu-Be flexures, we proceeded to study a simple pendulum suspended from a flexure strip. The pivot element was in fact two strips of 5 mm total width, 6 mm length and 100 μm thickness clamped in a manner similar to that shown in Figure 5(b). The total mass of the pendulum was close to 1 kg, most of which was in a spherical bob of duralumin at the lower end of a 20 cm duralumin rod (see Figure 3(c)).

A numerical calculation based on Haag's general formalism predicts f(W) = 0,01 Nm for this case, essentially the same value found using the simplified relation derived in Appendix I. In the absence of loss mechanisms other than anelasticity in the flexure, and taking $\Delta E/E$ as $4{,}7 \times 10^{-5}$ independent of frequency, we predict that the logarithmic decrement of small-angle pendulum oscillations should be about $2{,}4 \times 10^{-7}$ with an uncertainty of about 30 % stemming from the uncertainty in the thickness of the flexure. The value actually measured was $2{,}5 \times 10^{-7}$ equivalent to a Q of 4×10^{6}, see Figure 10.

A 5,5 kg simple pendulum of about the same dimension suspended from the same flexure should have a logarithmic decrement some five times smaller than that of the 1 kg pendulum. In fact we found that such a pendulum, Figure 3(c) had a logarithmic decrement that was larger but strongly amplitude dependent, see Figure 10. We conclude that in this case the flexure supports are not sufficiently tightly fixed and stick/slip damping is occuring.

In considering the behaviour of simple pendulums supported on flexure strips, it is worth noting once again that there is no need to increase the length of the flexure beyond that for which α L is greater than about 20. The bending of the flexure is then all taking place close to one end and further increases in length have little or no significant effect on its stiffness but increase the likelyhood of other modes of oscillation being excited.

Since the thermoelastic effect is a well-understood source of anelasticity in flexures [6], it is useful to estimate its expected contribution. In the usual notation, the thermoelastic contribution to $\Delta E/E$ is given by

$$\frac{\Delta E}{E} = \Delta \frac{\omega \tau_L}{1 + (\omega \tau_L)^2} \tag{14}$$

where Δ and τ_L have the following definitions :

$$\Delta = \frac{E \alpha_L^2 T}{\rho C} \quad ; \quad \tau_L = \frac{t^2 \rho C}{\pi^2 k}$$

Here α_L is the linear coefficient of thermal expansion for Cu-Be, ρ is its density, C its heat capacity (at constant volume), k its thermal conductivity and T is the absolute temperature. Using handbook values for Cu-Be and 100 μm for t, the flexure thickness, gives us the following parameter values :

$$\Delta = 0{,}0034 \quad ; \quad \tau_L = 3{,}0 \times 10^{-5}\ \text{s}.$$

Note that when, as in our experiments, $(\omega\ \tau_L) << 1$, eq. (14) becomes

$$\frac{\Delta E}{E} = \frac{E \alpha_L^2 T t^2}{\pi^2 k} \cdot \omega \tag{15}$$

Thus the thermoelastic contribution to $\Delta E/E$ at an oscillation frequency of 1 Hz is $6{,}5 \times 10^{-7}$. This is insignificant compared with the frequency-independent component of $\Delta E/E$.

6. Conclusions

We have shown that the modulus defect of a precipitation hardened Cu-Be alloy is insensitive to stress up to at least 95 % of the yield stress. Its value can be expressed by the relation

$$\frac{\Delta E}{E} = 4{,}3 \times 10^{-5} + 1{,}7 \times 10^{-23} \sigma^2 \tag{16}$$

where the uncertainty in the coefficients is about 10 %.

No significant difference in modulus defect is observed between the same Cu-Be alloy in the form of a flexure strip, torsion wire and torsion strip. The modulus defect does not, therefore, appear to be strongly dependent on grain structure nor on the degree of work hardening as these parameters are significantly different in the different configurations. The most accurate measurements appear to be those made using the torsion strip. This is principally because the stress is uniformly distributed along the strip and, in contrast to the case of the flexure strip, no geometrical factor calling for accurate knowledge of the shape and dimensions of the strip is required to derive the modulus defect from the measured logarithmic decrement. In the present measurements also, the apparent additional damping in the pendulum and in the mounting of the loads complicated the analysis and degraded the accuracy of the flexure measurements of the modulus defect although the experiments themselves were instructive.

The absence of significant stress dependency of the modulus defect is unexpected and is not consistent with our hypothesis given in our earlier paper of a probable stress dependency proportional to the ratio of σ to the yield stress σ shown in Equation 1.

The frequency-independent damping apparently originating at the clamping interface between the flexure and its support in certain configurations, can be explained on the basis of a stick/slip mechanism described in terms of a self-organized critical process. While its frequency independence does not allow it to be distinguished from the intrinsic damping of the Cu-Be, its presence is betrayed by a strong dependence on amplitude of oscillation.

We derive in Appendix I a new expression for the damping of a loaded flexure strip that, for the first time, correctly accounts for the elastic and gravitational potential energies.

The agreement between the measured and calculated damping in a simple pendulum hanging from a Cu-Be flexure strip is good evidence that we now understand the principal sources of damping and that the measured value of the modulus defect of Cu-Be is correct.

Acknowledgements The authors are pleased to acknowledge the essential contribution made by Mr J. Sanjaime, head of the BIPM mechanical workshop, who made the flexure and torsion specimens used in this work.

Load dependence of flexure damping

Consider an unforced pendulum undergoing simple harmonic motion through a small angular displacement θ about the vertical. The equation of motion is

$$- I \omega^2 + C_o = 0 \qquad \text{(A1)}$$

where I and ω are previously defined and c_0 is the total stiffness of the pendulum from all sources. In fact, c_0 has three components :

$$C_o = Wa + C_{grav} + C_{elas} = Wa + C_f$$

where W is the load, a is the distance of the centre of mass of the pendulum below the free end of the flexure, c_{grav} is a loss-free component of the flexure restoring force (i.e. simply due to changes of the height of the free end with respect to rotation in a gravitational field) and c_{elas} is the component due to changes in the elastic energy of the flexure with respect to rotation.

Anelasticity is seen when the Young's modulus contains a "modulus defect" ΔE that produces a phase change between the stress and strain of the flexure. In this case eq. (A1) can be written

$$- I \omega^2 + C_o + i\, f(W) \frac{\Delta E}{E} = 0 \qquad \text{(A2)}$$

where, assuming ΔE << E,

$$f(W) = E \frac{\partial C_{elas}}{\partial E}$$

and it follows that the logarithmic decrement η becomes

$$\eta = \frac{f(W)}{I \omega^2} \frac{\Delta E}{E} \qquad \text{(A3)}$$

We have proposed a model [1] to account for our observation that ΔE/E is independent of frequency over the range of our measurements. We now proceed to a derivation of f(W).

The elastic energy V_{elas} stored in a flexure bent through an angle θ is given by

$$V_{elas} = \frac{1}{2EH} \int_0^L M^2(x)\, dx$$

where M(x) is the bending moment given in [5]:

$$M(x) = (WEH)^{1/2} \frac{\cosh \alpha x}{\sinh \alpha L} \theta$$

The geometry is shown in figure 7 for the special case a = 0. Hence

$$V_{elas} = \frac{W}{2} \theta^2 \frac{1}{2\alpha} \left[\cosh \alpha L + \frac{\alpha L}{\sinh^2 \alpha L} \right] \qquad \text{(A4)}$$

Haag [4], in a very useful monograph, derived a more complicated and precise relation for V_{elas} that takes full account of the geometry and dynamics of the entire oscillating system. It is important therefore to note that eq. (A4) gives only the leading terms in Haag's more general (though less didactic) result. We have carried out numerical calculations which show that the consequences of the terms missing from eq. (A4) are negligible in all the cases we discuss in this report. For other geometries, however, the additional terms given by Haag may indeed become important.

From V_{elas} it is a simple matter to derive c_{elas} :

$$c_{elas} = \frac{\partial^2 V_{elas}}{\partial \theta^2} = \frac{W}{2\alpha} \left[\coth \alpha L + \frac{\alpha L}{\sinh^2 \alpha L} \right] \qquad \text{(A5)}$$

and, finally, f(W):

$$f(W) = \frac{W}{4\alpha} \left(\coth \alpha L + \frac{\alpha L}{\sinh^2 \alpha L} + \frac{2\alpha^2 L^2 \coth \alpha L}{\sinh^2 \alpha L} \right) \qquad \text{(A6)}$$

This derivation of c_{elas} differs from that previously presented in [8] and later cited in [1]. The author of [8] is due the credit for the new derivation given above.

Because we previously misidentified c_{elas}, we believe it useful to demonstrate that the present approach is the correct one. To do this, we first derive c_{grav} from the change in potential energy with respect to θ of a mass attached to the bottom of the flexure, again for the case of simple bending. Clearly

$$V_{grav} = W\,\Delta L$$

where ΔL is the vertical displacement of the end of the flexure as a function of θ. The formula for ΔL may be found in standard texts [9]:

$$\Delta L = \frac{1}{2}\int_0^L (y')^2\,dx = \frac{1}{2}\left(\frac{\tau}{W}\right)^2 \alpha^2 \int_0^L \left(\frac{\sinh \alpha x}{\sinh \alpha L}\right)^2 dx$$

resulting in

$$c_{grav} = \frac{W}{2\alpha}\left[\coth \alpha L - \frac{\alpha L}{\sinh^2 \alpha L}\right]$$

Note that

$$c_f = c_{grav} + c_{elas} = (W/\alpha)\coth(\alpha L), \qquad \text{(A7)}$$

the well-known result for the total stiffness of a flexure in simple bending. In addition, it can be shown that the following inequality is always satisfied

$$1/2 < c_{elas}/c_f < 1$$

A traditional method of deriving the right-hand-side of eq. (A7) was reproduced in [5]. In this derivation, based on a calculation of torques instead of energies, the torque at the fixed end of the flexure is given by

$$\tau_{fixed} = \frac{W}{\alpha}\frac{\theta}{\sinh \alpha L}$$

from which

$$c_{fixed} = \frac{W}{\alpha}\frac{1}{\sinh \alpha L}$$

And the torque at the free end is given as $Wr\theta$, from which it follows that

$$c_{free} = Wr,$$

where $r\theta$ is the horizontal displacement y_0 of the end of the flexure and

$$r = \frac{1}{\alpha}\tanh\frac{\alpha L}{2}$$

This approach also calculates c_f correctly. In our subsequent articles [1, 8], however, we mistook c_{fixed} for c_{elas}. In fact, it can be shown that c_{fixed} generally contains both elastic and gravitational components and c_{free}, though purely gravitational in origin, is thus incomplete as a description of the gravitational stiffness.

One further point is illuminated by our new derivation. Although $r\theta$ is merely the horizontal displacement of the flexure tip, we have [5, 8], as have others before us, referred to r as the "effective radius" of the flexure, thereby creating a false impression that the tip of the flexure actually traces out the arc of a circle of radius r. Our derivation of ΔL shows that the change in height of the free end of the flexure is not in general equal to $r(1-\cos\theta)$ to second order in θ.

The arc traced by the tip of a flexure is, perhaps, most easily observed for a very long, unloaded strip; a steel meter stick, for example. The above equations predict that in this special case, and for small angular displacements, $r = 500$ mm and $\Delta L = y_0^2/(3r)$, i.e. the arc of a circle of radius $(3/2)r$ to second order in θ.

The derivation of eq. (A7) assumes that the flexure is subjected to simple bending; that is, the bending moment has been assumed to be independent of position on the flexure. This is only an approximation, as we have mentioned in reference to Haag's more complete analysis. We thus expect there to be additional shear forces given by

$$F = \frac{\partial}{\partial x} M(x)$$

Including these forces in the calculation of flexure bending gives an additional term in the stiffness which can be included as a correction to H, the second moment of area. Thus

$$H' = H\left(1 + \frac{W}{Gbt}\right)$$

where G is the shear modulus given by

$$G = \frac{E}{2(1+\mu)}$$

and μ is the Poisson ratio. All our results for stiffness and damping in flexures can thus be modified easily to include the effects of shear by merely changing H to H'. In all of the flexure geometries which we employ, the effects (including damping) due to shear are negligible.

Some remarks on the comparative properties of a torsion fibre and a torsion strip

It is usually understood that the optimum configuration of a torsion pendulum used as a detector in gravitational experiments, is the one in which the torsion fibre is as thin as possible and the supported test masses correspondingly small. This conclusion is reached because the restoring force c_o of a torsion fibre is proportional to the fourth power of its radius and the load it can support is proportional to its radius squared. For a pendulum supported on a fibre of radius r, of length L and where G is the shear modulus,

$$c_o = \frac{\pi}{2} r^4 \frac{G}{L} \tag{1}$$

In very thin fibres loaded close to their yield stress the problem of zero drift is a serious one and is often a limiting factor in the accuracy of the experiment.

A torsion pendulum employing a torsion strip, rather than a torsion wire, has two advantages. One of these is obvious and is related to the geometry but the other is perhaps less obvious and was observed in this work. We observed that even after applying loads up to 95 % of the yield stress, the subsequent zero drift of the pendulum over some days was below our level of detection, namely about 50 µrad. The torsion strip appears to have a natural zero stability which is absent in the torsion wire.

The geometrical advantage of a torsion strip stems from its stiffness depending mostly on the thickness while the load it can support depends additionally on the width which can be varied independently. We have for a strip of width b and thickness t and length L

$$c_o = \frac{bt^3}{16} \left(\frac{16}{3} - 3{,}36 \frac{t}{b} \right) \frac{G}{L} \tag{2}$$

This is an approximation that is sufficiently close for all practical purposes to the exact expression given, for example in [A1]. For strips having $b >> t$, this expression differs very little from

$$c_o = \frac{bt^3}{3}\frac{G}{L} \qquad (3)$$

A comparison of equations (1) and (3) shows that for a given stress in the fibre and strip, i.e. we set $bt = \pi r^2$, the stiffness of the strip can be made much less than that of the fibre. The ratio of the two stiffnesses is given by

$$\frac{c_o\ (\text{fibre})}{c_o\ (\text{strip})} = \frac{3}{2\pi}\frac{b}{t}$$

so that for a strip having a ratio of b to t of 100 the sensitivity of the strip is fifty times that of the fibre. If in addition, the zero drift of such a strip is small, there may be experiments in gravitation for which a torsion strip is significantly more advantageous than a torsion wire.

That the torsion strips used in the present experiments were well behaved is further supported by the following observation. The value of c_o for the strip of width 1 mm, length 100 mm and thickness 90 µm was found to be
$(1{,}30 \pm 0{,}01) \times 10^{-4}$ Nm rad^{-1} and was independent of W over the entire range of loads from 3,5 N up to 77 N and is in good agreement with the value calculated from equations 2 or 3 taking G = 50 GPa. Had the torsion element been made from two thin parallel flexible wires of length L and separation b, however, the torsion stiffness, due simply to geometry, would have been

$$c_o' = Wb^2/(4L)$$

which for b = 1 mm, L = 100 mm, W = 77 N gives a value for c_o of $1{,}9 \times 10^{-4}$ Nm/rad. Since the measured value of c_o did not change by more than $0{,}01 \times 10^{-4}$ Nm/rad over the whole range of loads, we can infer that $c_o' \leq 0{,}01 \times 10^{-4}$ Nm/rad.

More generally, if a rotation θ raises the centre of mass by an amount z, then the potential energy changes by

$$Wz = \frac{1}{2}\, c_o'\, \theta^2.$$

We have just inferred, however, that $c_o' \leq 0{,}01 \times 10^{-4}$ Nm/rad at W = 77 N. For an amplitude θ = 10 mrad and W = 77 N, this implies that $z < 10^{-12}$ m. Conclusion : the strip is behaving rigorously as a torsion strip, the centre neutral fibre is neither shortening nor lengthening during oscillation by more than about one part in 10^{11} of its length !

Table I

W/N	b/mm	σ/MPa	$\frac{I\omega^2\eta}{f(W)}$ /10^{-5}	$\frac{A}{f(W)}$ /10^{-5}	$\frac{\Delta E}{E}$/10^{-5}
14,7	35	4,9	7,5	1,5	5,0
	24	7,2	7	2,2	4,8
	10	17,3	10,4	5,3	5,1
	5	34,6	15	9,6	5,4,
	5*	34,6	13	9,6	3,4
25,3	5*	59,5	17	6,9	10,1
54,7	5*	129	16	6,0	10,0
95,2	5*	224	9,2	3,7	5,5

$$\frac{\Delta E}{E} = \frac{I\,\omega^2\,\eta}{f(W)} - \frac{A}{f(W)}$$

Where A = 4,8 x 10^{-7} Nm

Average value for W = 14,7 N :

$$\frac{\Delta E}{E} = 4{,}7 \times 10^{-5}$$

* Measurements made with screwed-down loads and strengthened pendulum.

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Captions to figures

1. Vertical pendulum made from duralumin showing additional loads.

2. Vertical pendulum made from duralumin showing masses for adjusting the period.

3. Torsion and simple pendulums.

4. Cu-Be flexure strip assemblies
   (a) flexure cut from solid block of Cu-Be ;
   (b) composite flexure made from 80 μm Cu-Be sheet clamped between steel blocks ;
   (c) composite flexure made from 1,4 mm thick Cu-Be sheet with a 90 μm thick flexure cut in it and steel clamping blocks.

5. Cu-Be torsion strip assemblies
   (a) 200 mm long strip cut from 80 μm thick sheet and either soldered or crimped into copper end lugs ;
   (b) 100 mm long strip, 90 μm thick and 1 mm wide cut from 1,4 mm thick sheet to give integral 10 mm wide end plates for clamping in steel blocks ;
   (c) 100 mm long strip, cut from 80 μm thick sheet having 10 mm wide ends for clamping between steel blocks.

6. Measured values of logarithmic decrement, η, as a function of amplitude of pendulum oscillation for a range of pendulum loads : ⊙ flexure made from 80 μm thick sheet directly clamped between steel blocks (Figure 4(b)) ; Δ flexure made from 1,4 mm thick sheet having an integral 90 μm thick flexure (Figure 4(c)). Inset : values of η obtained by extrapolating these curves to zero amplitude plotted as a function of stress.

7. The shape of a flexure strip under a load W applied at the lower end of the flexure. Inset illustrates the shear stresses acting at the interfaces between the strip and the clamping blocks.

8. The quantity $\eta I\omega^2/f(W)$ plotted as a function of stress : O , □, Δ and ⊙ for the flexure of Figure 4(c) progressively reduced in width for b = 35, 24, 10 and 5 mm respectively, all with the unloaded pendulum, W = 14,7 N ; X, the same

flexure with a width of 5 mm after strengthening the pendulum and at loads up to 95 N as indicated ; ● and ✖ values corrected for the constant A following Equation 13. Data taken from Table 1.

9. The modulus defect, $\Delta E/E$, for two Cu-Be strips measured as a function of stress up to the breaking points using a torsion pendulum having periods from 28 s to 73 s. The first strip, ⊙ , was 90 μm thick, 1 mm wide and 100 mm long and cut from 1,4 mm Cu-Be sheet (see Figure 5(b)) ; it failed at a load of 950 MPa was being applied ; the second, □, was 80 μm thick and cut from 80 μm thick sheet (see Figure 5(c)), it failed a few minutes after a load of 1150 MPa had been applied.

10. Measured and predicted values of the logarithmic decrement, η, and Q of simple pendulums. X, a 1 kg bob having a period of 1,05 s with predicted values of $\eta = 2{,}4 \times 10^{-7}$ and $Q = 4{,}2 \times 10^{6}$; ⊙ , a 5,5 kg bob having a period of 1,05 s with predicted values of $\eta = 4{,}4 \times 10^{-8}$ and $Q = 2{,}3 \times 10^{7}$.

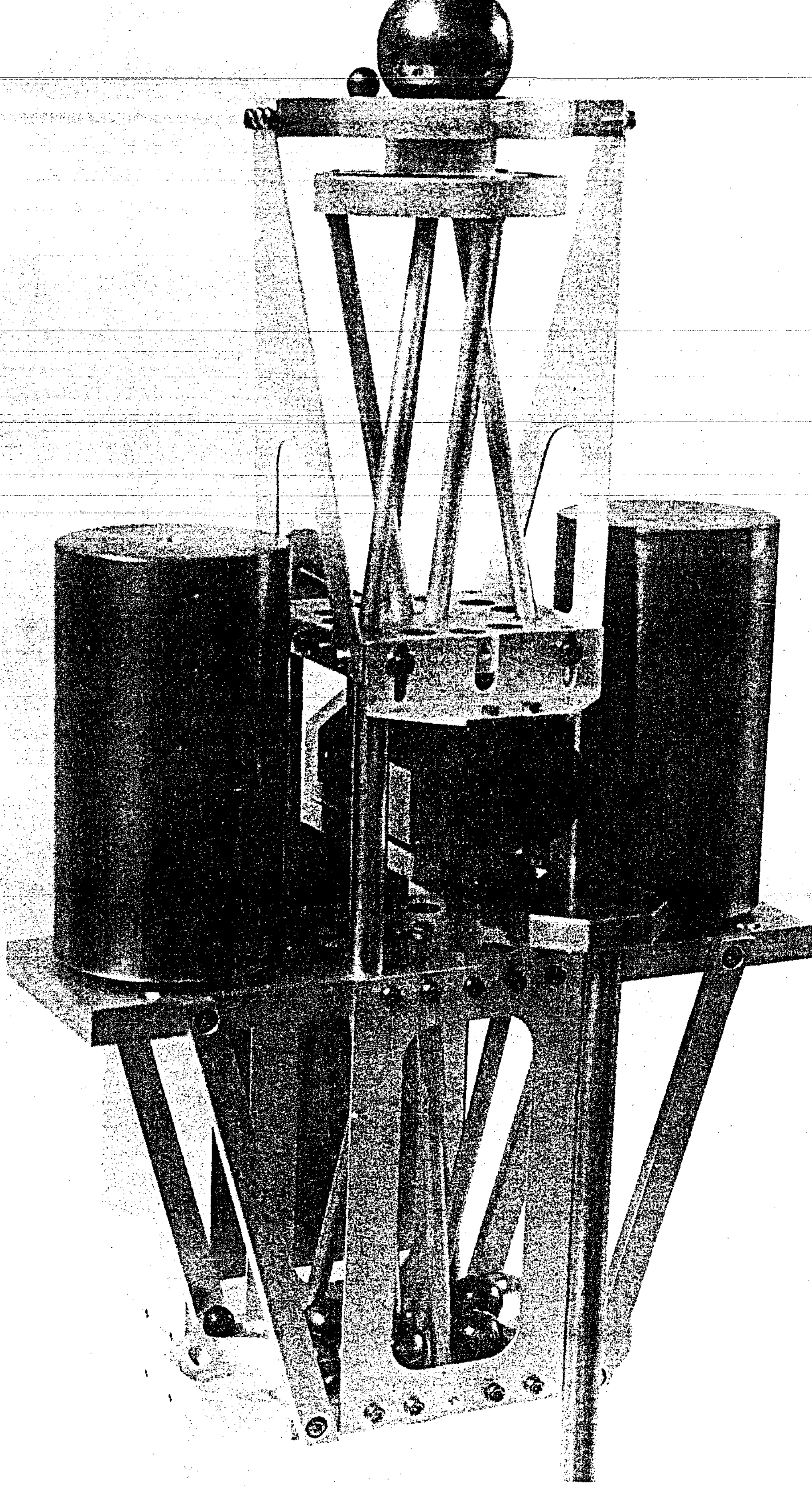


Figure - 1

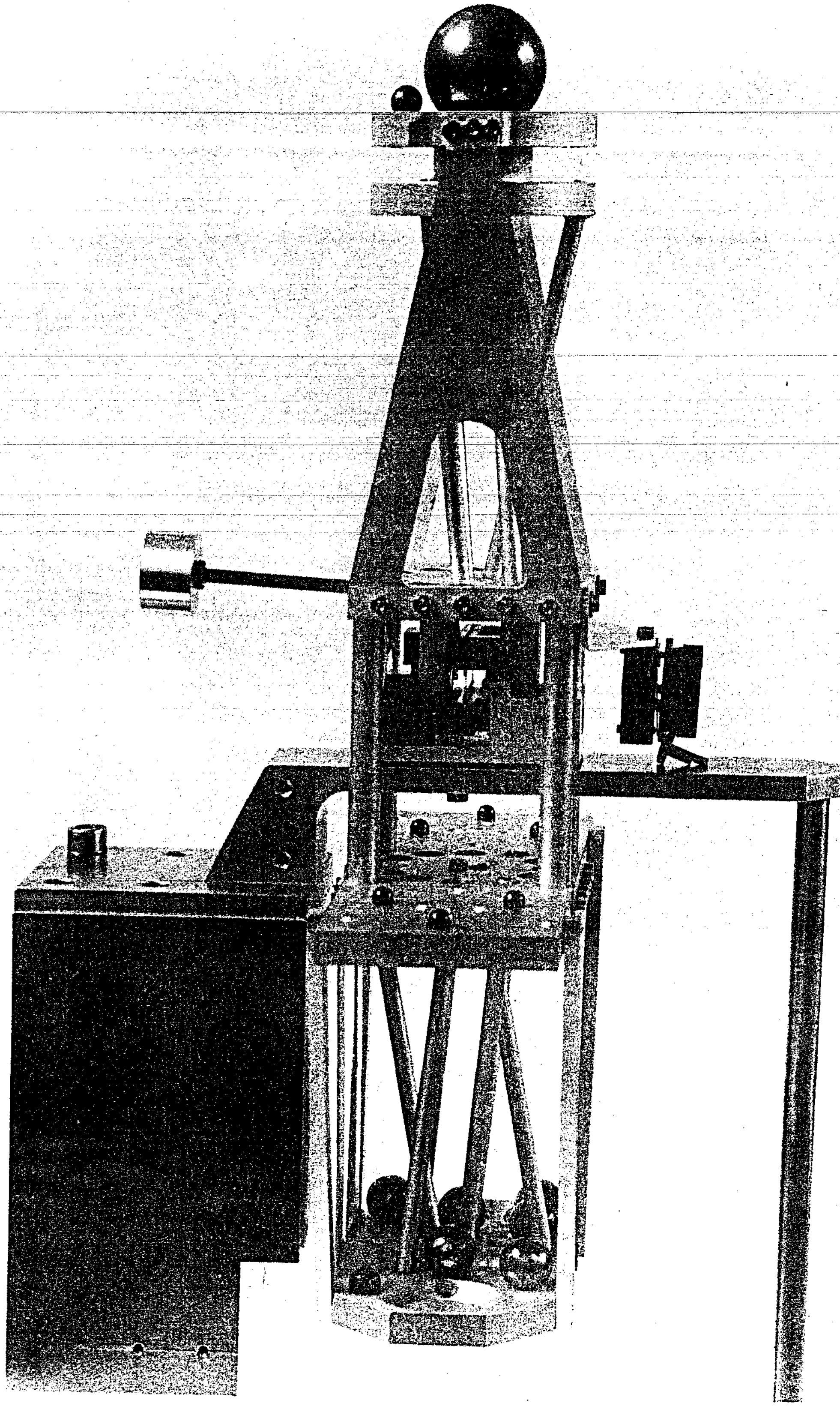


Figure - 2

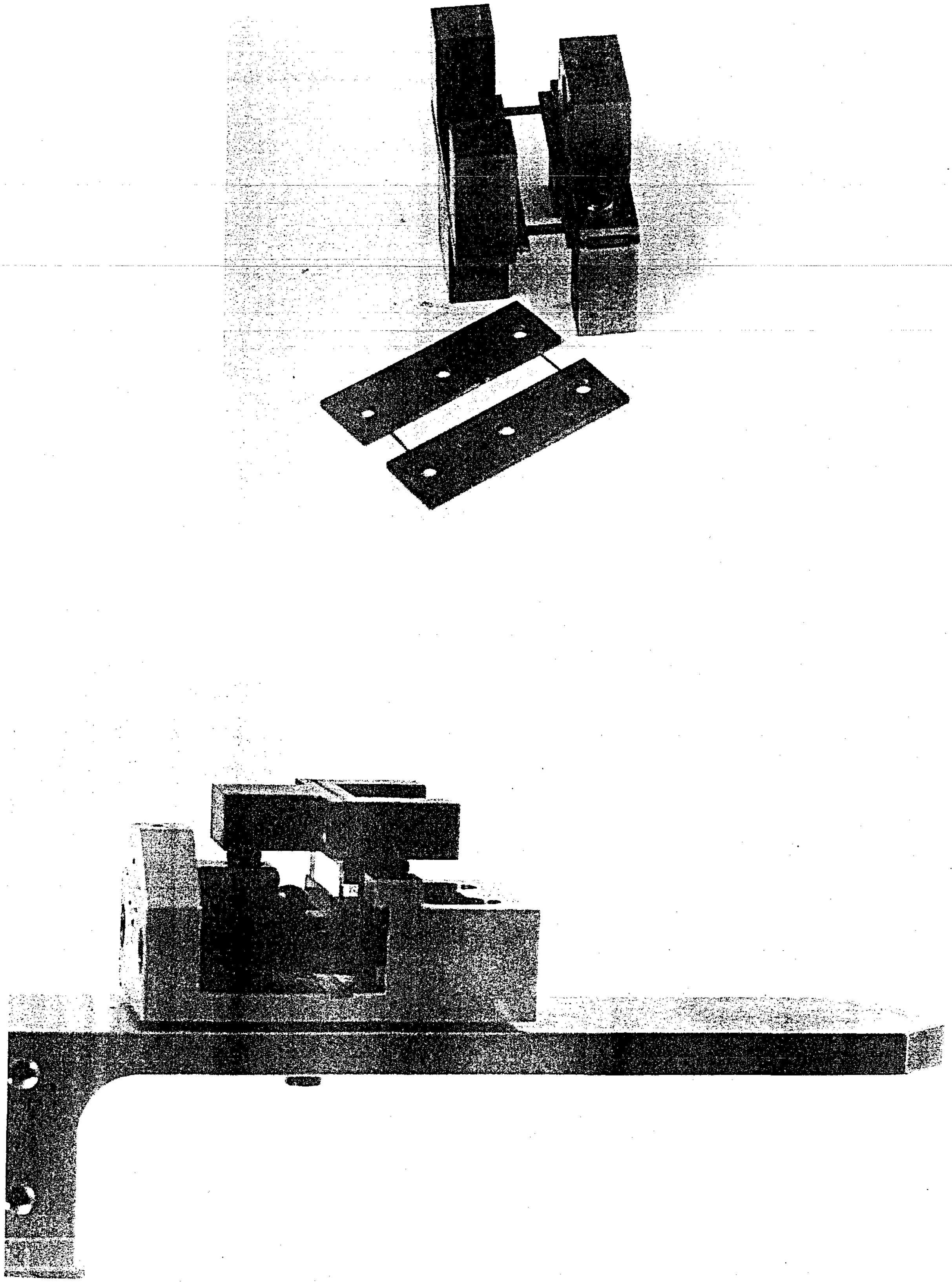


Figure - 2a

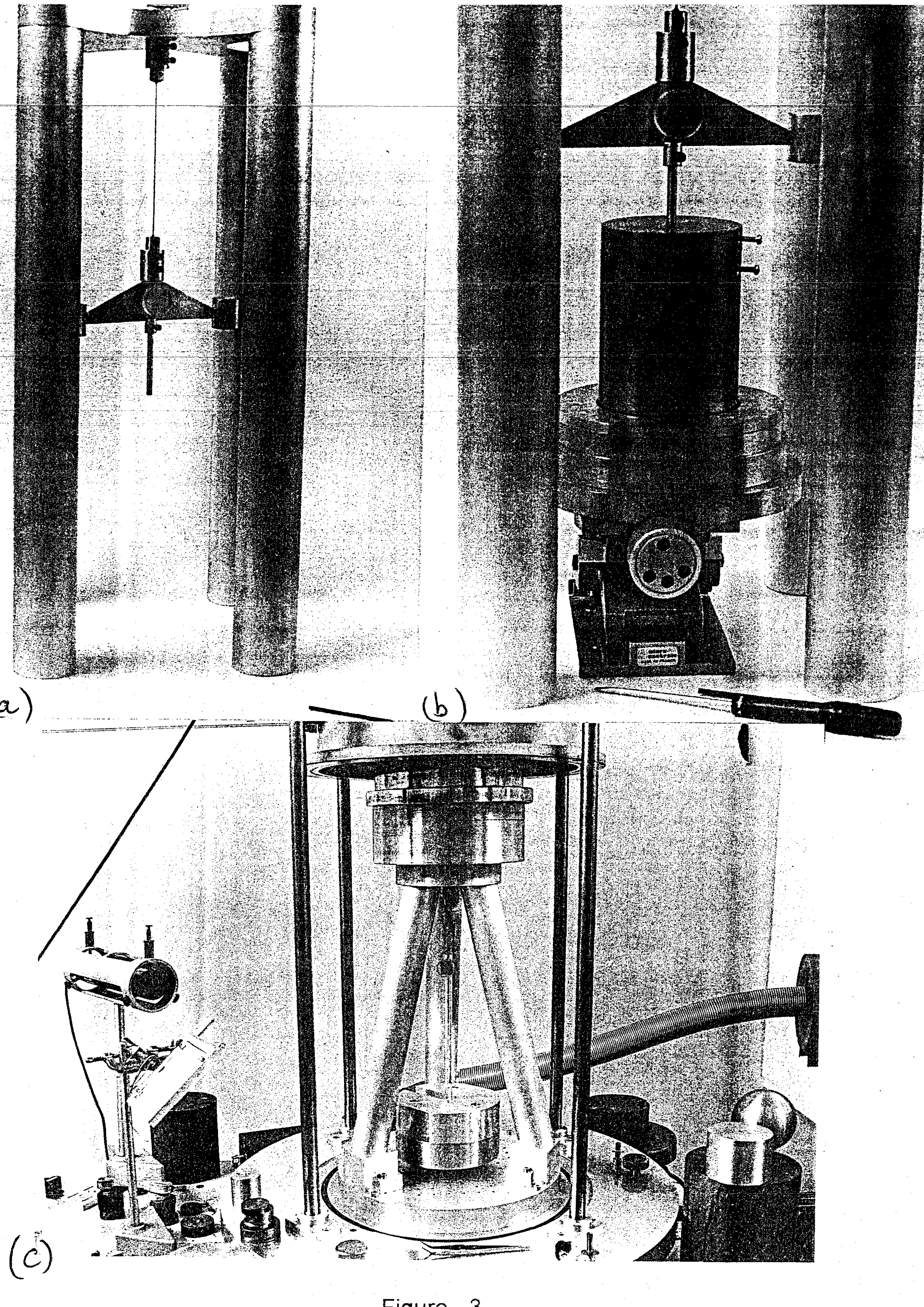


Figure - 3

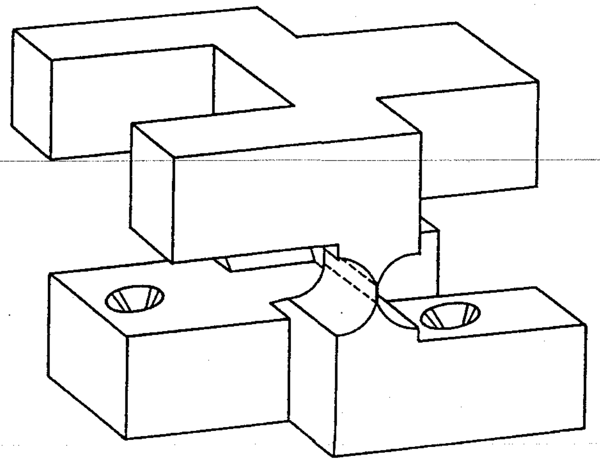


Figure 4(a)

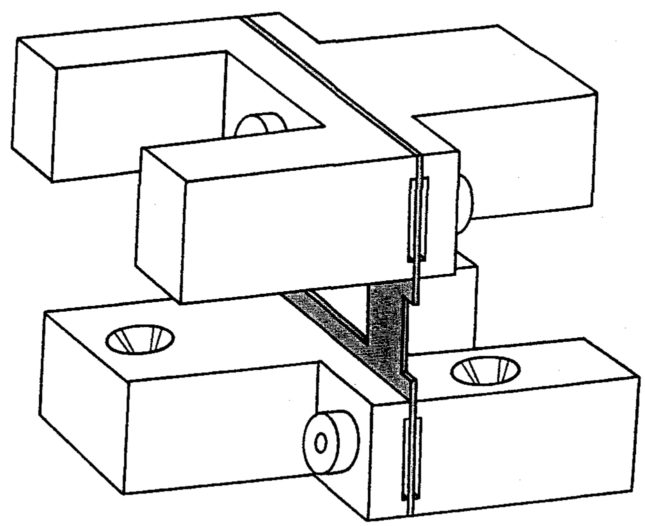


Figure 4(b)

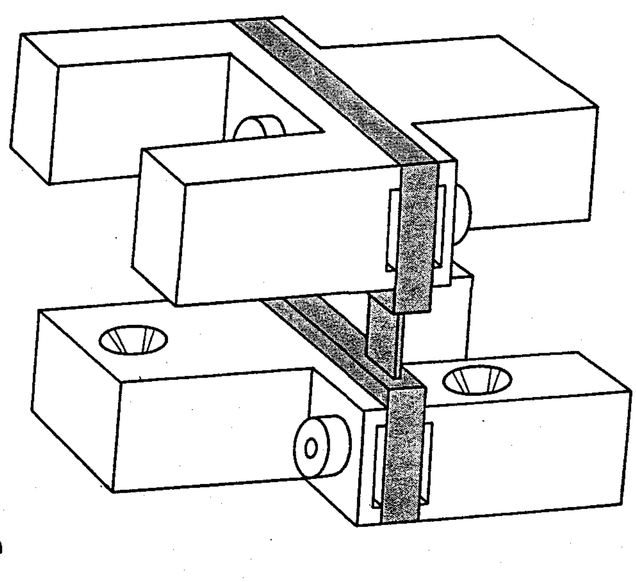


Figure 4(c)

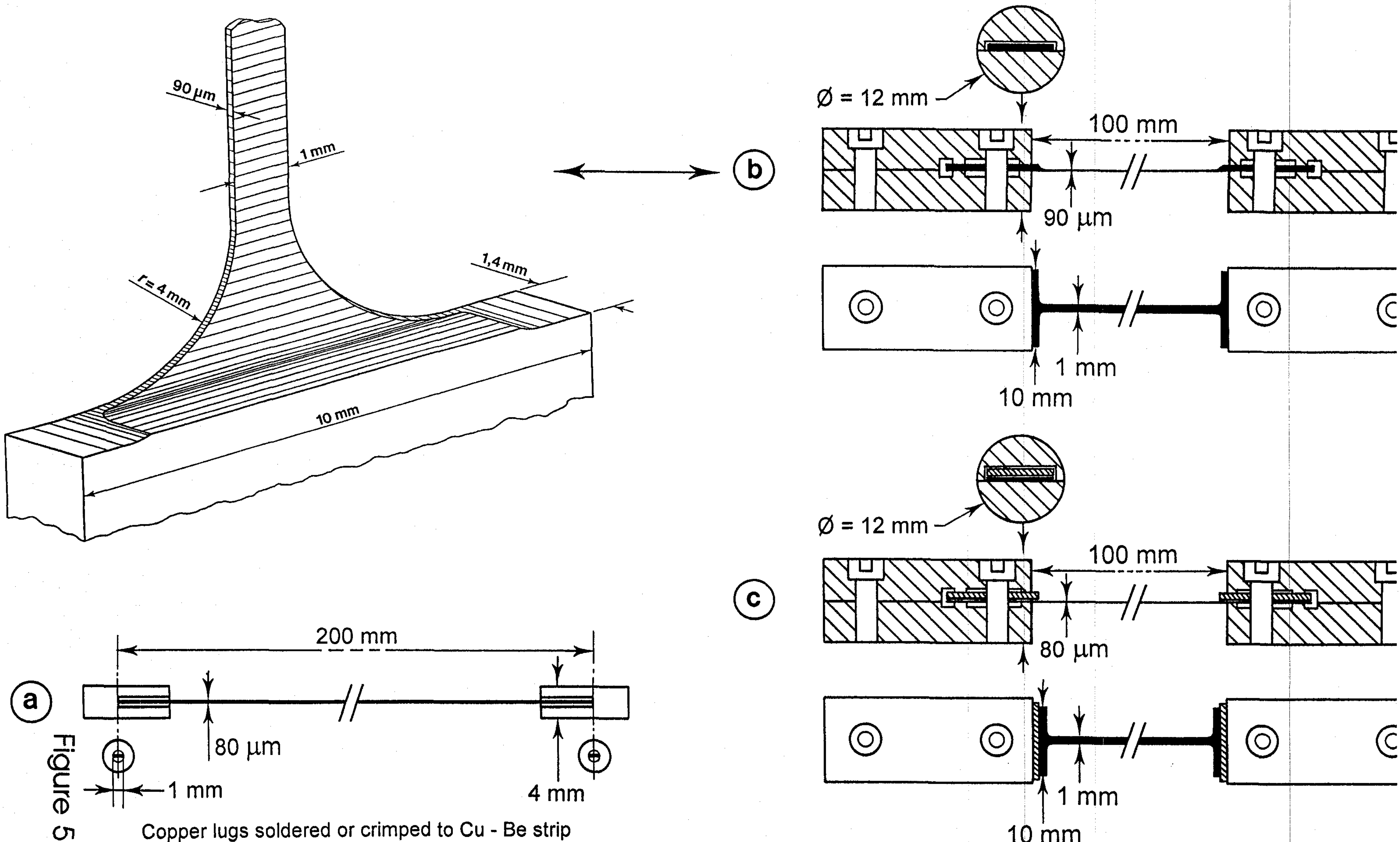
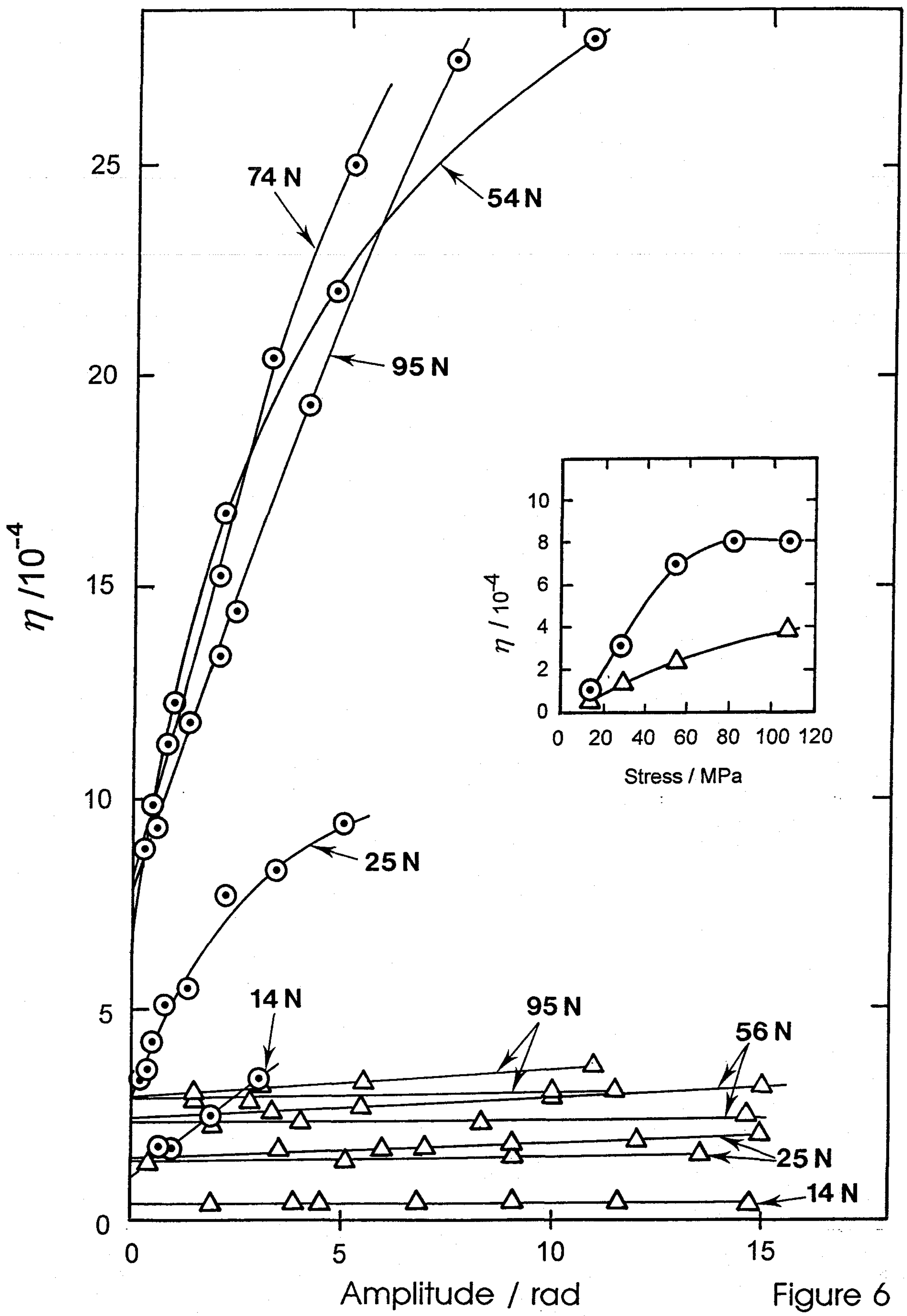

90 µm
1 mm
r = 4 mm
1,4 mm
10 mm
b
Ø = 12 mm
100 mm
90 µm
1 mm
10 mm
c
Ø = 12 mm
100 mm
80 µm
1 mm
10 mm
a
200 mm
80 µm
1 mm
4 mm
Copper lugs soldered or crimped to Cu - Be strip

Figure 5



η /10^-4
Amplitude / rad
74 N
54 N
95 N
25 N
14 N
95 N
56 N
25 N
14 N
0
5
10
15
20
25
0
5
10
15
η / 10^-4
Stress / MPa
0
2
4
6
8
10
0
20
40
60
80
100
120

Figure 6

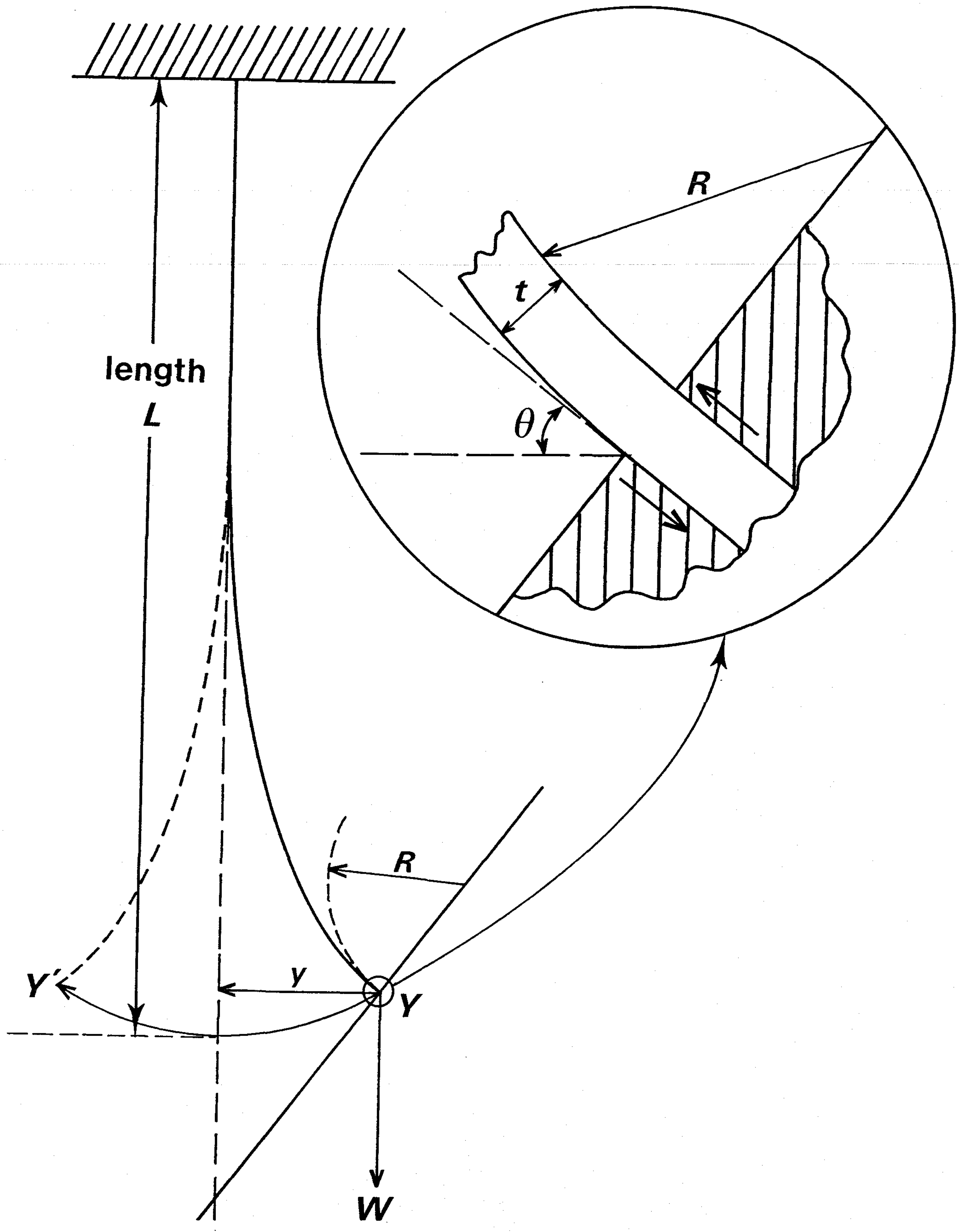


Figure 7

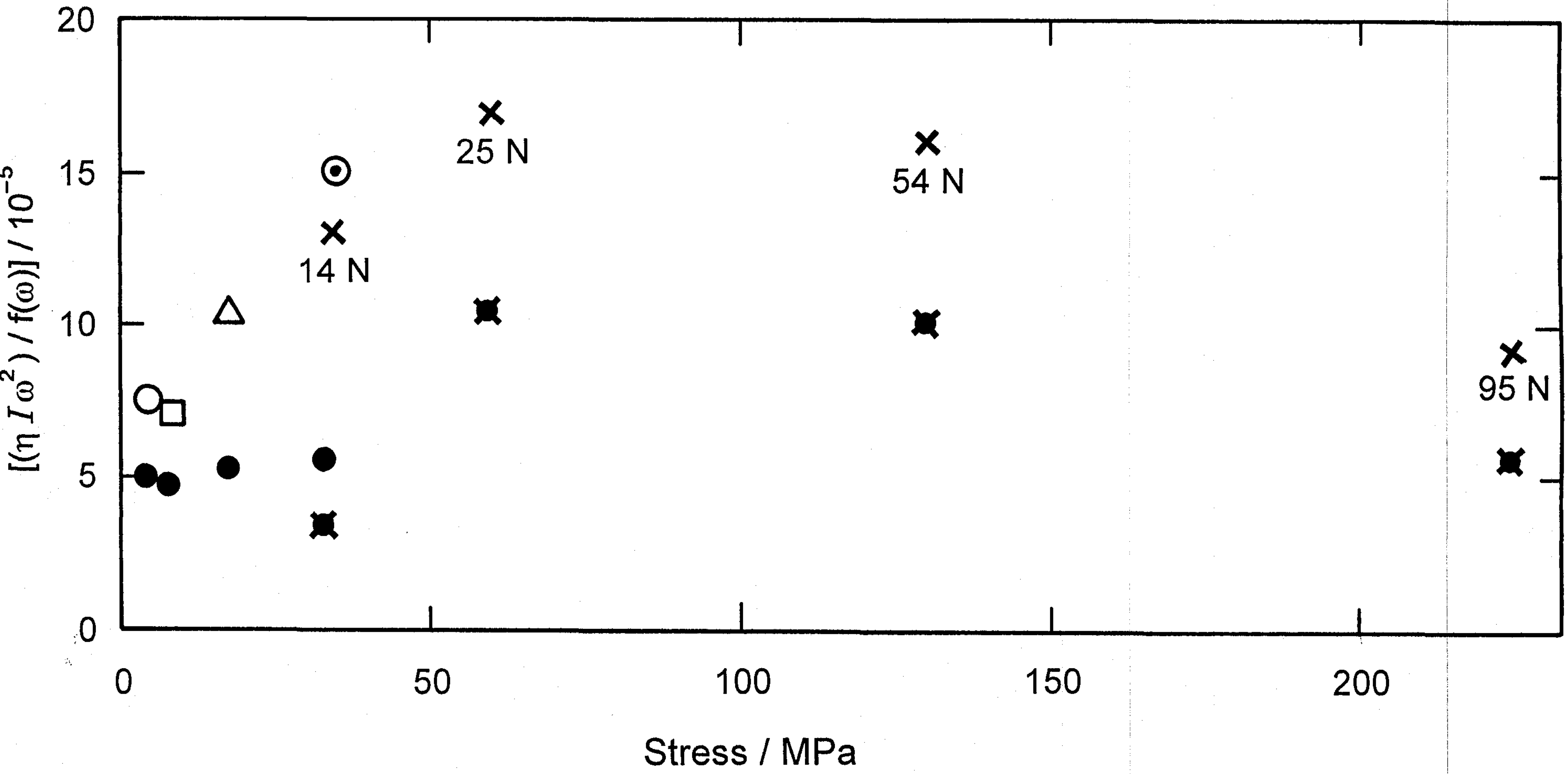
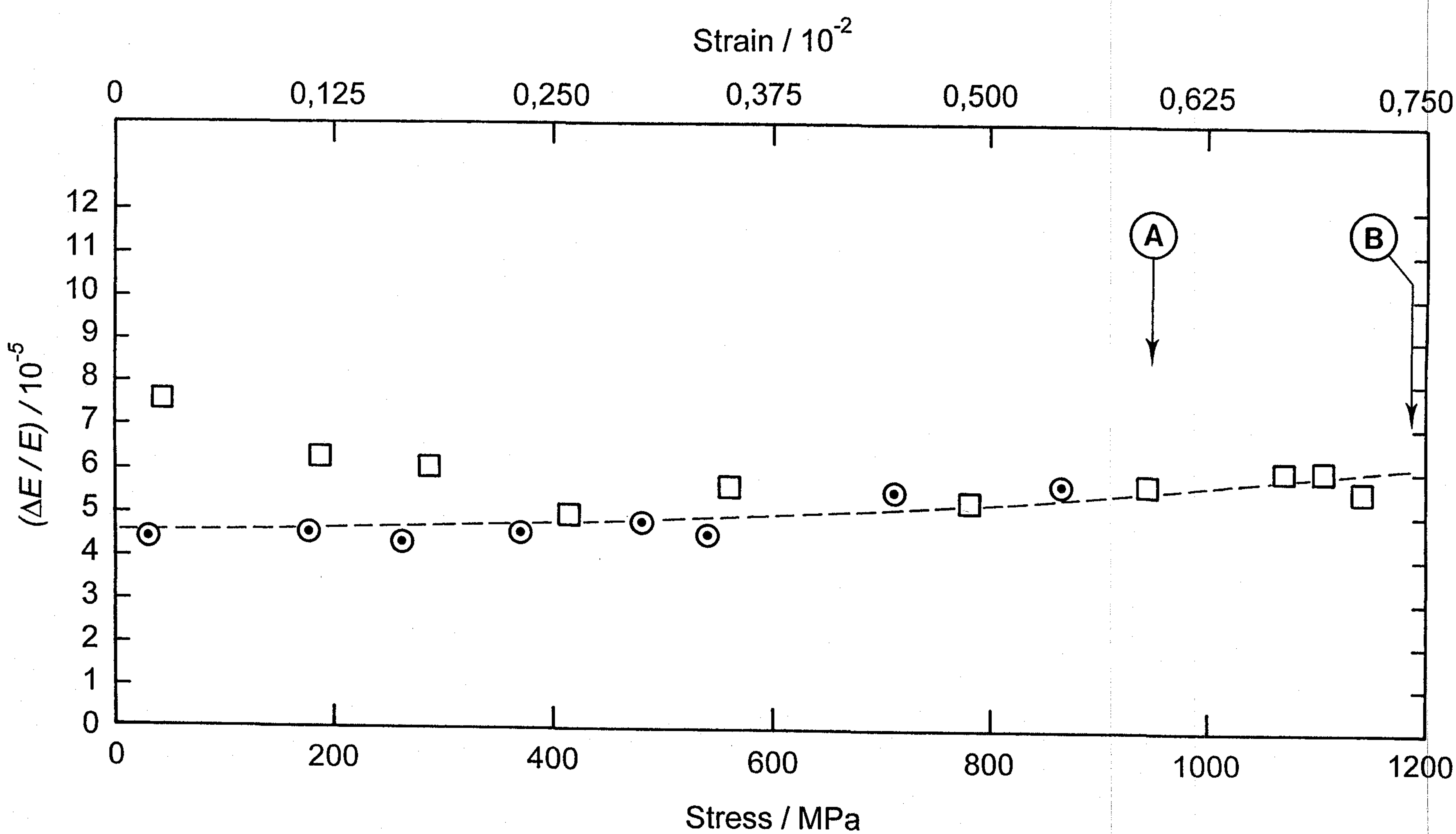

20
15
10
5
0
0
50
100
150
200
Stress / MPa
[(η I ω2) / f(ω)] / 10−5
25 N
54 N
14 N
95 N

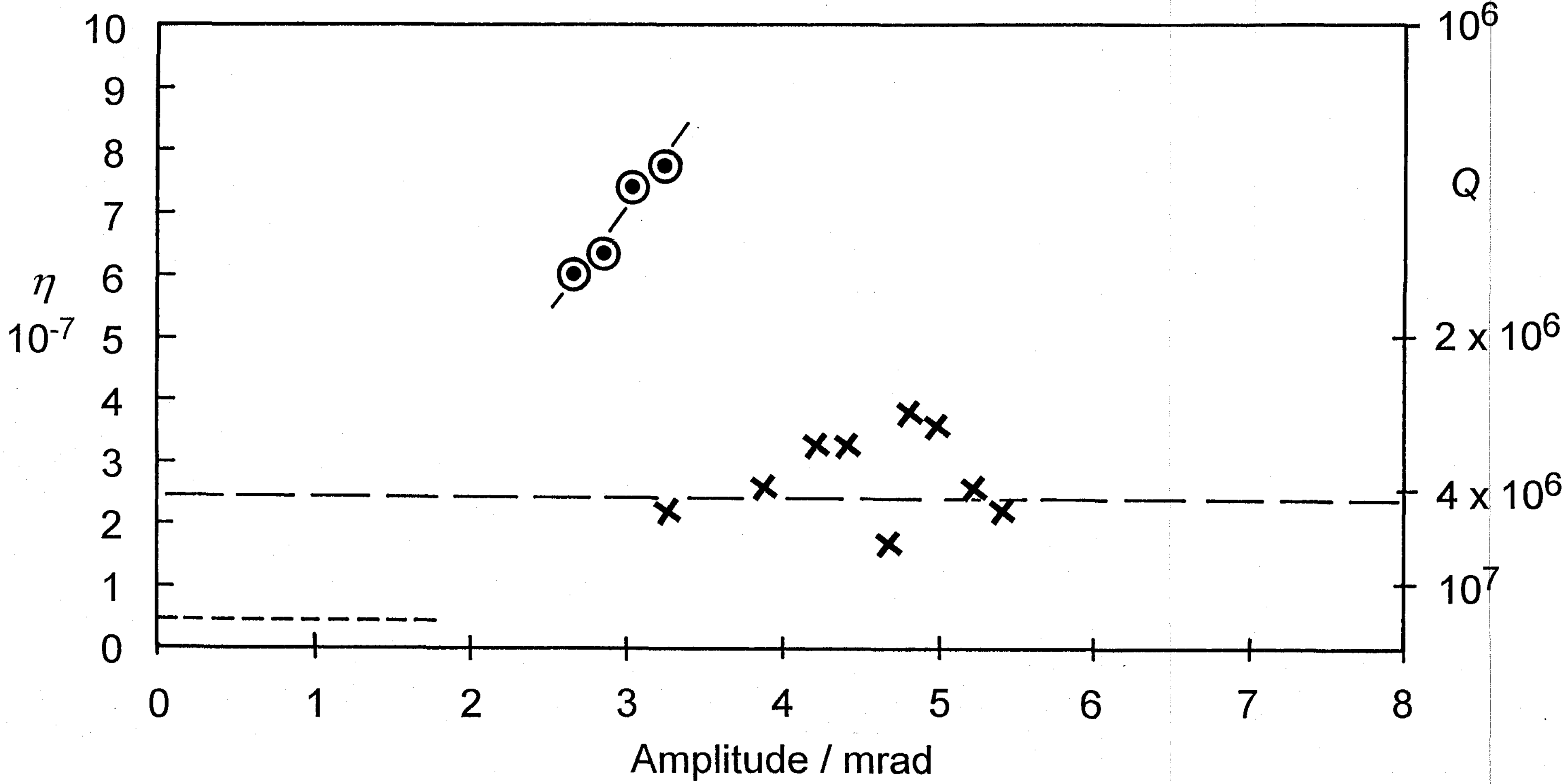
Figure 8



Strain / 10^-2
0
0,125
0,250
0,375
0,500
0,625
0,750
12
11
10
9
8
7
6
5
4
3
2
1
0
(ΔE / E) / 10^-5
A
B
0
200
400
600
800
1000
1200
Stress / MPa

Figure 9

Figure 10 Logarithmic decrement, η, and Q of simple pendulums



× 1 kg bob, T = 1,05 s predicted values $\eta = 2{,}4 \times 10^{-7}$ $Q = 4{,}2 \times 10^{6}$

⊙ 5,5 kg bob, T = 1,05 s predicted values $\eta = 4{,}4 \times 10^{-8}$ $Q = 2{,}3 \times 10^{7}$