LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Technical Note **LIGO-T960005-00-R** May, 1996

PRINCIPLES OF CALCULATING ALIGNMENT SIGNALS IN COMPLEX RESONANT OPTICAL INTERFEROMETERS

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PRINCIPLES OF CALCULATING ALIGNMENT SIGNALS IN COMPLEX RESONANT OPTICAL INTERFEROMETERS

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May 8, 1996

ABSTRACT

In the large laser interferometers presently under construction to measure gravitational waves the alignment of the optical components with respect to the incoming laser beam is crucial to maintain maximum phase sensitivity. We present a basic formalism to analytically calculate the effects of misalignment and beam distortions in an arbitrary complex optical system, including coupled cavities and Michelson interferometer configurations coupled with cavities. The electromagnetic field is decomposed into a superposition of higher-order Gaussian modes, while misaligned and distortive optical components along with free space propagators are represented by matrix operators that act on the state vectors in this basis. We show how to generally deduce useful alignment signals, in order to design angular control systems.

KEYWORDS

Gravitational-wave observatories, laser interferometer, misalignment of Gaussian beams, mode decomposition.

1 INTRODUCTION

Gravitational radiation from astrophysical sources produces a strain in space transverse to the direction of propagation. Since this strain has opposite signs along orthogonal axes, variants of a Michelson interferometer have been proposed to measure it. A typical interferometer configuration, currently planned for the Laser Interferometric Gravitational-wave Observatory (LIGO) interferometer [1], is comprised of a Michelson interferometer with partially transmitting mirrors as inputs to Fabry-Perot optical cavities in each arm and a partially transmitting "recycling" mirror between the input laser and the beamsplitter. The Fabry-Perot arm cavities increase the changes in phase of the light due to arm length changes, and the recycling mirror enhances the signal-to-noise ratio by increasing the circulating light in the interferometer. The optical lengths in the interferometer can be held on resonance using various locking schemes [2],[3]. They are based either on the Pound-Drever-Hall reflection locking technique which was first proposed to hold a Fabry-Perot cavity on resonance [4] or on the Schnupp modulation technique [5] which requires the implementation of a length asymmetry into the Michelson interferometer.

While maintaining the longitudinal separations between the test mass mirrors to within 10^{-12} meters over a 4 km baseline is crucial, the sensitivity of the interferometer to gravitational-wave strain also depends on the angular alignment of the interferometer with respect to the incoming laser beam. Misalignment of the optical components of the interferometer with respect to the incident laser light causes light power in the fundamental mode of the interferometer to be coupled into higher-order modes. This reduces the amount of power circulating in the arm cavities due to diminished coupling of light into them, resulting in reduced phase sensitivity [6]. The higher-order modes leaking out the signal extraction port of the interferometer result in increased photocurrent and thus shot-noise, and effectively lower the contrast of the interferometer, which, in turn, lowers the recycling gain. Both effects compromise the signal-to-noise ratio of the gravitational-wave signal readout. Since misalignments affect the two arms independently, we get a loss of common mode rejection of fluctuations in the laser power and laser frequency, and of input beam jitter. Furthermore, the longitudinal control system can be corrupted by noise due to coupling between transverse and longitudinal degrees of freedom. For these reasons, it is crucial to maintain tight angular control of the interferometer mirrors.

In this paper, we present a formalism which can be used to study the problem of misalignment in the LIGO interferometer, with the goal of characterizing the sensitivity of the interferometer to angular misalignment and designing a dynamic scheme for maintaining alignment.

The general concept of using the transverse (off-axis) modes of an optical resonator to detect misalignment and mismatch was proposed by Anderson [7]. Variations of this idea in conjunction with phase sensitive detection have been proposed and experimentally demonstrated for simple two mirror resonators [8], [9]. Specifically, Morrison *et al.* [9] use the Pound-Drever-Hall reflection-locking technique [4] for alignment sensing. This is particularly advantageous since light which is already circulating in the interferometer is used to sense both longitudinal and transverse degrees of freedom, which also ensures that the interferometer is aligned relative to the input light beam. Morrison *et al.* investigated a flat-curved Fabry-Perot interferometer, where they showed that for a infinitely long cavity an angular misalignment of the input (rear) mirror produces an alignment signal in the near (far) field of the light reflected from the input mirror.

The present work extends this concept by developing a formalism to analyze more complex optical systems comprising of an arbitrarily complicated setup of optical elements. In particular, it can be applied to systems consisting of cavities placed inside of other cavities. The field circulating in the misaligned or distorted optical system is decomposed into a superposition of the eigenmodes of the unperturbed system; the scale of the imperfections or misalignments determines the number of eigenmodes needed for an accurate description. In this approach misaligned or distortive optical components are represented as operators in the basis of these eigenmodes. This approach relies on analytical methods, which gives it tremendous advantage over currently used numerical methods, e.g. Fast Fourier Transform [10], which are computationally demanding. For simplicity and tractability, we first apply our model to the ubiquitous Fabry-Perot cavity, and then build up more complex optical configurations by cascading more optical components in the train. We thereby emphasize the generality of the model as an analytical tool for analyzing field distortions in any complex optical system.

The formalism for calculating misalignment and distortion operators in the modal space is presented in section 2. The detection scheme for misalignments is described in section 3. Section 4 outlines some applications for more complex optical systems like coupled cavities and recycled Michelson interferometer configurations.

2 FORMALISM FOR MODE DECOMPOSITION

The x- and y- axes of the coordinate system are chosen to be transverse to the beam propagation or optic axis of the perfectly aligned and undistorted system (z-axis). Using the paraxial approximation [11] one can quite generally expand the electromagnetic field of a light beam as a superposition of orthonormal Gaussian modes in the form

$$E(x, y, z) = \sum a_{mn} U_{mn}(x, y, z) \tag{1}$$

where a_{mn} is a vector in the modal space. The $U_{mn}(x,y,z)$ are Gaussian modes, which may be Hermite-Gaussian functions, as in Appendix A. Our goal is to compute the eigenfunctions $U_{mn}(x,y,z)$ only once for the perfectly aligned and undistorted system and then treat any angular misalignment or distortion as a perturbation which transfers energy between transverse modes only. In other words, the perturbation can be expressed as a matrix operator acting on a complex vector space (the modal space), and the solutions to the paraxial wave equation of the misaligned or distorted system can be calculated through a perturbation series approach from the solutions of the unperturbed system.

If $M(x,y,z_2,z_1)$ is an operator which transforms the electromagnetic field of a **misaligned or distorted** optical system at position z_1 into a field at position z_2, i.e.

$$E'(x, y, z_2) = M(x, y, z_2, z_1) \otimes E(x, y, z_1) \tag{2}$$

its representation $M_{mn,kl}(z_2,z_1)$ in modal space can be written as

$$M_{mn,\,kl}(z_2, z_1) = \int_{-\infty}^{\infty}\!\!\int U^{\dagger}{}_{mn}(x, y, z_2) M(x, y, z_2, z_1) U_{kl}(x, y, z_1) dx\, dy \tag{3}$$

where the functions $U_{mn}(x,y,z)$ are the eigenmodes of the **unperturbed** system.

An important simplification in calculating the modal space representation of these operators for a real physical system can be obtained by entirely separating the longitudinal propagation from misalignment and distortion effects caused by lenses and mirrors, which affect the wavefront at a fixed longitudinal position only.

Since the $U_{mn}(x, y, z)$ are the vacuum eigenmodes, the modal space representation of the propagation operator must be diagonal. In the Hermite-Gaussian basis, e.g., the propagator simplifies to

$$P_{mn,\,kl}(\eta) = \delta_{mk}\delta_{nl}\, e^{-ik(z_2 - z_1)} e^{i(m + n + 1)\eta} \tag{4}$$

where $\eta = \eta(z_2) - \eta(z_1)$ is the Guoy phase shift between position z_2 and z_1 (see Appendix A). We have separated the rapid longitudinal (z-coordinate) variation in the propagator of the Gaussian beam. The propagator is the only operator which retains a significant z-dependence. Hence, for lenses and mirrors eqn. (3) reduces to

$$M_{mn,\,kl} = \langle mn|M(x, y)|kl\rangle \tag{5}$$

where the braket-product is defined as the integration over the transverse degrees of freedom and where $\langle mn|$ and $|kl\rangle$ are the Gaussian eigenmodes with all z-dependence due to propagation removed.

Having calculated the modal space representation of both the propagation and the effects of misalignment and distortion, it is now possible to calculate the eigenmodes of any misaligned or distorted configuration of the optical system by means of linear algebra only, without repeatedly resolving the paraxial wave equation.

Consider a slightly imperfect and slightly misaligned mirror[1] (see Fig. 1). Spatial variations over the mirror surface cause each part of the wavefront which is incident at a lateral offset (x,y) to acquire an additional phase shift due to a local displacement in the z-direction. The parameters of

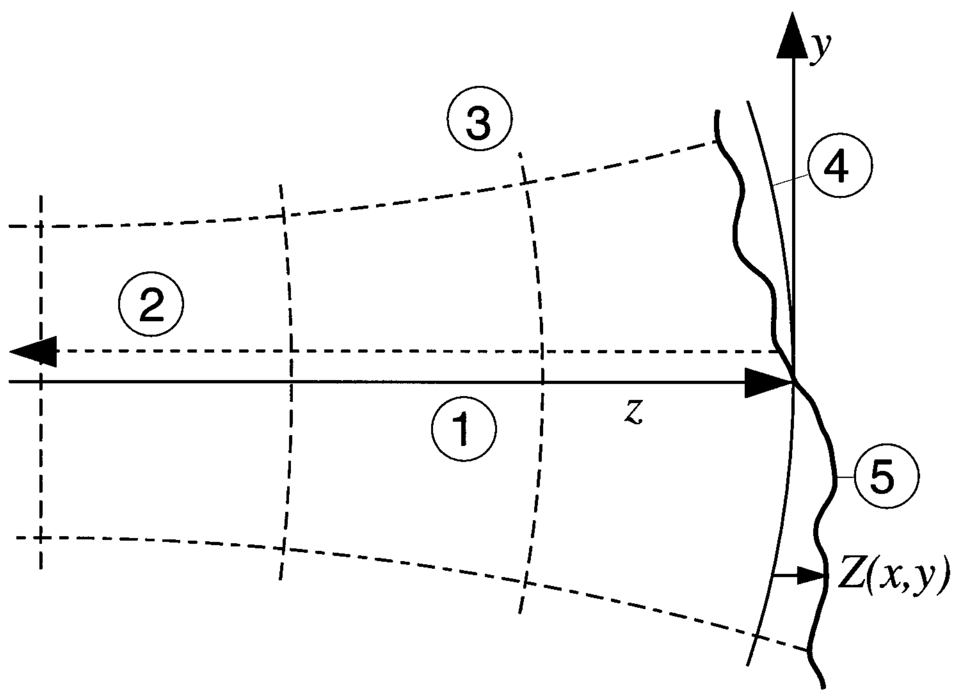

Figure 1: Reflection on an imperfect and misaligned mirror surface.
① direction of incoming laser beam, ② reflected beam, ③ undistorted wavefront, ④ ideal mirror surface, ⑤ physical mirror surface and *Z(x,y)* deviation from ideal mirror surface.

1. The validity of this approximation will become clear later-on.

the unperturbed Gaussian beam are matched to the ideal mirror surface and all deviations from the ideal surface are contained in the function $Z(x,y)$. The mirror distortion operator can then be written as[2]

$$M(x, y) = e^{-2ikZ(x, y)} \tag{6}$$

This operator is unitary and, thus, conserves energy[3]. The modal space representation then becomes [12]:

$$M_{mn,kl} = \langle mn| e^{-2ikZ(x,y)} |kl\rangle = \langle mn| \exp\left(-2ik \sum_{op,qr} |op\rangle Z_{op,qr} \langle qr|\right) |kl\rangle \tag{7}$$

Since the lateral size of the Gaussian beam is fully described by the spot size, $\omega(z)$, which is simply a transverse scaling factor, the z-dependence of $Z_{op,qr}$ in eqn. (7) can be removed by the variable substitutions $x' = \sqrt{2}x/\omega(z)$ and $y' = \sqrt{2}y/\omega(z)$. Expanding $Z(x,y)$ in a series of orthonormal polynomials H_i (such as the Hermite or Zernike polynomials) of the form

$$-2kZ(x, y) = \sum_{i,j} c_{ij} H_i(x') H_j(y') \tag{8a}$$

one obtains:

$$-2kZ_{op,qr} \equiv \sum_{i,j} c_{ij} T^{ij}_{op,qr} = \sum_{i,j} c_{ij} \langle op| H_i(x) H_j(y) |qr\rangle|_{z=0} \tag{8b}$$

The $T^{ij}_{op,qr}$ are Hermitian operators which serve as generators for the unitary transformation that describes the wavefront distortion. A detailed calculation of these generators in the Hermite-Gaussian basis are presented in Appendix A.

The physical meaning of the individual terms in the expansion of eqn. (8a) — using Hermite polynomials in the expansion — can be understood as follows: the constant term, c_{00}, corresponds to longitudinal displacements of the mirror; terms proportional in $H_1(x)$ or $H_1(y)$ and, hence, linear in x or y correspond to yaw and pitch, respectively, where the coefficients c_{10} and c_{01} are proportional to the rotation angle; the terms proportional to H_2 correspond to deviations in the curvature of the spherical phase front; and so on.

For a small rotation about the y-axis $Z(x,y)$ can be written as $Z(x, y) = \theta_x x$. By substituting $\Theta_x = \theta_x \pi\omega(z)/\lambda$, which is the rotation angle normalized to the divergence angle of the beam, the operator becomes:

$$-2kZ(x, y) = -\sqrt{8}\Theta_x \frac{\sqrt{2}x}{\omega(z)} = -\sqrt{2}\Theta_x H_1(x') \tag{9}$$

In the Hermite-Gaussian basis, the recursion relationship for the Hermite polynomials simplifies the generator T^{10} to

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2. Generally, one has to multiply the right hand side of eqn. (6) by the reflection coefficient to obtain the true reflected field of the mirror.
3. The distortion operator which describes reflection from the rear surface of the mirror is given by $\overline{M}(x, y) = e^{2ikZ(x, y)} = M^{\dagger}(x, y)$.

$$T^{10}_{mn,\,kl} = \frac{1}{\sqrt{2}}\delta_{nl}(\sqrt{k}\,\delta_{m,\,k-1} + \sqrt{m}\,\delta_{m,\,k+1}) \tag{10}$$

The Hermite-Gaussian basis is particularly well-suited to describe small misalignments, as indicated by the simple form of eqn. (10). Notably, if the generator in eqn. (10) is applied to the fundamental TEM_{00} mode, the TEM_{10} is the only resulting mode. Any other higher-order mode which can be excited by a misaligned mirror is introduced by the matrix exponential of eqn. (7) only; in other words, they are not significant, if the normalized rotation angle is small compared to 1.

A similar problem is the distortion of the wavefront when it passes through an optical element such as a partially transmitting mirror or a lens. If the deviation in thickness of the optical element from its ideal shape is denoted by *d(x,y)*, then the distortion operator can be written as

$$M(x, y) = e^{-i(n-1)k\,d(x,\,y)} \tag{11}$$

where n is the refractive index of the optical medium. This operator is of the same form as the distortion operator in reflection with one exception: the linear term in *d(x,y)* does not account for an angular tilt, but instead describes a wedge. In fact, an angularly misaligned optical element shifts the beam laterally. For a small tilt θ_x about the y-axis, the shift in the x-direction can be calculated using Snell's law:

$$\Delta(x) \approx \theta_x\left(1 - \frac{1}{n}\right)d \tag{12}$$

where n and d are the index of refraction and the thickness of the optical element, respectively. A detailed calculation of the lateral shift operator is presented in Appendix A.2.

With regard to optical cavities, tilting the partially transmitting input mirror of the cavity has two effects: first, the wavefront of the field inside the cavity is distorted by the tilt of the reflecting surface, and second, the incident beam is shifted laterally upon transmission through the input coupler. Comparing eqn. (12) with eqn. (9), the effect of lateral shift turns out to be much smaller, roughly by a factor of d/z_0, which is negligible for most practical cavity configurations.

3 A DETECTION SCHEME IN THE MODAL PICTURE

The Pound-Drever-Hall reflection-locking scheme [4] is a powerful technique for holding optical cavities and interferometers on resonance. When phase modulated light is incident on an optical cavity, then deviation from the resonance length causes amplitude modulation to the light reflected from the cavity. Close to the resonant state, the modulation depth is proportional to the deviation from resonance and, hence, can be used as an error signal to hold the cavity on resonance.

The same technique can be used for spatial sensing of the wavefront [9]. If one of the mirrors of a resonant optical system is misaligned, the higher-order transverse modes excited can be detected by measuring the amplitude modulation on a segmented photodetector (see Fig. 2). The misalignment signal is essentially the interference between the fundamental TEM_{00} mode of the

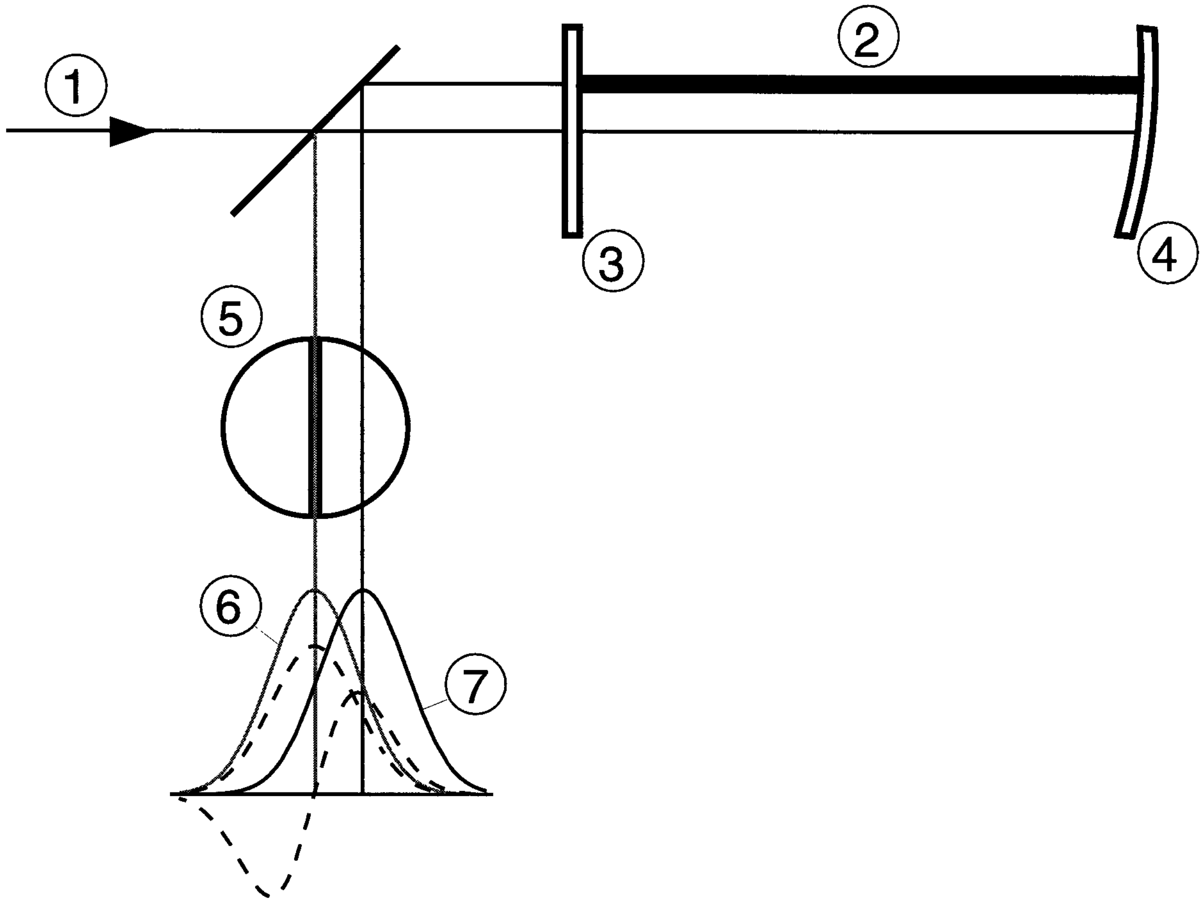

Figure 2: Alignment Signals from a misaligned Fabry-Perot interferometer
① incoming laser beam, ② resonant cavity mode, ③ partially transmitting input mirror, ④ tilted back mirror, ⑤ segmented photodetector, ⑥ reflected sideband (gray curve) and ⑦ reflected carrier light (solid curve) with its modal decomposition (dashed curve).

carrier (sideband) and the higher-order modes — e.g. TEM_{01} — of the sideband (carrier). Since the different higher-order modes propagate with different Guoy phases, the detected modulation depth also depends on the distance between the photodetector and the cavity or interferometer output. Higher-order modes generated by tilted optical elements which have a longitudinal separation — the input and rear mirrors of a cavity, for example — have to travel different distances to the photodetector and can thus be distinguished by using two segmented photodetectors at two different locations.

In this section an expression for the demodulated signal is derived for a misaligned optical system in the modal space. If $O(k)$ is an operator which depicts an entire optical system and if the incoming laser beam consists of a carrier and a pair of phase modulation sidebands, the field at the output of the system can be written as

$$E_{out} = E^{CR} + E^{SB+} + E^{SB-} \tag{13a}$$

$$E^{CR} = O(k_{CR})E_{inp} \quad \text{and} \quad E^{SB} = i\Gamma O(k_{CR} \pm \Delta\omega/c)E_{inp} \tag{13b}$$

where k_{CR} corresponds to the wavelength of the carrier, $\Delta\omega$ is the frequency of modulation, Γ is the modulation depth and E_{inp} is the incident field in the modal space (usually the fundamental mode).

The total light power on the photodetector which is placed a distance η in Guoy phase away from the output of the optical system is

$$\bar{S} = \frac{1}{2}[P(\eta)E_{out}]^{\dagger}D^{\Omega}[P(\eta)E_{out}] \tag{14}$$

where D^{Ω} is an operator that accounts for the physical dimensions of the photodetector. If $\bar{S}$ is demodulated with the modulation frequency $\Delta\omega$, then only terms which have an $e^{-i\Delta\omega t}$ dependence on the modulation frequency remain in the baseband:

$$S = \frac{1}{2}(E^{CR})^{\dagger}P^{\dagger}(\eta)D^{\Omega}P(\eta)E^{SB-} + \frac{1}{2}(E^{SB+})^{\dagger}P^{\dagger}(\eta)D^{\Omega}P(\eta)E^{CR} \tag{15}$$

In Appendix A.3, we calculate the demodulation operator D^{Ω} in the modal space for arbitrarily shaped detectors. From the form of the propagator, see eqn. (4), it can be seen that the η dependence in S can be expressed in terms of a sine and cosine series:

$$S = \sum_{s=0}^{\infty} d_s \cos(s\eta) + \sum_{s=0}^{\infty} e_s \sin(s\eta) \tag{16}$$

where the coefficients d_s and e_s are complicated functions of the detector shape and the output fields. They are completely independent of the detector position if the ratio of the beam spot size over the detector diameter is held constant.

The above expansion is helpful in understanding the Guoy phase dependence of the demodulated signal. For instance, a single photodetector which covers the full cross-section of the beam has only one non-zero coefficient, d_0. Similarly, d_1 and e_1 are the only significant coefficients for small angular misalignments measured by a half-plane detector which is split along one axis.

4 APPLICATION TO GENERAL OPTICAL SYSTEMS

The above technique is applied to calculate the alignment signals of a Fabry-Perot interferometer (see Figs. 2 and 3a). The cavity round-trip propagation in matrix form reads:

$$P_{rt} = (-r_1)(-r_2)M_1 P M_2 P \tag{17}$$

where M_1 and M_2 are the misalignment matrices of the front and the rear mirror, respectively, P denotes the propagator between the two mirrors[4] and r_1 and r_2 are the amplitude reflection coefficients of the two mirrors[5]. The steady-state equation for the field inside the cavity, E_{ins}, can then be written as

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4. Note that both the plane wave phase factor and the Guoy phase shift are exactly the same for both beam directions.
5. We follow the convention that if the light is reflected from a mirror surface which is drawn as bold curve in Fig. 3, an additional factor of (–1) has to be taken into account.

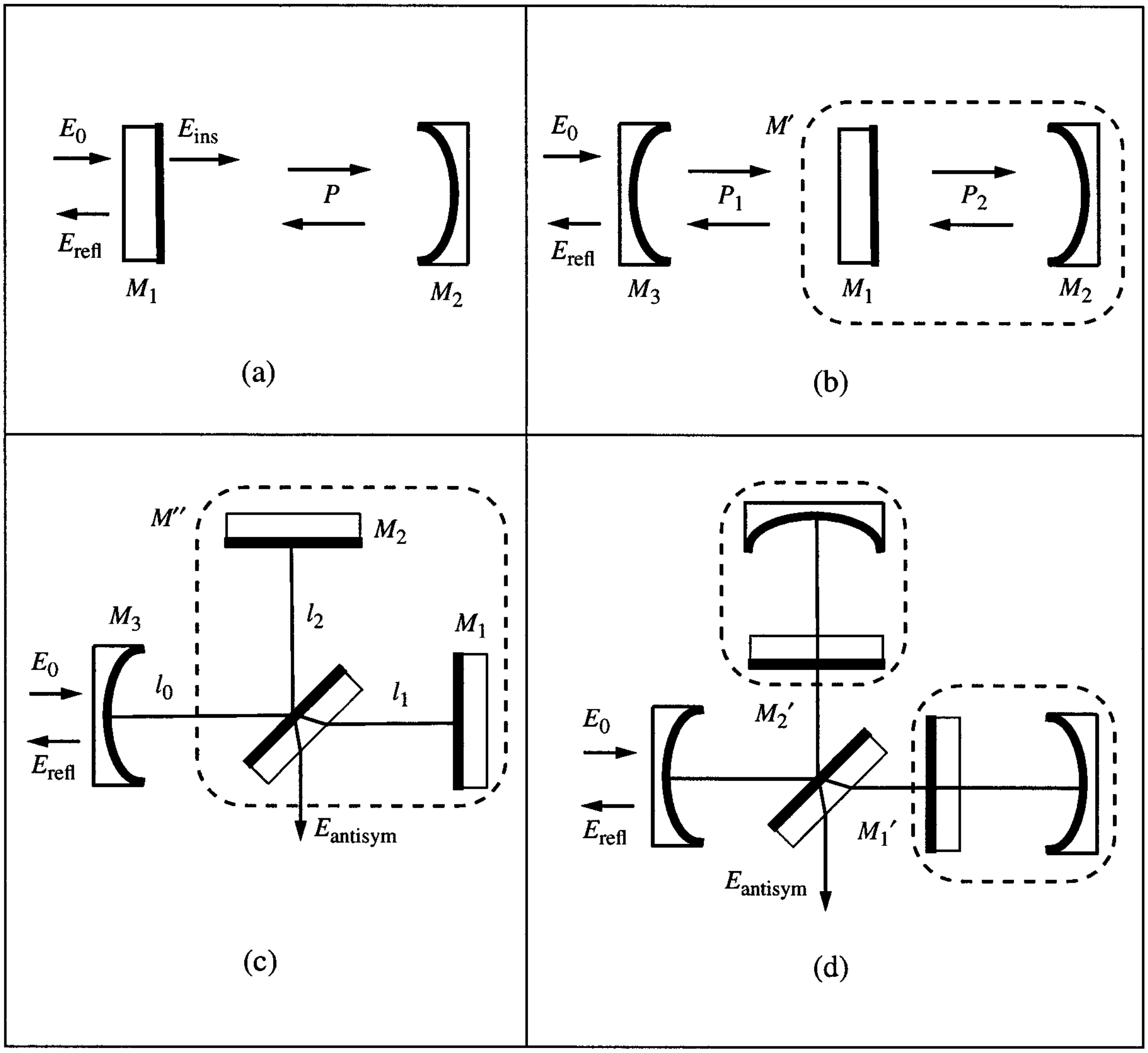

Figure 3: Setup of (a) a Fabry-Perot interferometer, (b) a coupled pair of cavities, (c) a recycled Michelson interferometer and (d) a recycled Michelson interferometer with arm cavities.

$$E_{ins} = P_{rt}E_{ins} + t_1 E_0 = t_1 (I - P_{rt})^{-1} E_0 \quad (18)$$

and the reflected field becomes:

$$E_{refl} = r_1 M_1{}^{\dagger} E_0 - t_1 r_2 P M_2 P E_{ins} = r_1 M_1{}^{\dagger}\left(I - \frac{r_1^2 + t_1^2}{r_1^2} P_{rt}\right)(I - P_{rt})^{-1} E_0 \equiv M' E_0 \quad (19)$$

where t_1 is the amplitude transmission coefficient of the input mirror, E_0 is the input field and I is the identity matrix.

For small misalignments the only important modes are the fundamental TEM_{00} mode and the lowest order transverse (Hermite-Gaussian) modes, TEM_{10} and TEM_{01}. Using the notation to write all three modes as the components of a single vector, an electromagnetic field in modal space can be written as:

$$E = \left[TEM_{00}\ TEM_{10}\ TEM_{01}\right]^T \tag{20}$$

Making use of eqns. (4) and (10) and including terms to first order in Θ_x and Θ_y only, the propagator and the mirror misalignment matrices become

$$P(z_2, z_1) = e^{-ik(z_2 - z_1)}\begin{bmatrix} e^{i\eta} & 0 & 0 \\ 0 & e^{2i\eta} & 0 \\ 0 & 0 & e^{2i\eta} \end{bmatrix} \quad \text{and} \quad M(\Theta_x, \Theta_y) \approx \begin{bmatrix} 1 & -2i\Theta_x & -2i\Theta_y \\ -2i\Theta_x & 1 & 0 \\ -2i\Theta_y & 0 & 1 \end{bmatrix} \tag{21}$$

Assuming that the incoming light E_0 is a pure phase modulated TEM_{00} mode with a carrier which is resonant in the Fabry-Perot interferometer and sidebands which are exactly anti-resonant, then the alignment signal S measured with a half-plane detector — as defined in eqns. (A.14) and (A.16) — can be obtained from eqns. (15), (16) and (19). To first order in Θ S is given by

$$S = S_0 \Gamma \frac{E_0^2}{2}(\Theta_{1x}\cos(\eta_0 + \bar{\eta}) + \Theta_{2x}\cos(\eta_0 + \bar{\eta} + \eta)) \tag{22}$$

where $\bar{\eta}$ is the Guoy phase shift that the TEM_{00} mode of the reflected beam acquires between the input mirror and the photodetector. For a Fabry-Perot interferometer S_0 and $\bar{\eta}$ are complicated functions of r_1, t_1, r_2 and η. An attractive feature of eqn. (22) is that if additional higher-order modes are included in its derivation, they contribute to the order of Θ^3 or higher only. This is illustrated in Fig. 4, where the angular error signal S and the electromagnetic field strength of the modes inside the cavity are plotted against the misalignment angle of the front mirror Θ_{1x}. The calculations were made with one transverse degree of freedom only. It can be seen that the first order approximation (2 modes) is in good agreement with the 'exact' solution (22 modes) up to angles of 0.3. The cavity parameters were the ones of a LIGO arm cavity with a cavity length of 4 km, radii of curvature for the front and rear mirror of −14500 km and 7400 km, respectively, a perfectly reflecting rear mirror and a front mirror with a power transmission of 3%.

The Guoy phase η is pivotal to the mode decomposition technique. Each non-degenerate mode of the field has a different propagation phase associated with it, and it is precisely this property that allows us to infer which optical component in the optical train is misaligned. From eqn. (22) it can be seen that the Guoy phase difference of the misalignment signal, generated by the front and back mirrors of a Fabry-Perot interferometer, is the Guoy phase shift acquired by the fundamental mode when propagating from one mirror to the other. Unfortunately, this means that for a highly degenerate cavity the misalignment signals from the front and rear mirrors become indistinguishable.

The single cavity equations can be used to calculate the fields propagating in a pair of coupled cavities. Referring to figure 1(b), one can calculate the field reflected from mirror 3 by replacing

M_1 and M_2 and their corresponding mirror reflectivities in eqns. (17) and (18) with M_3 and M', given in eqn. (19), i.e.

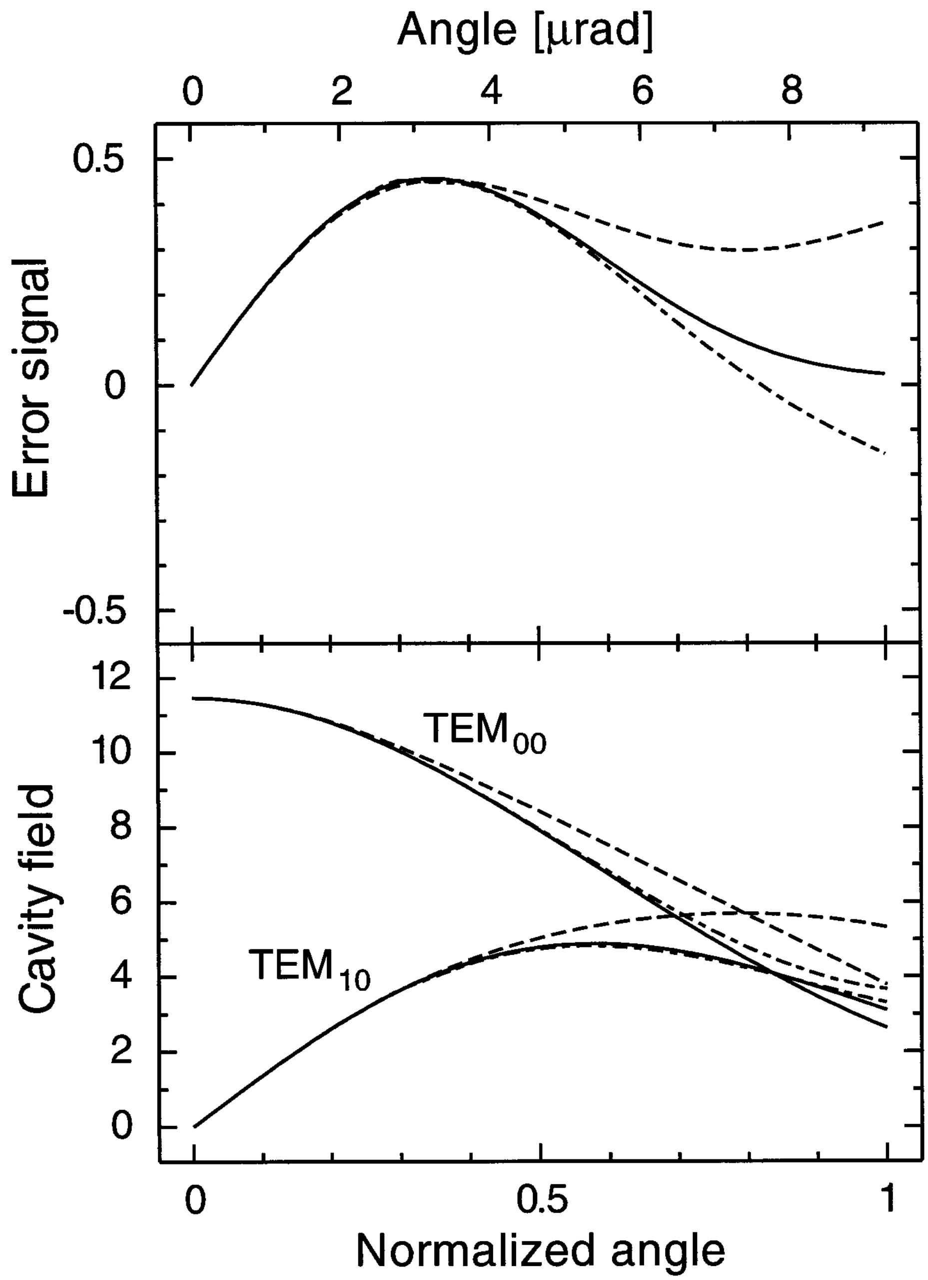

Figure 4: Angular error signal (top) and mode decomposition (bottom) as a function of the misalignment angle of the front mirror in a resonant Fabry-Perot interferometer. The calculations were made with one transverse degree of freedom only, using 2 modes (dashed line), 4 modes (dash-doted line) and 22 modes (solid line), respectively. The cavity fields are given in units of the input field, whereas the error signal is given relative to the input power and the modulation depth, calculated for the Guoy phase of the detector which gives the maximum signal at small angles. The cavity has a finesse of about 200; its parameters are given in the text.

$$E_{refl} = r_3 M_3^{\dagger}\left(I + \frac{r_3^2 + t_3^2}{r_3} M_3 P_1 M' P_1\right)(I + r_3 M_3 P_1 M' P_1)^{-1} E_0 \tag{23}$$

where M_3 and t_3 are the mirror matrix and amplitude transmission coefficient for mirror *3*, respectively, and P_1 and P_2 are the free space propagators corresponding to the length of each cavity.

Similarly, for a Michelson interferometer with a partially transmitting mirror at the input, as shown in figure 1(c), all the complexity of the Michelson interferometer can be contained in a matrix operator M''. The field reflected from the input mirror is then given by

$$E_{refl} = r_3 M_3^{\dagger}\left(I + \frac{r_3^2 + t_3^2}{r_3} M_3 P_0 M'' P_0\right)(I + r_3 M_3 P_0 M'' P_0)^{-1} E_0 \tag{24a}$$

$$\text{with} \quad M'' = -t_{BS}^2 r_1 P_1 M_1 P_1 - r_{BS}^2 r_2 P_2 M_2 P_2 \tag{24b}$$

and the field at the antisymmetric output of the interferometer becomes

$$E_{antisym} = t_3 r_{BS} t_{BS}(-r_1 P_1 M_1 P_1 + r_2 P_2 M_2 P_2)(I + r_3 M_3 P_0 M'' P_0)^{-1} E_0 \tag{25}$$

where r_{BS} and t_{BS} are the amplitude reflection and transmission coefficients of the beamsplitter.

In this manner, we can piece together any optical system and extract the spatially varying fields at any transverse plane along the direction of propagation of the wavevector. Figure 1(d) shows the optical layout for the complete LIGO interferometer. Fields for this system can be calculated by replacing M_1 and M_2 in the equations for the simple Michelson interferometer with matrices M_1' and M_2' which are similar to the cavity operator M' of Figure 1(b).

5 CONCLUSIONS

We have presented the theory of mode decomposition in complex resonant optical interferometers using a formalism which is versatile, flexible and easily extensible to include an arbitrary number of higher-order modes. This formalism is an elegant analytical modeling tool which can be easily adapted for different optical configurations. There is no intrinsic limit on the number of optical components, since the algorithms described in this work are computationally very efficient.

The wavefront sensing scheme does not require an external reference point (as is the case with auxiliary optical levers, for example), which have the potential to drift relative to the interferometer. Furthermore, the interferometer light already carries phase modulation used for the length sensing system. All cases examined to date support the conclusion that if a particular modulation configuration gives reliable information to sense the longitudinal degrees of freedom for the optical system, then probing the spatial dependence of the same light must give equally robust alignment information. However, the signals originating from the two mirrors forming a single cavity may be indistinguishable, since for certain special cases — such as a highly degenerate cavity — the Guoy phase difference of the two alignment signals can be insignificant.

We have used the modal model described in this work as a design and analysis tool to study various configurations of interferometric gravitational-wave detectors, including the complete LIGO interferometer[6]. In particular, we have studied the degradation in sensitivity to gravitational-waves of a proposed LIGO interferometer due to angular misalignments [6]. Simulations carried out using this model will play an important role in converging on the optimal modulation configuration for LIGO. A table-top scale prototype interferometer experiment is underway to test an alignment sensing scheme which was designed using the present model.

We thank our colleagues on the LIGO project who helped us with many useful suggestions and comments. This work is supported by the NSF grant PHY-9210038. One of us (D.S.) is supported by the Swiss National Science Foundation.

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APPENDIX A FORMULAE

In the paraxial approximation, the solutions to the scalar wave equation in one dimension can be expressed as a superposition of Hermite-Gaussian modes [11]:

$$U_m(x, z) = \left(\frac{1}{2}\right)^{1/4}\left(\frac{1}{2^m m!\,\omega(z)}\right)^{1/2} H_m\left(\frac{\sqrt{2}x}{\omega(z)}\right)$$
$$\exp\left(-x^2\left(\frac{1}{\omega^2(z)} + \frac{ik}{2R(z)}\right)\right)\exp(i(m + 1/2)\eta(z)) \tag{A.1}$$

where the z-axis points in the beam propagation direction and where $\eta(z)$, $\omega(z)$ and $R(z)$ are the mode-dependent Guoy phase shift, the spot size and the curvature of the phase front at position z, respectively,

$$\eta(z) = \tan^{-1}\left(\frac{z}{z_0}\right),\quad \omega(z) = \omega_o\sqrt{1 + \left(\frac{z}{z_o}\right)^2}\quad \text{and}\quad R(z) = z + \frac{z_o^2}{z} \tag{A.2}$$

where the Rayleigh length, z_0, is given by $z_0 = \pi\omega_0^2/\lambda$ with ω_o the waist size. $H_m(x)$ is the Hermite polynomial of order m. The following relations are used repeatedly in the calculations which follow:

$$\int_{-\infty}^{\infty} U^{\dagger}_m(x, z)\, U_n(x, z)dx = \delta_{mn} \tag{A.3a}$$

$$2xH_m(x) = H_{m+1}(x) + 2mH_{m-1}(x) \tag{A.3b}$$

$$\frac{d}{dx}H_m(x) = 2mH_{m-1}(x) \tag{A.3c}$$

$$\int_{-\infty}^{\infty} U^{\dagger}_m(x, 0)\frac{H_i(\sqrt{2}x/\omega_0)}{H_k(\sqrt{2}x/\omega_0)}U_k(x, 0)dx = \sqrt{\frac{2^i i!}{2^k k!}}\delta_{mi} \tag{A.3d}$$

where eqn. (A.3a) is the orthonormality condition; eqns. (A.3b) and (A.3c) are recursion relations to be used to derive Hermite polynomials of any order, beginning with $H_0(x) = 1$.

In two dimensions the Hermite-Gaussian modes are given by

$$U_{mn} = U_m(x, z)U_n(y, z)e^{-ikz} \tag{A.4}$$

with the plane wave phase shift factor included for completeness [11].

APPENDIX A.1 The wavefront distortion operator

The generator for the wavefront distortion operator is given in eqn. (8b). If we now expand part of the integrand as a series of Hermite polynomials, we obtain

$$H_i(x)H_j(y)H_k(x)H_l(y) = \sum_{rs} h^{ij}_{rs,\,kl}H_r(x)H_s(y) \tag{A.5}$$

Multiplying the operator in eqn. (8b) by $H_k(x)H_l(y)/H_k(x)H_l(y)$ and using eqns. (A.5) and (A.3d) gives

$$T^{ij}_{mn,kl} = h^{ij}_{mn,kl}\sqrt{\frac{2^m m!2^n n!}{2^k k!2^l l!}} \tag{A.6}$$

In particular, for a simple tilt around the *y*-axis we get eqn. (10).

APPENDIX A.2 The Lateral shift operator

If a beam is laterally shifted with respect to the direction of propagation, the shift operator $O(\Delta x,\Delta y)$ is defined as

$$E(x+\Delta x, y+\Delta y, z) = O(\Delta x, \Delta y) \otimes E(x, y, z) \tag{A.7}$$

Expanding the left hand side in a Taylor series about x and y gives

$$O(\Delta x, \Delta y) = \exp\left(i\left(\Delta x\frac{1}{i}\frac{d}{dx} + \Delta y\frac{1}{i}\frac{d}{dy}\right)\right) \tag{A.8}$$

In the modal basis, this unitary operator can be written as

$$O_{mn,kl}(\Delta x, \Delta y) = \langle mn|\exp\left(i\sum_{op,qr}|op\rangle\left(\frac{\sqrt{2}\Delta x}{\omega(z)}T^x_{op,qr} + \frac{\sqrt{2}\Delta y}{\omega(z)}T^y_{op,qr}\right)\langle qr|\right)|kl\rangle \tag{A.9a}$$

$$\text{with} \quad T^x_{op,qr} = \delta_{pr}\int_{-\infty}^{\infty} dx\, U^\dagger_o(x,z)\frac{\omega(z)}{i\sqrt{2}}\frac{d}{dx}U_r(x,z) \tag{A.9b}$$

where we already performed the straightforward integration over *y*. $T^y_{op,qr}$ can be deduced from the right hand side of eqn. (A.9b) by replacing *x* with *y* and by interchanging the indices *o,p* and *q,r*. Once again, the recursion relations of eqn. (A.3) are used to solve the integration:

$$H_r\left(\frac{\sqrt{2}x}{\omega(z)}\right)\frac{\omega(z)}{i\sqrt{2}}\frac{d}{dx}U_r(x,z) = \left(\frac{1}{2}H_{r+1}\left(\frac{\sqrt{2}x}{\omega(z)}\right)t + rH_{r-1}\left(\frac{\sqrt{2}x}{\omega(z)}\right)t^*\right)U_r(x,z) \tag{A.10a}$$

$$\text{where} \quad t = i - \frac{\omega^2(z)k}{2R(z)} = i - \frac{z}{z_0} \tag{A.10b}$$

The generator of shifts in *x* is obtained using eqn. (A.3d):

$$T^x_{mn,kl} = \frac{1}{\sqrt{2}}\delta_{nl}(\sqrt{m}\,t\,\delta_{m,k+1} + \sqrt{k}\,t^*\,\delta_{m,k-1}) \tag{A.11}$$

APPENDIX A.3 The demodulation operator

If Ω is the topology of the photodetector, then the demodulated signal is given by:

$$S = \frac{1}{2}\int\int_{\Omega} dx\,dy[P(\eta, x, y)E^{CR}(x, y)]^{\dagger}[P(\eta, x, y)E^{SB-}(x, y)] \qquad (A.12)$$
$$+ \frac{1}{2}\int\int_{\Omega} dx\,dy[P(\eta, x, y)E^{SB+}(x, y)]^{\dagger}[P(\eta, x, y)E^{CR}(x, y)]$$

where $E^{CR}(x,y)$ and $E^{SB\pm}(x,y)$ are the field amplitudes at the output of an optical system for the carrier and sidebands of the modulated light, respectively. $P(\eta,x,y)$ is the propagator between the output and the photodetector. In the modal space coordinates the first term in eqn. (A.12) becomes

$$S_1 = \frac{1}{2}\sum_{mn,\,kl}(E^{CR}_{mn})^{\dagger}e^{i(m+n+1)\eta}\int\int_{\Omega} dx\,dy\,U^{\dagger}_{mn}(x, y, z)\,U_{kl}(x, y, z)e^{i(k+l+1)\eta}E^{SB-}_{kl} \qquad (A.13)$$

Comparing this expression with the definition of the demodulation operator in the modal space in eqn. (15) gives

$$D^{\Omega}_{mn,\,kl} = \int\int_{\Omega} dx\,dy\,U^{\dagger}_{mn}(x, y, z)U_{kl}(x, y, z) \qquad (A.14)$$

When the photodetector size is constant relative to the beam spot size, the demodulation operator is independent of the position of the photodetector. For a full-plane detector which covers the entire beam, eqn. (A.14) reduces to the orthonormality condition of eqn. (A.3a), i.e.

$$D^{full-plane}_{mn,\,kl} = \delta_{mk}\delta_{nl} \qquad (A.15)$$

whereas for a half-plane detector which is split along the y-axis and where the signals from the two half-planes are subtracted from each other, the integral becomes

$$\int\int dx\,dy \rightarrow \int_{-\infty}^{\infty} dy\left\{\int_{0}^{\infty} dx - \int_{-\infty}^{0} dx\right\} \qquad (A.16)$$