LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
- LIGO -
CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Document Type	**LIGO-T960019-00-**	**D**	2/20/96

Frequency, Intensity and Oscillator Noise in the LIGO

Jordan Camp

Distribution of this draft:

xyz

This is an internal working note
of the LIGO Project..

California Institute of Technology
LIGO Project - MS 51-33
Pasadena CA 91125
Phone (818) 395-2129
Fax (818) 304-9834
E-mail: info@ligo.caltech.edu

Massachusetts Institute of Technology
LIGO Project - MS 20B-145
Cambridge, MA 01239
Phone (617) 253-4824
Fax (617) 253-7014
E-mail: info@ligo.mit.edu

WWW: http://www.ligo.caltech.edu/

1 INTRODUCTION

This document presents an analysis of the propagation of noise on the input light to the LIGO interferometer to the output gravity wave port. The general approach is to describe the noise as audio sidebands about the light carrier, append RF sidebands to the carrier and its audio side band frequencies through phase modulation and then propagate the resultant set of (9) frequencies through the interferometer, taking into account the lengths traversed, and the imperfections of mismatched arm storage times and arm cavity deviations from resonance. Finally, the fields are added at the photodetector and demodulated at the mixer output to give the resultant signal, which is then compared to an actual signal produced by a shaking of an arm cavity.

2 PHASE MODULATION

The laser electric field of amplitude E_L is phase modulated to produces RF sidebands:

$$E = E_L \, e^{i\Gamma \cos\Omega t}$$

$$= E_L \left[1 + \frac{i\Gamma}{2}(e^{i\Omega t} + e^{-i\Omega t})\right] \quad \text{for } \Gamma << 1$$

3 AUDIO SIDEBANDS ON LIGHT

In each case the laser E-field (which includes the audio sidebands from the considered noise source) is phase modulated.

3.1. Laser frequency noise

writing frequency noise as: $\nu = \nu_0 + \delta\nu \cos \omega t$

The field phase variation is: $\phi(t) = 2\pi\upsilon_0 t + \frac{2\pi\delta\upsilon}{\omega}\sin\omega t$

and we have $E_L = E_0 \left[e^{i\omega_0 t} + \frac{\pi\delta\upsilon}{\omega}(e^{i(\omega_0+\omega)t} - e^{i(\omega_0-\omega)t})\right]$

thus E, the electric field incident on the IFO,

$$= E_0 \left[e^{i\omega_0 t} + \frac{\pi\delta\upsilon}{\omega}(e^{i(\omega_0+\omega)t} - e^{i(\omega_0-\omega)t})\right] \left[1 + \frac{i\Gamma}{2}(e^{i\Omega t} + e^{-i\Omega t})\right]$$

$$= E_0 \quad [(e^{i\omega_0 t} + \frac{\pi\delta\upsilon}{\omega}(e^{i(\omega_0+\omega)t} - e^{i(\omega_0-\omega)t}) \quad + \quad \frac{i\Gamma}{2}(e^{i(\omega_0+\Omega)t} + e^{i(\omega_0-\Omega)t})$$

$$+ \frac{i\pi\delta\upsilon\Gamma}{2\omega}(e^{i(\omega_0+\omega+\Omega)t} - e^{i(\omega_0-\omega+\Omega)t} + e^{i(\omega_0+\omega-\Omega)t} - e^{i(\omega_0-\omega-\Omega)t})]$$

3.2. Laser intensity noise

writing intensity noise as: $E = E_0 + \delta E \cos \omega t$

we have $E_L = E_0 \left[e^{i\omega_0 t} + \frac{\delta E}{2E_0}(e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t})\right]$

$$E= E_0 \left(e^{i\omega_0 t} + \frac{\delta E}{2E_0}(e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t}) + \frac{i\Gamma}{2}(e^{i(\omega_0+\Omega)t} + e^{i(\omega_0-\Omega)t})\right.$$

$$\left. + \frac{i\Gamma\delta E}{4E_0}(e^{i(\omega_0+\omega+\Omega)t} + e^{i(\omega_0-\omega+\Omega)t} + e^{i(\omega_0+\omega-\Omega)t} + e^{i(\omega_0-\omega-\Omega)t})\right]$$

3.3. Oscillator phase noise

we write $A_{osc} = \Gamma \cos(\Omega t + a_o \cos \omega t)$

$$= \Gamma [\cos \Omega t - a_o \sin \Omega t \cos \omega t]$$

$$= \Gamma [\cos \Omega t - (a_o/2) [\sin (\Omega+\omega)t + \sin (\Omega-\omega)t]]$$

Thus $E = E_L\ e^{i\Gamma\left[\cos\Omega t - \frac{a_0}{2}(\sin((\Omega+\omega)t + \sin(\Omega-\omega)t))\right]}$

$$= E_0 \left(e^{i\omega_0 t} + \frac{i\Gamma}{2}(e^{i(\omega_0+\Omega)t} + e^{i(\omega_0-\Omega)t})\right.$$

$$\left. - \frac{i\Gamma a_o}{4}(e^{i(\omega_0+\omega+\Omega)t} + e^{i(\omega_0-\omega+\Omega)t} - e^{i(\omega_0+\omega-\Omega)t} - e^{i(\omega_0-\omega-\Omega)t}\right]$$

3.4. Oscillator amplitude noise

we write $Aosc = \Gamma (1 + (\delta A/A)\cos\omega t) \cos \Omega t$

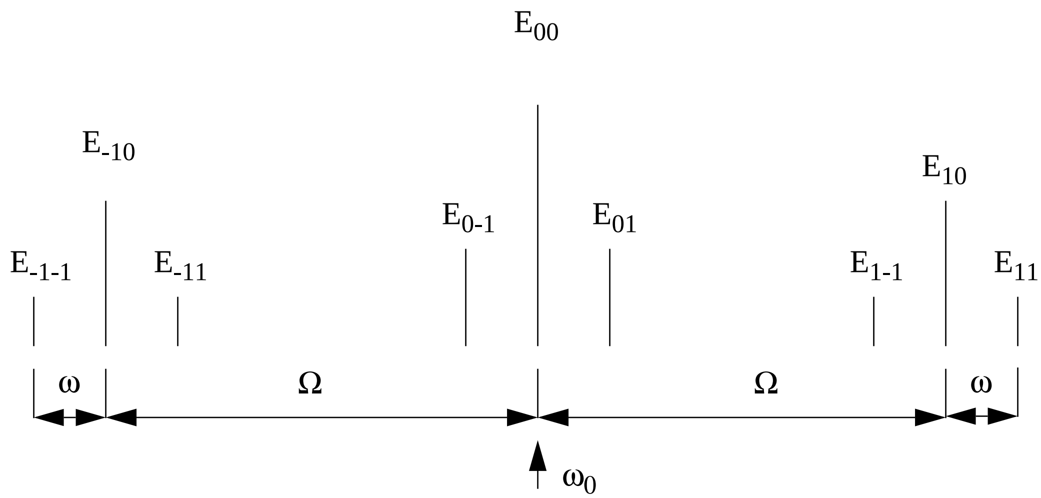
Thus $E = E_L\ e^{i\Gamma\cos\left(\Omega t + \frac{\delta A}{2A}(\cos(\Omega+\omega)t + \cos(\Omega-\omega)t)\right)}$

$$= E_0 \left(e^{i\omega_0 t} + \frac{i\Gamma}{2}(e^{i(\omega_0+\Omega)t} + e^{i(\omega_0-\Omega)t})\right.$$

$$\left. + \frac{i\Gamma\delta A}{4A}(e^{i(\omega_0+\omega+\Omega)t} + e^{i(\omega_0-\omega+\Omega)t} + e^{i(\omega_0+\omega-\Omega)t} + e^{i(\omega_0-\omega-\Omega)t})\right]$$

3.5. Summary of sidebands

These sets of sidebands, RF and Audio, may be summarized with the following diagram and table.



In this notation, E_{ab} refers to the electric field component of RF index *a* (where 0 connotes the carrier) and audio index *b*. In the following table, all amplitudes are multiplied by the factor E_0.

Table 1: Audio sidebands on light

	E_{-1-1}	E_{-10}	E_{-11}	E_{0-1}	E_{00}	E_{01}	E_{1-1}	E_{10}	E_{11}
Laser ν noise	$\frac{-i\Gamma\pi\delta\upsilon}{2\omega}$	$\frac{i\Gamma}{2}$	$\frac{i\Gamma\pi\delta\upsilon}{2\omega}$	$\frac{-\pi\delta\upsilon}{\omega}$	1	$\frac{\pi\delta\upsilon}{\omega}$	$\frac{-i\Gamma\pi\delta\upsilon}{2\omega}$	$\frac{i\Gamma}{2}$	$\frac{i\Gamma\pi\delta\upsilon}{2\omega}$
Laser amp noise	$\frac{i\Gamma\delta E}{4E_0}$	$\frac{i\Gamma}{2}$	$\frac{i\Gamma\delta E}{4E_0}$	$\frac{\delta E}{2E_0}$	1	$\frac{\delta E}{2E_0}$	$\frac{i\Gamma\delta E}{4E_0}$	$\frac{i\Gamma}{2}$	$\frac{i\Gamma\delta E}{4E_0}$
Osc. ν noise	$\frac{\Gamma a_o}{4}$	$\frac{i\Gamma}{2}$	$\frac{\Gamma a_o}{4}$		1		$\frac{-\Gamma a_o}{4}$	$\frac{i\Gamma}{2}$	$\frac{-\Gamma a_o}{4}$
Osc. amp. noise	$\frac{i\Gamma\delta A}{4A}$	$\frac{i\Gamma}{2}$	$\frac{i\Gamma\delta A}{4A}$		1		$\frac{i\Gamma\delta A}{4A}$	$\frac{i\Gamma}{2}$	$\frac{i\Gamma\delta A}{4A}$
Signal		$\frac{i\Gamma}{2}$		ik_0X	1	ik_0X		$\frac{i\Gamma}{2}$	

4 PROPAGATION OF LIGHT TO DARK PORT

In this section we calculate the transfer function of the E-field components from the IFO input to the dark port.

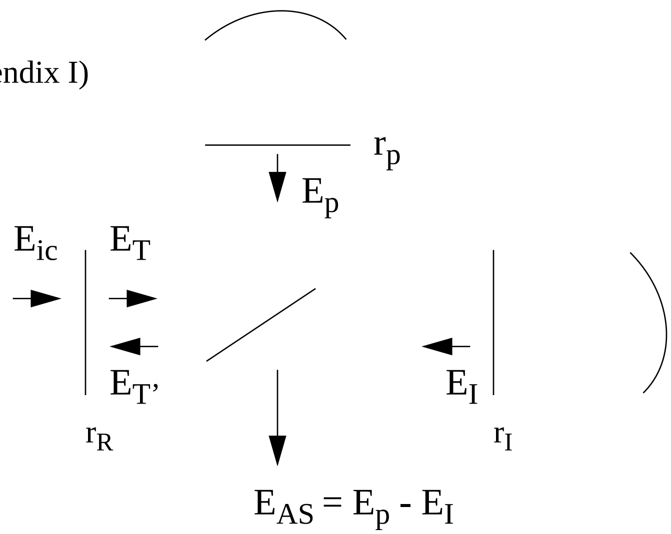
4.1. Carrier

r_P, r_I : complex arm reflectivities (appendix I)

r_R : recycling mirror reflectivity

E_{AS} : carrier field at dark port

$l_I(\phi_{IC})$, $l_P(\phi_{PC})$: round-trip length (phase) in recyc cavity for carrier



we have
$$E_{T'} = \frac{E_T}{2}\,(r_P e^{-i\phi_{PC}} + r_I e^{-i\phi_{IC}})$$

and
$$E_T = E_{ic} t_R + \left(-\frac{E_T}{2}\right) r_R (r_P e^{-i\phi_{PC}} + r_I e^{-i\phi_{IC}})$$

so that
$$E_T = \frac{E_{ic} t_R}{1 + \frac{r_R}{2}(r_P e^{-i\phi_{PC}} + r_I e^{-i\phi_{IC}})}$$

now
$$\phi_{PC} = 2\,k\,l_P$$
$$= 2\left(\frac{2\pi\upsilon_0 + \omega}{c}\right)\left(\frac{\lambda_M}{4} - \frac{\delta}{2}\right)$$

(where λ_M is the modulation wavelength, and δ is the asymmetry)

$$= \frac{2\omega}{c}\left(\frac{\pi c}{2\Omega} - \frac{\delta}{2}\right)$$

likewise, $$\phi_{IC} = \frac{2\omega}{c}\left(\frac{\pi c}{2\Omega} + \frac{\delta}{2}\right)$$

We write: $$\phi_{I,P} = \omega\varepsilon_{I,P}$$

Then $$E_{AS} = \frac{E_T}{2}\left(r_P e^{-i\phi_{PC}} - r_I e^{-i\phi_{IC}}\right)$$

$$= \frac{\frac{E_{ic}}{2} t_R (r_P(1 - i\varepsilon_P\omega) - r_I(1 - i\varepsilon_I\omega))}{1 + \frac{r_R}{2}(r_P(1 - i\varepsilon_P\omega) + r_I(1 - i\varepsilon_I\omega))}$$

$$\sim \frac{\frac{E_{ic}}{2} t_R (\Delta r - r_I i\omega\Delta\varepsilon)}{1 + \frac{r_R \Sigma r}{2}(1 - i\bar{\varepsilon}\omega)}$$

where $\Delta r = r_P - r_I$, $\Sigma r = r_P + r_I$, $\Delta\varepsilon \sim \varepsilon_P - \varepsilon_I$, $\bar{\varepsilon} \sim (1/2)(\varepsilon_P + \varepsilon_I)$

The expression for the arm reflectivity is (see appendix I):

$$r = \frac{r_0 + i r_F^{-1}\frac{\omega}{\omega_c} + ix}{1 + \frac{i\omega}{\omega_c}}$$

where r_0 is the arm reflectivity on resonance, ω_c is the cavity pole, r_f is the front mirror reflectivity, and x expresses the rms offset from fringe center. The expressions for Δr and Σr may be obtained from the following table, which assumes input mirror energy transmission of 3% and a 100 ppm difference in loss in the mirrors between the inline and perpendicular cavities:

Table 2: Typical LIGO arm cavity parameters

cavity	mirror loss	r_F	t_F	r_B	r_0	ω_c
1 (I)	100 ppm	.98484	.1731	.9999	-.9870	580.7
2 (P)	200 ppm	.98478	.1731	.9998	-.9806	584.5

We can then obtain: $$\Sigma r = 2r$$

$$\Delta r \sim \frac{10^{-2}\left(r_0 + i r_F^{-1}\frac{\omega}{\omega_c}\right) + ix}{1 + \frac{i\omega}{\omega_c}}$$

Thus the denominator term: $\dfrac{1}{1+\dfrac{r_R\Sigma r}{2}(1+i\bar{\varepsilon}\omega)}$

$$= \frac{1+i\dfrac{\omega}{\omega_c}}{1+i\dfrac{\omega}{\omega_c}+r_Rr_0+r_Rr_F^{-1}i\dfrac{\omega}{\omega_c}+r_Rx - i\bar{\varepsilon}\omega r_Rr_0+\bar{\varepsilon}r_Rr_F^{-1}\dfrac{\omega^2}{\omega_C}+r_Rix\bar{\varepsilon}\omega}$$

$$= \frac{1+i\dfrac{\omega}{\omega_c}}{(1+r_Rr_0)\left(1+\dfrac{i\omega}{\omega_{cc}}+\dfrac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}$$

where $\omega_{cc} = \dfrac{1+r_Rr_0}{1+r_Rr_F^{-1}}\omega_c$ and small terms have been neglected.

now the numerator term: $(\Delta r + r_I i\omega\Delta\varepsilon)$

$$\sim \frac{10^{-2}\left[r_0+ir_F^{-1}\dfrac{\omega}{\omega_c}\right]+ix+ir_I\omega\Delta\varepsilon - r_I\dfrac{\Delta\varepsilon\omega^2}{\omega_c}}{1+\dfrac{i\omega}{\omega_c}}$$

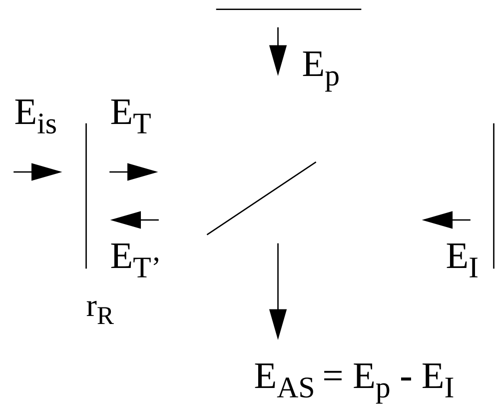
$$\sim \frac{10^{-2}\left[r_0+ir_F^{-1}\dfrac{\omega}{\omega_c}\right]+ix}{1+\dfrac{i\omega}{\omega_c}}$$

Thus E_{AS} = (numerator/denominator)

$$= \frac{\dfrac{E_{ic}}{2}t_R\left(10^{-2}\left[r_0+ir_F^{-1}\dfrac{\omega}{\omega_c}\right]+ix\right)}{(1+r_Rr_0)\left(1+\dfrac{i\omega}{\omega_{cc}}-\dfrac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}$$

which is the carrier field at audio sideband ω at the dark port.

4.2. RF Sidebands



We consider the sidebands to be antiresonant in the arm cavities, so that the reflectivity of the (compound) arm mirrors is +1.

As before (but for sideband fields, where the subscripts I and P now refer to the RF sideband phases):

$$E_{AS} = \frac{\frac{E_{is}t_R}{2}\left(e^{-i\phi_P} - e^{-i\phi_I}\right)}{1 + \frac{r_R}{2}\left(e^{-i\phi_P} + e^{-i\phi_I}\right)}$$

Now $\phi_I = 2\,k\,l_I$

$$= 2\left(\frac{2\pi\nu_0 + \Omega + \omega}{c}\right)\left(\frac{\lambda_M}{4} + \frac{\delta}{2}\right)$$

$$= \pi + \frac{\pi\omega}{\Omega} + \frac{\Omega\delta}{c} + \frac{\omega\delta}{c}$$

and $\phi_P = \pi + \frac{\pi\omega}{\Omega} - \frac{\Omega\delta}{c} - \frac{\omega\delta}{c}$

With $\alpha = \frac{\Omega\delta}{c}$ and $\beta = \frac{\omega\delta}{c}$:

$$E_{AS} = \frac{-E_{is}t_R i\sin(\alpha+\beta)\left(1 - \frac{i\pi\omega}{\Omega}\right)}{1 - r_R\cos\alpha\left(1 - \frac{i\omega}{2\Omega}\right)}$$

$$\sim \frac{-E_{is}t_R i(\alpha+\beta)}{1 - r_R\cos\alpha}$$

where we have neglected the recycling cavity pole $f_{rc} = (1-r_R)(\pi\Omega) \sim 60$ kHz.

5 PHOTODETECTOR CURRENTS, MIXER OUTPUTS

5.1. Photodetector current

If $E_1(\omega_1+\omega_0)$, E_2 $(\omega_2+\omega_0)$ are two electric fields incident on the photodetector where $\omega_1>\omega_2$ and ω_1, $\omega_2 << \omega_0$, the optical frequency, we can write:

$$E_p = Re[(E_1 e^{i\omega_1 t} + E_2 e^{i\omega_2 t})e^{i\omega_0 t}]$$

$$= Re(Ae^{-i\omega_0 t})$$

and $i_p = \overline{EE^*} = AA^* = [E_1 E_2{}^* e^{i(\omega_1-\omega_2)t} + cc] = 2Re[E_1 E_2{}^* e^{i(\omega_1-\omega_2)t}]$ where i_p is the photodetector current and we have omitted terms which do not mix the two frequencies.

Refering to figure 1 and writing the contributions to i_p for all the fields separated by $\Omega \pm \omega$ which will give a demodulated in-band output we have:

$$i_p =$$

$$2Re[E_{00}E^*_{-1-1}e^{i(\Omega+\omega)t} + E_{00}E^*_{-11}e^{i(\Omega-\omega)t} + E_{11}E^*_{00}e^{i(\Omega+\omega)t} + E_{1-1}E^*_{00}e^{i(\Omega-\omega)t}]$$

$$+ \quad 2Re[E_{0-1}E^*_{-10}e^{i(\Omega-\omega)t} + E_{10}E^*_{0-1}e^{i(\Omega+\omega)t} + E_{01}E^*_{-10}e^{i(\Omega+\omega)t} + E_{10}E^*_{01}e^{i(\Omega-\omega)t}]$$

The 1st 4 terms represent mixing of the carrier with the RF audio sidebands

2nd 4 terms " " carrier audio sidebands with the RF sidebands

5.2. Demodulation

The dark port mixer output is the product of the photodetector signal with a quadrature phase reference:

$$V_{out} = \int i_p \sin\Omega t dt$$

now $\frac{1}{T}\int_t^{(t+T)} Re[ae^{i(\Omega+\omega)t}]\sin\Omega t dt = -Re(a)\sin\omega t - Im(a)\cos\omega t$ (T = modulation period)

$$= \int Re[-a^* e^{i(\Omega-\omega)t}]\sin\Omega t dt \quad = \quad -Im[ae^{i\omega t}]$$

Thus we can write:

$$V = -Im\Big(E_{00}(E^*_{-1-1} - E^*_{1-1}) + E^*_{00}(E_{11} - E_{-11})$$

$$+ E_{01}(E^*_{-10} - E^*_{10}) + E^*_{0-1}(E_{10} - E_{-10})\Big)e^{i\omega t}$$

and therefore $|V| =$

$$\left|E_{00}(E^*_{-1-1} - E^*_{1-1}) + E^*_{00}(E_{11} - E_{-11}) + E_{01}(E^*_{-10} - E^*_{10}) + E^*_{0-1}(E_{10} - E_{-10})\right|$$

Carrier x RF audio sidebands   Carrier audio sidebands x RF

where the carrier and RF fields were earlier derived:

<u>RF</u>	<u>Carrier</u>
$E_{sb} = \dfrac{E_{is}t_R i(\alpha + \beta)}{F_R}$	$E_c = \dfrac{E_{ic}t_R(a + ib\omega + ix)}{2F_{CC}\left(1 + \dfrac{i\omega}{\omega_{cc}} - \dfrac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}$
$\alpha = \dfrac{\Omega\delta}{c}$	$a = 10^{-2}\, r_0$
$\beta = \dfrac{\omega\delta}{c}$	$b = 10^{-2}\, \dfrac{r_F}{\omega_c}$
$F_R = -(1 - r_R\cos\alpha)$	$x = \dfrac{\omega_0 \Delta x_{rms}}{L\omega_c}$
	$F_{CC} = 1 + r_R r_0$
	$\bar{\varepsilon} = 4 \times 10^{-8}$

and the field amplitudes incident on the recycling mirror, E_{is} and E_{ic}, are given in table I.

6 OUTPUT NOISE AND COMPARISON WITH SIGNAL

In this section we compare the noise sources with an arm cavity signal.

6.1. Arm Cavity Signal

With the carrier field of appendix II we can immediately write the mixer output as:

$$V = \frac{E_0^2 t_R^2}{F_R F_{CC}} \cdot \frac{\pi \nu_0 X}{L\sqrt{\omega^2 + \omega_c^2}} \cdot \left(\frac{\Gamma}{2}\alpha + \frac{\Gamma}{2}\alpha\right)$$

$$= \frac{E_0^2 t_R^2}{F_R F_{CC}} \cdot \frac{\pi \nu_0 X}{L\sqrt{\omega^2 + \omega_c^2}} \cdot \Gamma\alpha$$

6.2. Laser frequency noise

For this and the following noise sources we use the carrier field of section 5.2.

6.2.1. Carrier audio sidebands x RF sidebands

we have
$$V = \frac{\Gamma\pi\delta\upsilon}{\omega}\left|\frac{E_0^2 t_R^2 (a + ib\omega + ix)}{2F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right| \cdot (-(-\alpha) - (-\alpha))$$

$$+ \left(-\frac{\Gamma\pi\delta\upsilon}{\omega}\right)\left|\frac{E_0^2 t_R^2 (a + ib\omega - ix)}{2F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right|((-\alpha) - (-(-\alpha)))$$

$$= \frac{\Gamma\pi\delta\upsilon}{\omega}\left|\frac{E_0^2 t_R^2 (a + ib\omega)}{F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right|(\alpha)$$

Here the carrier audio sidebands undergo filtering by the coupled (recycling-arm) cavity.

6.2.2. Carrier x RF audio sidebands

$$V = \left(\frac{\Gamma\pi\delta\upsilon}{2\omega}\right)\left[\frac{E_0^2 t_R^2 (a + ix)}{2F_R F_{CC}}\right]((-\alpha - \beta) - (\alpha - \beta))$$

$$+\left(-\frac{\Gamma\pi\delta\upsilon}{2\omega}\right)\left[\frac{E_0^2 t_R^2 (a - ix)}{2F_R F_{CC}}\right]((-(\alpha + \beta)) - (-(-\alpha + \beta)))$$

$$= \frac{\pi\delta\upsilon}{\omega}\ \frac{E_0^2 t_R^2 a}{F_R F_{CC}}\ \Gamma\alpha$$

and we see that the RF audio sidebands do not receive filtering in the recycling cavity.

6.2.3. Frequency noise sum and comparison to signal

we sum the above outputs:

$$V = \frac{\pi\delta\upsilon}{\omega}\ \frac{E_0^2 t_R^2}{F_R F_{CC}} \left| \frac{(a + ib\omega)}{\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)} - a \right| \Gamma\alpha$$

$$\sim\ \pi\frac{10^{-2} E_0^2 t_R^2}{F_R F_{CC}}\ \frac{\delta\upsilon}{\sqrt{\omega_{cc}^2 + \omega^2}}\ \Gamma\alpha$$

and comparison of the noise to the signal (6.2.1) gives:

$$\frac{V_{\delta\nu}}{V_{signal}} = 10^{-2}\ \frac{\delta\nu}{\nu_0}\ \frac{L}{X}\ \sqrt{\frac{\omega^2 + \omega_c^2}{\omega^2 + \omega_{cc}^2}}$$

$$\sim 10^{-2}\ \frac{\delta\nu}{\nu_0}\ \frac{L}{X}$$

over most of the frequency range of interest.

6.3. Laser Intensity noise

6.3.1. Carrier audio sidebands x RF sidebands

$$\mathrm{V} = \frac{\Gamma\delta E}{2E_0}\left|\frac{E_0^2 t_R^2 (a + ib\omega + ix)}{2F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right| \cdot (\alpha) + \frac{\Gamma\delta E}{2E_0}\left|\frac{E_0^2 t_R^2 a + ib\omega - ix}{2F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right| \cdot (-\alpha)$$

$$= \frac{\Gamma\delta E}{2E_0}\left|\frac{E_0^2 t_R^2 ix}{F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right| \cdot \Gamma\alpha$$

6.3.2. Carrier x RF audio sidebands

$$\mathrm{V} = \frac{\Gamma\delta E}{4E_0}\frac{E_0^2 t_R^2}{2F_R F_{CC}}\left|(a + ix)[-(-\alpha-\beta) - (-(\alpha-\beta))] + (a - ix)[-(\alpha+\beta) - (-(-\alpha+\beta))]\right|$$

$$= \frac{\delta E}{2E_0}\left|\frac{E_0^2 t_R^2}{F_R F_{CC}} ix\right| \cdot \Gamma\alpha$$

6.3.3. Intensity noise sum and comparison to signal

$$\mathrm{V} = \left|\frac{\delta E}{2E_0}\frac{E_0^2 t_R^2 \left(2ix - x\frac{\omega}{\omega_{cc}}\right)}{F_R F_{CC}\left(1 + \frac{i\omega}{\omega_{cc}} - \frac{\bar{\varepsilon}\omega^2}{\omega_{cc}}\right)}\right| \cdot (\Gamma\alpha)$$

$$\sim \frac{\delta E}{2E_0}\, x \left(\frac{E_0^2 t_R^2}{F_R F_{CC}}\frac{\omega}{\sqrt{\omega_{cc}^2 + \omega^2}}\right)(\Gamma\alpha)$$

$$\sim \frac{\delta E}{2E_0}\, x \left(\frac{E_0^2 t_R^2}{F_R F_{CC}}\right) \cdot (\Gamma\alpha)$$

and comparison of the noise to the signal (6.2.1) gives:

$$\frac{V_{\delta E}}{V_{signal}} = \frac{\delta E}{2E_0}\,\frac{\Delta x_{rms}}{X}\,\frac{\omega}{\omega_c}$$

6.4. Oscillator Phase Noise

6.4.1. Carrier x RF audio sidebands

$$V = (a + ix) \cdot \frac{\Gamma}{4} \cdot a_o[(-i)(-\alpha - \beta) - (i)(\alpha - \beta)] + (a - ix) \cdot \frac{\Gamma}{4} \cdot a_o[(-i)(\alpha + \beta) - (i)(-\alpha + \beta)]$$

$$= a_O \cdot \frac{E_0^2 t_R^2 x}{F_R F_{cc}} \cdot \Gamma\beta$$

6.4.2. Carrier x RF sidebands x demodulation audio sidebands

In the case of oscillator phase noise, the demodulation signal also has audio sidebands, which give an inband signal when mixed with the carrier and RF sidebands. Since the oscillator audio sidebands are out of phase with their carrier, the resultant demodulation is In-Phase with the optical phase modulation.

The oscillator spectrum is :

$A = \Gamma\ [\cos\Omega t - (a_o/2)(\sin(\Omega+\omega)t + \sin(\Omega-\omega)t)]$

The quadrature spectrum is:

$A = \Gamma\ [\sin\Omega t + (a_o/2)(\cos(\Omega+\omega)t + \cos(\Omega-\omega)t)]$

which shows the audio sidebands In-Phase with the original optical PM.

Then the mixer output is:

$$V = \int Re[(E_{00}E_{-10}{}^* + E_{10}E_{00}{}^*)e^{i\Omega t}]\frac{a_o}{2}(\cos(\Omega + \omega)t + \cos(\Omega - \omega)t)dt$$

$$= Re(E_{00}E_{-10}{}^* + E_{10}E_{00}{}^*)\frac{a_o}{2}\cos\omega t$$

Now $$(E_{00}E_{-10}{}^* + E_{10}E_{00}{}^*) = (a + ix)\frac{\Gamma(-\alpha)}{2} + (a - ix)\frac{\Gamma\alpha}{2}$$

$$= -ix\Gamma\alpha$$

$$\text{thus } Re(E_{00}E_{-10}{}^* + E_{10}E_{00}{}^*) = 0$$

so that $V_{out} = 0$ for this path.

6.4.3. Comparison of Oscillator phase noise with Laser frequency noise

In this case we compare the oscillator phase noise to the laser frequency noise, since the oscillator noise should be limited to below this level.

$$\frac{V_a}{V_{\delta\upsilon}} = \frac{a_o x \beta\omega}{\pi\alpha 10^{-2}\delta\upsilon}$$

$$= \frac{10^2 a_o x \omega^2}{\pi\Omega\delta\upsilon}$$

6.5. Oscillator amplitude noise

A look at Table I shows that oscillator and laser amplitude noise may be handled the same way. The demodulation signal is also the same because the limiter amplifier removes the AM side-bands. Thus we can immediately write:

$$\frac{V_{\delta A}}{V_{signal}} = \frac{\delta A}{2A}\frac{\Delta x_{rms}}{X}\frac{\omega}{\omega_c}$$

6.5.1. Comparison of Oscillator amplitude noise with Laser amplitude noise

We have:

$$\frac{V_{\delta A}}{V_{\delta E}} = \frac{\delta A}{A} \Big/ \frac{\delta E}{E}$$

7 SUMMARY AND CONCLUSIONS

We may summarize the above findings as follows:

- Laser frequency, laser amplitude, oscillator phase and oscillator amplitude noise can be seen to produce audio sidebands on the laser carrier and its RF sidebands (Table I).
- The audio sidebands on the carrier are filtered by the coupled cavity of the recycling mirror and the arm cavity (see pg. 7).
- The audio sidebands on the RF sidebands are not filtered and dominate the noise (pg. 8).
- Laser frequency noise couples to a difference in arm cavity storage time ($\Delta\tau$) while laser amplitude and oscillator amplitude couple to an arm cavity deviation from exact resonance (Δx_{rms}).
- Oscillator phase noise couples to an arm cavity deviation from resonance through a detuning of the audio sidebands from resonance in the recycling cavity.

We restate the noise calculations and draw conclusions on LIGO requirements. The laser frequency and amplitude noise sources (of magnitude $\delta\nu$ and δE) are compared to a signal produced by an arm cavity motion of amplitude X, while the oscillator noise (of magnitude a_0 and δA) is compared to the corresponding laser noise.

In the following, L is the arm cavity length, ω_c is the arm cavity pole frequency, Δx_{rms} is the cavity offset from resonance, ν_0 and E_0 are the laser frequency and intensity, and Ω is the oscillator frequency.

In the expressions below we take:

X (100 Hz) = 1×10^{-19} m / rHz and X (10 kHz) = 4×10^{-18} m / rHz

$\nu_0 = 3 \times 10^{14}$ Hz

$\omega_c = 600$ Hz.

$\Omega = 2\pi \times 12 \times 10^6$ Hz.

$L = 4 \times 10^3$ m

7.1. Laser frequency noise

$$\frac{V_{\delta\nu}}{V_{signal}} \sim 10^{-2} \frac{\delta\nu}{\nu_0} \frac{L}{X}$$

Thus to suppress frequency noise to 10% of signal level, we get $\delta\nu$ (100 Hz) ~ 1×10^{-7} / rHz and $\delta\nu$ (10 kHz) ~ 4×10^{-6} / rHz..

7.2. Laser amplitude noise

$$\frac{V_{\delta E}}{V_{signal}} = \frac{\delta E}{2E_0}\frac{\Delta x_{rms}}{X}\frac{\omega}{\omega_c}$$

Here we take $(\delta E/E) \sim 2 \times 10^{-8}$ / rHz, which is ~20 dB above the shot noise limited amplitude noise of 100 mW of light. To keep amplitude noise at 10% signal level at 100 Hz (and above), we need $\Delta x_{rms} \sim 10^{-12}$ m.

7.3. Oscillator phase noise

$$\frac{V_a}{V_{\delta\nu}} = \frac{10^2 a_o x \omega^2}{\pi\Omega\delta\upsilon}$$

$$\sim \frac{3 \cdot 10^3 a_0 \Delta x_{rms} \omega^2}{\delta\nu}$$

Taking $\delta\nu$ (100 Hz) = 10^{-7} Hz / rHz (see above), $\Delta x_{rms} \sim 10^{-12}$ m, and requiring $V_o \sim V_{\delta\nu}$, we have:

$a_o \sim 10^{-4}$ (or -80 dBc / rHz) at 100 Hz. We also find $a_0 \sim$ (-130 dBc / rHz) at 10 kHz.

7.4. Oscillator amplitude noise

$$\frac{V_{\delta A}}{V_{\delta E}} = \frac{\delta A}{A}\frac{E}{\delta E}$$

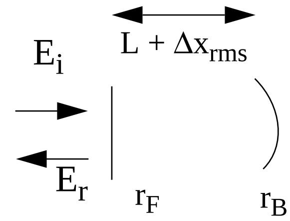
The requirement that $V_{\delta A} \sim V_{\delta E}$ gives $(\delta A/A) \sim$ (-155 dBc / rHz) at f = 100 Hz and above.

8 ACKNOWLEDGEMENTS

This treatment closely follows the approach used in M. Regher’s thesis “Signal Extraction and Control for an Interferometric Gravitational Wave Detector”. The calculation was undertaken at the suggestion of S. Whitcomb. A review of the calculation was provided by H. Yamamoto.

APPENDIX 1 ARM CAVITY REFLECTIVITY

$$r_c = \frac{E_r}{E_i}$$



with the usual circulating field analysis we can write:

$$r_c = r_F + \frac{t_F^2 r_B e^{-i\Phi}}{1 + r_F r_B e^{-i\Phi}}$$

where $$\Phi = 2k(L + \Delta x_{rms}) = 2\left(\frac{2\pi\nu_0 + \omega}{c}\right)(L + \Delta x_{rms})$$

$$= \pi + \frac{2L\omega}{c} + 2k_0\Delta x_{rms}$$

Thus $$r_c = r_F - \frac{t_F^2 r_B\left(1-i\left(\frac{2L\omega}{c} + 2k_0\Delta x_{rms}\right)\right)}{1-r_F r_B\left(1-i\left(\frac{2L\omega}{c} + 2k_0\Delta x_{rms}\right)\right)}$$

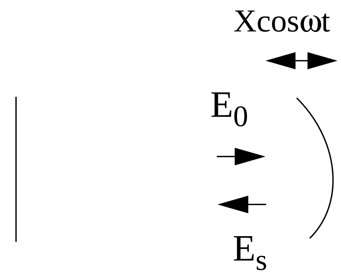
$$= \frac{r_0 + r_F^{-1} i\left(\frac{\omega}{\omega_c} + \frac{\omega_0\Delta x_{rms}}{\omega_c L}\right)}{1 + i\left(\frac{\omega}{\omega_c} + \frac{\omega_0\Delta x_{rms}}{\omega_c L}\right)}$$

where $\omega_c = \frac{c}{2L}\cdot\frac{1 - r_F r_B}{r_F r_B}$, $\omega_0 = ck_0$, and $r_0 = r_c(\Phi = \pi)$

Also with $\Delta x_{rms} < 10^{-11}$ m and writing $x = \frac{\omega_0\Delta x_{rms}}{\omega_c L}$ we have:

$$r_c = \frac{r_0 + r_F^{-1} i\left(\frac{\omega}{\omega_c} + x\right)}{1 + i\frac{\omega}{\omega_c}}$$

APPENDIX 2 ARM CAVITY SIGNAL AT GW PORT

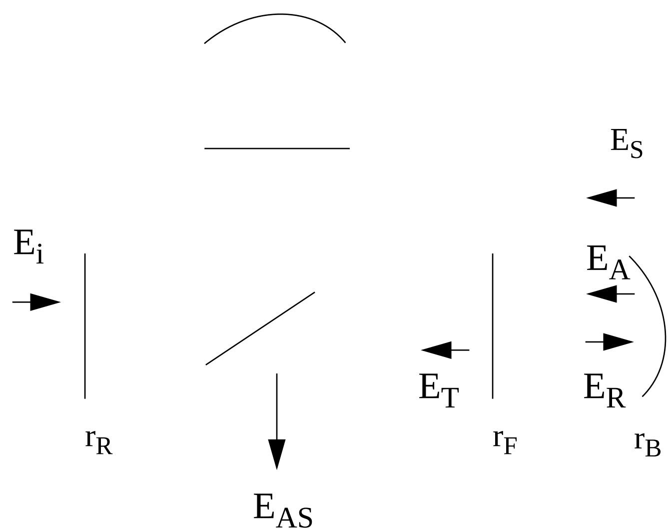


We look at the shaking of a cavity mirror as a phase modulation of the cavity carrier light which produces audio sidebands on the carrier field:

$$E_s = E_c e^{i(\omega_0 t - 2k_0 X \cos\omega t)}$$

$$= E_c[e^{i\omega_0 t} - ik_0 X(e^{i(\omega + \omega_0)t} + e^{i(\omega - \omega_0)t})]$$

and again we solve for the circulating field in the recycled IFO, where the audio sidebands are seen as a source field to be added to the circulating field:



then $$E_A = E_S + r_B E_R$$

$$= E_s + r_B E_A e^{-i2kL}(-r_F) \qquad \text{where } k = \frac{2\pi\nu_0 + \omega}{c}$$

so that $$E_A = \frac{E_s}{(1 - r_F r_B)\left(1 + \frac{i\omega}{\omega_c}\right)}$$

and $$E_T = \frac{E_s t_F}{(1 - r_F r_B)\left(1 + \frac{i\omega}{\omega_c}\right)}$$

$$= E_{AS}$$

now E_s = source field of audio sidebands

$= E_c$(in arm) ik$X$

$$= E_0\left(\frac{t_R}{1 + r_R r_0}\right)\left(\frac{t_F}{1 - r_F}\right) ikX$$

Thus $$E_{AS} = \frac{E_0}{2}\left(\frac{t_R}{1 + r_R r_0}\right)\left(\frac{t_F^2}{1 - r_F}\right)\frac{cikX}{2L\omega_c\left(1 + \frac{i\omega}{\omega_c}\right)}$$

$$\sim E_0\left(\frac{t_R}{1 + r_R r_0}\right)\frac{\pi\nu_0 iX}{L\omega_c\left(1 + \frac{i\omega}{\omega_c}\right)}$$

This expression for the carrier field at the gravity wave port contains the arm cavity pole and the amplification of the field from the recycling cavity.