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This transmission consists of 12 pages, including this cover sheet.

Message:

BRUCE,

HAVE A LOOK THROUGH THIS CALCULATION. THEN WE CAN MORE EASILY DISCUSS THE DETAILS OVER THE PHONE

ROBERT

In the Minkowski coordinate system

$$h_{rr}^{k\ell m} = \sqrt{\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{2k^4}}\ \frac{e^{-ikt}}{\sqrt{2k}}\ \sqrt{\frac{2}{\pi}}\ \frac{k}{r^2}\ j_\ell(kr)\ Y_{\ell m}(\Omega)$$

so that

$$\hat{h}_{rr} = \int_0^\infty dk \sum_{\ell m} \left(\hat{a}_{k\ell m}\, h_{rr}^{k\ell m} + \text{h.c.} \right)$$

I may write $e^{-ikt} j_\ell(kr)/r^2$ in Milne coordinates as

$$e^{-ikt} j_\ell(kr) = \frac{i^\ell}{2\sqrt{2\pi}}\ \frac{e^{-\eta}}{k} \int_{-\infty}^{\infty} dp \left(\Gamma(ip+\ell+1)\, e^{\pi p/2}\ e^{-ip\eta}\ \frac{P_{ip-1/2}^{-(\ell+1/2)}(\cosh\chi)}{\sqrt{\sinh\chi}} \right) k^{-ip}$$

I know (from experience) that the k^{-2} term in the normalization of $h_{\mu\nu}$ may give me trouble later (a difficult integration).

I can avoid this by the following trick

$$\frac{e^{-ikt}}{k^2} = \int_{t-i\infty}^{t-i0} dt' \int_{t'-i\infty}^{t'-i0} dt''\ e^{-ikt''}$$

So I want to simplify

$$f = \int^{t} dt' \int^{t'} dt'' \quad e^{-\eta} \; e^{-ip\eta} \; \frac{P^{-(l+\frac{1}{2})}_{ip-\frac{1}{2}}(\cosh\chi)}{\sqrt{\sinh\chi}}$$

S&T have already done such a simplification

1. At fixed r, $\quad dt = \frac{r}{\sinh^2\chi}\, d\chi$

2. $\quad e^{-(1+ip)\eta} = \left(\frac{\sinh\chi}{r}\right)^{(1+ip)}$

So we get

$$f = \int^{\chi} \frac{d\chi'}{\sinh^2\chi'}\, r \int^{\chi'} \frac{d\chi''}{\sinh^2\chi''}\, r \left(\frac{\sinh\chi''}{r}\right)^{1+ip} \frac{P^{-(l+\frac{1}{2})}_{ip-\frac{1}{2}}(\cosh\chi'')}{\sqrt{\sinh\chi''}}$$

$$= r^2\, r^{-(1+ip)} \int^{\chi} d\chi' \,(\sinh\chi')^{-2} \int^{\chi'} d\chi'' \,(\sinh\chi'')^{ip-1} \frac{P^{-(l+\frac{1}{2})}_{ip-\frac{1}{2}}(\cosh\chi'')}{\sqrt{\sinh\chi''}}$$

$$= r^2\, e^{-(1+ip)\eta} (\sinh\chi)^{-(1+ip)} \int^{\chi} d\chi' \,(\sinh\chi')^{-2} \int^{\chi'} d\chi'' \,(\sinh\chi'')^{ip-1} \; P^{-(l+\frac{1}{2})}_{ip-\frac{1}{2}}(\cosh\chi'') \big/ \sqrt{\sinh\chi''}$$

$$h_{rr}^{klm} = \sqrt{\frac{(l-1)l(l+1)(l+2)}{2\pi}}\ \sqrt{k}\ Y_{lm}(\Omega)\cdot$$

$$\frac{i^{l}}{2\sqrt{\pi}}\ \frac{\cancel{\ }}{k}\int_{-\infty}^{\infty} dp\ k^{-ip}\,\Gamma(ip+l+1)\,e^{\pi p/2}\ \times$$

$$\left\{ e^{-(1+ip)\eta}\,(\sinh\chi)^{-(1+ip)} \int^{\chi} d\chi'\,(\sinh\chi')^{-2} \int^{\chi'} d\chi''\,(\sinh\chi'')^{ip-\frac{3}{2}}\ P^{-(l+\frac{3}{2})}_{ip-\frac{1}{2}}(\cosh\chi'') \right\}$$

Before I evaluate or manipulate the χ-integrals, I will substitute

$$\psi_p = e^{-(1+ip)\eta}\ \frac{e^{\pi p/2}}{2\sqrt{p\sinh\pi p}}$$

as the Minkowski mode function in Milne coordinates.

$$h_{rr}^{klm} = \frac{\sqrt{(l-1)l(l+1)(l+2)}}{2\pi}\ i^{l}\,Y_{lm}(\Omega) \int_{-\infty}^{\infty} dp\ k^{-(ip+\frac{1}{2})}\,\Gamma(ip+l+1)\sqrt{p\sinh\pi p}\ \ \psi_p(\eta)\ \times$$

$$\left\{ (\sinh\chi)^{-(1+ip)} \int^{\chi} d\chi'\,(\sinh\chi')^{-2} \int^{\chi'} d\chi''\,(\sinh\chi'')^{ip-\frac{3}{2}}\ P^{-(l+\frac{3}{2})}_{ip-\frac{1}{2}}(\cosh\chi'') \right\}$$

Before I proceed, I will point out two things

1. when I go to evaluate an expectation $\langle h_{rr} h_{rr} \rangle$

I will have terms like

$$\left\langle \int dk \int dk' \, \hat{a}_{k\ell m} \, \hat{a}^{\dagger}_{k'\ell' m'} \int dp \; k^{-i(p+\xi)} \; k'^{\,i(p'-\xi)} \; \text{etc} \right\rangle$$

$\downarrow$ depends on p & p'

$\simeq$ after integrating out the $\delta(k-k')$

$$\simeq \int dk \int dp \int dp' \; \underbrace{\frac{1}{k} \; k^{i(p'-p)}}_{\text{This can be recast to produce } \delta(p-p')} \; \text{etc}$$

$$\simeq \int dp \int dp' \int_0^{\infty} \frac{dk}{k} \; e^{i \ln k (p'-p)} \; \text{etc}$$

$$\simeq \int dp \int dp' \int_{-\infty}^{\infty} dx \; e^{ix(p'-p)} \; \text{etc}$$

$$\simeq \int dp \; \text{etc}$$

So the integration simplifies!

2. I hope (!?) that it will be convenient to evolve ψ, ψ' from Minkowski to deSitter, and then on to $k=-1$ FRW expansion era, while keeping the same ξ-dependence in h_{rr}.

According to S&T

$$Q := (\sinh x)^{-(1+ip)} \int^{x} dx' (\sinh x')^{-2} \int^{x'} dx'' (\sinh x'')^{ip-3/2} P^{-(\ell+1/2)}_{ip-1/2}(\cosh x'')$$

$$= \frac{1}{(ip+\ell)(ip+\ell-1)} \frac{1}{\sinh^{5/2} x} P^{-(\ell+1/2)}_{ip-5/2}(\cosh x)$$

Can I check this? I need to make use of

$$\to (\sinh x)^{\nu-1} P^{\mu}_{\nu}(\cosh x) = \frac{d}{dx}\left\{ \frac{(\sinh x)^{\nu}}{(\nu-\mu)} P^{\mu}_{\nu-1}(\cosh x) \right\}$$

$$= \frac{1}{\nu-\mu}\left\{ \nu (\sinh x)^{\nu-1} \cosh x\, P^{\mu}_{\nu-1} + (\sinh x)^{\nu} P^{\mu\,\prime}_{\nu-1} \right\}$$

s.t.
$$\frac{1}{\sinh x} P^{\mu}_{\nu} = \frac{1}{(\nu-\mu)}\left(\nu \frac{\cosh x}{\sinh x} P^{\mu}_{\nu-1} + P^{\mu\,\prime}_{\nu-1} \right)$$

$$(\nu-\mu) P^{\mu}_{\nu} - \nu z P^{\mu}_{\nu-1} = \frac{\cancel{\quad}}{(z^2-1)} \frac{d}{dz} P^{\mu}_{\nu-1}$$

$$(\nu-\mu+1) P^{\mu}_{\nu+1} - (\nu+1) z P^{\mu}_{\nu} = (z^2-1) \frac{d}{dz} P^{\mu}_{\nu} \quad \checkmark$$

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So then

$$\int^{\chi'} d\chi'' \, (\mathrm{sh}\chi'')^{ip-\frac{3}{2}} \, P^{-(l+\frac{1}{2})}_{ip-\frac{1}{2}}(\mathrm{ch}\chi'')$$

$\mu = -(l+\frac{1}{2})$

$\nu = ip-\frac{1}{2}$

$$= \int^{\chi'} d\chi'' \, (\mathrm{sh}\chi'')^{\nu-1} \, P^{\mu}_{\nu}(\mathrm{ch}\chi'')$$

$$= \frac{(\mathrm{sh}\chi')^{\nu}}{\nu-\mu} \, P^{\mu}_{\nu-1}(\mathrm{ch}\chi')$$

And the next integral

$$\int^{\chi} d\chi' \, (\mathrm{sh}\chi')^{-2} \, \frac{(\mathrm{sh}\chi')^{\nu}}{\nu-\mu} \, P^{\mu}_{\nu-1}(\mathrm{ch}\chi')$$

$$= \int^{\chi} d\chi' \, \frac{(\mathrm{sh}\chi')^{\nu-2}}{(ip-\frac{1}{2}+l+\frac{1}{2})} \, P^{\mu}_{\nu-1}(\mathrm{ch}\chi')$$

Now $\nu = ip-\frac{3}{2}$

$$= \int^{\chi} d\chi' \, \frac{(\mathrm{sh}\chi')^{\nu-1}}{(ip+l)} \, P^{\mu}_{\nu}(\mathrm{ch}\chi')$$

$$= \frac{1}{(ip+l)} \cdot \frac{1}{(ip+l-1)} \, (\mathrm{sh}\chi)^{ip-\frac{3}{2}} \, P^{-(l+\frac{1}{2})}_{ip-\frac{3}{2}}(\mathrm{ch}\chi)$$

So that Q (from page 5) is

$$Q = \frac{(\mathrm{sh}\chi)^{-\frac{3}{2}}}{(ip+l)(ip+l-1)} \, P^{-(l+\frac{1}{2})}_{ip-\frac{3}{2}}(\mathrm{ch}\chi) \quad \checkmark$$

Returning to h_{rr}^{klm}

$$h_{rr}^{klm} = \frac{i^{l}}{2\pi}\sqrt{(l-1)l(l+1)(l+2)}\; Y_{lm}(\Omega)\int_{-\infty}^{\infty} dp\; \frac{k^{-(ip+\frac{3}{2})}\,\Gamma(ip+l+1)\sqrt{p\sinh\pi p}}{(ip+l)(ip+l-1)\,\sinh^{3/2}\chi}\;\psi_p(r)\; P^{-(l+\frac{3}{2})}_{ip-\frac{1}{2}}(\cosh\chi)$$

[It is possible to express $P^{-(l+\frac{3}{2})}_{ip-\frac{1}{2}}$ as a combination of associated Legendre functions of lower index $ip-\frac{1}{2}$ and upper indices $-(l+\frac{1}{2})$, $-(l+\frac{3}{2})$, etc. However, I will not do so here since I prefer this compact form.]

$$\hat{h}_{rr} = \int dk \sum_{lm} \hat{a}_{klm}\left[\frac{i^{l}}{2\pi}\sqrt{(l-1)l(l+1)(l+2)}\; Y_{lm}(\Omega)\cdot \int_{-\infty}^{\infty} dp\; \frac{k^{-(ip+\frac{3}{2})}\,\Gamma(ip+l+1)\sqrt{p\sinh\pi p}}{(ip+l)(ip+l-1)\,\sinh^{3/2}\chi}\;\psi_p(r)\; P^{-(l+\frac{3}{2})}_{ip-\frac{1}{2}}(\cosh\chi)\right]$$

$$+\ \text{h.c.}$$

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Now

$$ds^2 = \underbrace{g_{\mu\nu}}_{\text{BACKGROUND METRIC}} dx^\mu dx^\nu + \underbrace{h_{\mu\nu}}_{\text{LINEARIZED GRAVITATIONAL WAVE PERTURBATION}} dx^\mu dx^\nu$$

in the MILNE coordinate system $\left\{ ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + \mathrm{sh}^2\chi \, d\Omega^2) + h_{\mu\nu} dx^\mu dx^\nu \right\}$

the Minkowski metric perturbation h_{rr} appears as

$$h_{\eta\eta} = a^2(\eta)\, \mathrm{sh}^2\chi \, h_{rr} \qquad a(\eta) = e^\eta$$

$$h_{\eta\chi} = a^2(\eta)\, \mathrm{sh}\chi \, \mathrm{ch}\chi \, h_{rr}$$

$$h_{\chi\chi} = a^2(\eta)\, \mathrm{ch}^2\chi \, h_{rr}$$

Since I know h_{rr}, I can easily obtain $h_{\eta\eta}, h_{\eta\chi}, h_{\chi\chi}$.

The Sachs-Wolfe effect is

$$\frac{\delta T}{T}(\hat{n}) = \frac{1}{2}\int_0^L d\lambda \left\{ \partial_\alpha h_{\beta\eta} + \partial_\beta h_{\alpha\eta} - \partial_\eta h_{\alpha\beta} \right\}\Big|_{\substack{\chi=\lambda \\ \eta=\eta_0-\lambda}} \Omega^\alpha \Omega^\beta$$

where $\Omega^\alpha = \frac{1}{a}(-1, 1, 0, 0)$

so that

$$\frac{\delta T}{T}(\Omega) = \frac{1}{2}\int_0^L d\lambda\ I(\lambda)$$

$$I(\lambda) = \frac{1}{a^2}\left[\partial_\eta (h_{\eta\eta} - h_{\chi\chi}) + 2\,\partial_\chi (h_{\eta\chi} - h_{\eta\eta}) \right]$$

$$= \frac{1}{a^2}\left[\partial_\eta (a^2 [\sinh^2\chi - \cosh^2\chi]\, H) + 2\partial_\chi (a^2 [\sinh\chi\cosh\chi - \sinh^2\chi]\, H) \right]$$

$$(\text{where } H := h_{rr})$$

$$= -2\frac{\dot{a}}{a} H - \dot{H} + 2e^{-2\chi} H + 2e^{-\chi}\sinh\chi\, H' \Big|_{\substack{\chi=\lambda \\ \eta=\eta_0-\lambda}}$$

with $\dot{} \equiv \frac{\partial}{\partial\eta}$, $' \equiv \frac{\partial}{\partial\chi}$

Next, make the definition

$$\hat{I}(\lambda) = \frac{i^{\ell}}{2\pi} \sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}\, Y_{\ell m}(\Omega) \cdot \int_{-\infty}^{\infty} dp \, \frac{k^{-(ip+\frac{3}{2})}\,\Gamma(ip+\ell+1)\sqrt{p \sinh \pi p}}{(ip+\ell)(ip+\ell-1)}\, F_{\ell p}(\eta,\chi)\Big|_{\substack{\chi=\lambda \\ \eta=\eta_0-\lambda}}$$

$$\left(\int dk \sum_{\ell m} \hat{a}_{k\ell m} \right) + h.c.$$

where $F_{\ell p}(\eta,\chi) \equiv \left(-2\frac{\dot{a}}{a} - \frac{\partial}{\partial \eta} + 2e^{-2\chi} + 2e^{-\chi}\sinh\chi \frac{\partial}{\partial \chi}\right)\left(\frac{\psi_p(\eta)}{\sinh^{1/2}\chi}\, P^{-(\ell+\frac{1}{2})}_{ip-\frac{1}{2}}(\cosh\chi)\right)$

So that

$$|a_{\ell m}|^2 = \frac{1}{4}\int_0^{\infty} dk \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \int_0^{L} d\lambda \int_0^{L} d\lambda' \Bigg[\frac{1}{4\pi^2}(\ell-1)\ell(\ell+1)(\ell+2) \times$$

$$\left[F^*_{\ell p'}(\eta_0-\lambda',\lambda')\, F_{\ell p}(\eta_0-\lambda,\lambda) \; \frac{k^{-(ip+\frac{3}{2})}\, k^{(ip'-\frac{3}{2})}\,\Gamma(ip+\ell+1)\,\Gamma(-ip'+\ell+1)\sqrt{pp'\sinh\pi p \sinh \pi p'}}{(ip+\ell)(-ip'+\ell)(ip+\ell-1)(-ip'+\ell-1)} \right] \Bigg]$$

$$= \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{16\pi^2} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \int_0^{L} d\lambda \int_0^{L} d\lambda'$$

$$\frac{\Gamma(ip+\ell+1)\,\Gamma(-ip'+\ell+1)\sqrt{pp'\sinh\pi p\sinh\pi p'}}{(ip+\ell)(-ip'+\ell)(ip+\ell-1)(-ip'+\ell-1)}\, F_{\ell p}(\eta_0-\lambda,\lambda)\, F^*_{\ell p'}(\eta_0-\lambda',\lambda')$$

$$\cdot \int_0^{\infty} dk \, \frac{1}{k}\, k^{i(p'-p)}$$

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$$|q_{\ell m}|^2 = \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{16\pi^2} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \int_0^L d\lambda \int_0^L d\lambda'$$

$$\frac{\Gamma(ip+\ell+1)\,\Gamma(-ip'+\ell+1)\sqrt{pp'\sinh\pi p\,\sinh\pi p'}}{(ip+\ell)(-ip'+\ell)(ip+\ell-1)(-ip'+\ell-1)} F_{\ell p}(\eta_0-\lambda,\lambda)\, F^*_{\ell p'}(\eta_0-\lambda',\lambda')$$

$$\cdot\ 2\pi\,\delta(p-p')$$

$$= \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{8\pi} \int_{-\infty}^{\infty} dp \left| \int_0^L d\lambda\ F_{\ell p}(\eta_0-\lambda,\lambda) \right|^2 \frac{p\sinh\pi p\,|\Gamma(ip+\ell+1)|^2}{(p^2+\ell^2)(p^2+(\ell-1)^2)}$$

$$\boxed{|q_{\ell m}|^2 = \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{8\pi} \int_{-\infty}^{\infty} dp\ \ p\sinh\pi p\,|\Gamma(ip+\ell-1)|^2 \left| \int_0^L d\lambda\ F_{\ell p}(\eta_0-\lambda,\lambda) \right|^2}$$

The above expression is certainly valid for Minkowski space-time, where

$$\boxed{F_{\ell p}(\eta,\chi) = \left\{ -2\frac{\dot a}{a} - \frac{\partial}{\partial\eta} + 2e^{-2\chi} + 2e^{-\chi}\sinh\chi\frac{\partial}{\partial\chi} \right\} \frac{\Psi_p(\eta)}{\sinh^{1/2}\chi}\, P^{-(\ell+1/2)}_{ip-1/2}(\cosh\chi)}$$