LIGO Laser Interferometer Gravitational-Wave Observatory

# Introduction to Fabry-Perot Cavities and 40m Power-Recycling Interferometer

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lecture notes from LIGO Science Seminar
March 1997

LIGO-T970262-00-D

# Overview

- The physics of Fabry Perot cavities
  - ›› setting the scene for recycling
- Recycling at the 40m: Optical Design and Choice of Parameters
  - ›› coupled cavities
  - ››choice of optical parameters
- Displacement Calibration and Shot Noise

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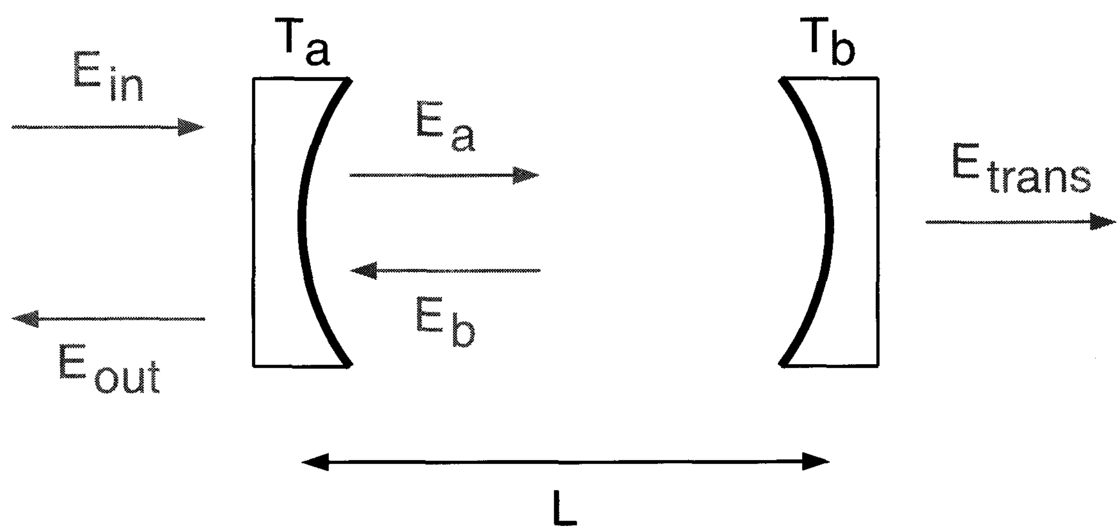
# The Physics of Fabry Perot Cavities

- definitions of standard parameters
- cavity DC response
- phase modulation
- reflection locking

# Fields in Fabry Perot Cavities

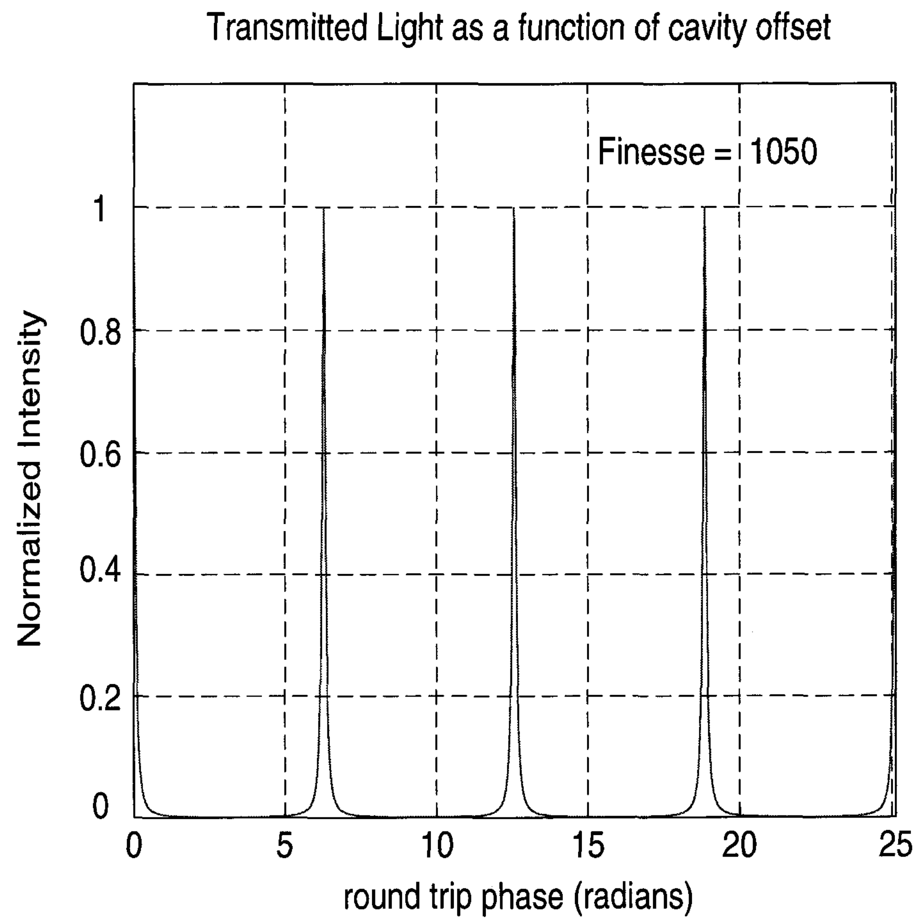



The transmitted light power

$$\frac{P_{trans}}{P_{in}} = \frac{T_a T_b}{(1 - r_a r_b)^2 + 4 r_a r_b \sin^2(\phi/2)}$$

Light in the cavity experiences a round trip phase of $\phi$. On RESONANCE

$$\phi = 2n\pi \text{ or equivalently } f = \frac{c}{2L}n$$




The transmission peaks are spaced in frequency by

$$\frac{c}{2L} = \text{Free Spectral Range of the Cavity}$$

The cavity finesse is defined to be

$$\text{Finesse} = \frac{\text{Separation of Adjacent Transmission Peaks}}{\text{Full width of peak at half maximum}} = \frac{\pi\sqrt{r_a r_b}}{(1 - r_a r_b)}$$

Another useful quantity is the Co-efficient of Finesse

$$F = \frac{4r_a r_b}{(1 - r_a r_b)^2} = \frac{4(\text{Finesse})^2}{\pi^2}$$

Some typical numbers:

| Arm cavity | FSR (MHz) | Finesse | $\delta f$ (Hz) |
|---|---|---|---|
| 40m recombined | 3.9 | 12,770 | 305 |
| 40m recycling | 3.9 | 1050 | 3700 |
| LIGO | 0.04 | 206 | 180 |

| Arm cavity | Ta | Tb | L |
|---|---|---|---|
| 40m recombined | 280 ppm | 12 ppm | 100 ppm |
| 40m recycled | 5750 ppm | 12 ppm | 100 ppm |
| LIGO | 0.03 | 10 ppm | 30 ppm |

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# Cavity Reflectivity

$$\frac{E_{out}}{E_{in}} = \frac{r_a - r_b(1 - L_a)e^{i\phi}}{1 - r_a r_b e^{i\phi}}$$

-> the cavity looks like a mirror of variable reflectivity. At resonance

$$r_{cavity} = \left(\frac{E_{out}}{E_{in}}\right)_{\phi = 0} = \frac{r_a - r_b(1 - L_a)}{1 - r_a r_b}$$

For small deviations from resonance

$$C_o = \left|\frac{\partial}{\partial\phi}\left(\frac{E_{out}}{E_{in}}\right)\bigg|_{\phi = 0}\right| = \frac{T_a r_b}{(1 - r_a r_b)^2}$$

This is the DC cavity response. Typical numbers:

| **Arm cavity** | $r_{cavity}$ | $C_o$ |
|---|---|---|
| 40m recombined | -0.138 | 4626 |
| 40m recycled | -0.929 | 645 |
| LIGO arm cavity | -0.995 | 130 |

# Coupling of Cavity

$$\left(\frac{E_{out}}{E_{in}}\right)_{\phi = 0} = \frac{r_a - r_b(1 - L_a)}{1 - r_a r_b}$$

Optimal Coupling occurs when $E_{out} = 0$. For small transmission and losses this occurs when

$$T_a = T_b + L_a + L_b$$

If $T_a < T_b + L_a + L_b$ then $E_{out}$ is +ve i.e. in phase with incident light - cavity is undercoupled.

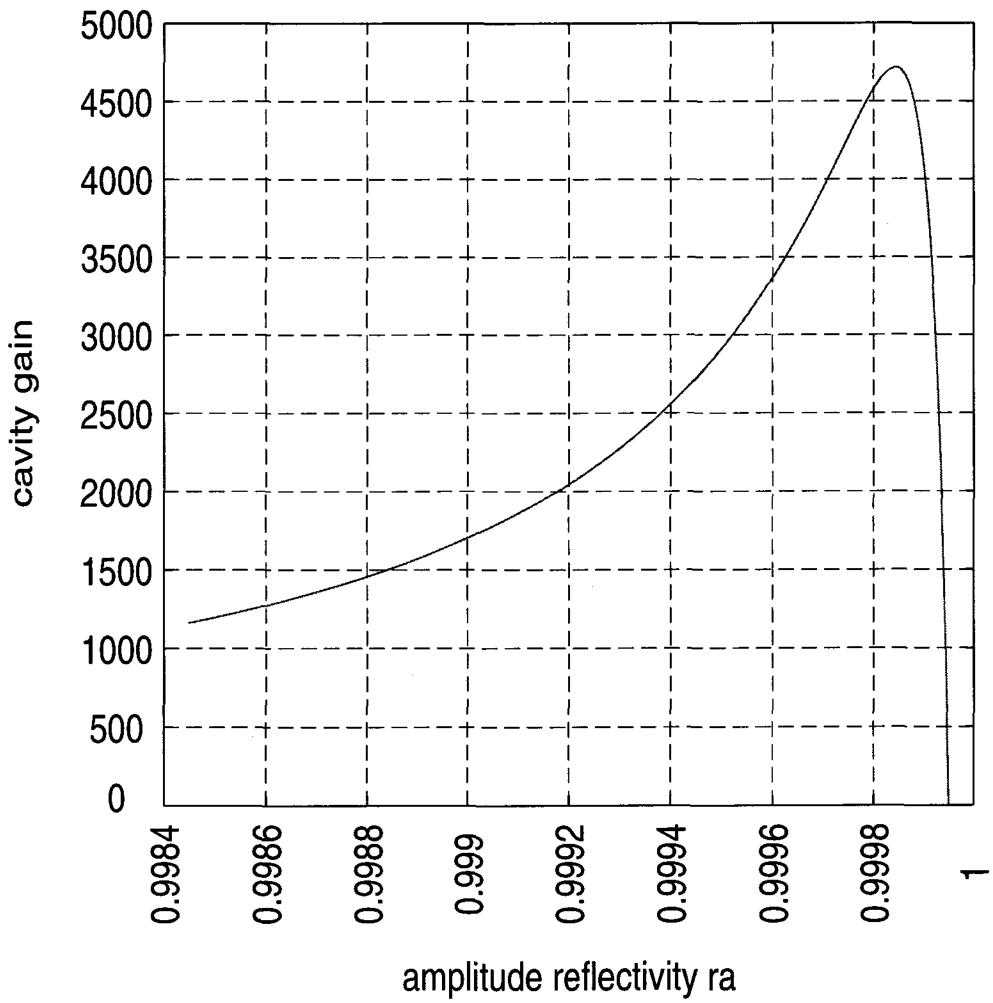
If $T_a > T_b + L_a + L_b$ then $E_{out}$ is -ve i.e. out of phase with incident light - cavity is overcoupled.

# Cavity Gain

On resonance the optical gain of the cavity is:

$$G = \left|\frac{E_a}{E_{in}}\right|^2 = \frac{T_a}{(1 - r_a r_b)^2}$$

Coupling of Power to Fabry–Perot Cavity



Optimal coupling => maximum cavity gain

$$G_{opt} \approx \frac{1}{T_b + L_a + L_b}$$

cavity response for small changes in $\phi$

$$C_o = G r_b$$

$\therefore$ for maximum cavity response want to be (close to) optimally coupled

# Cavity Visibility

Cavity visibility

$$V = 1 - \frac{P_{min}}{P_{max}}$$

where $P_{min}$ and $P_{max}$ is the in lock and out of lock power respectively reflected by the arm.

Note that since in general only a fraction *M* of the input laser light, $P_o$, is modematched to the cavity

$$P_{min} = R_{cavity} M P_o + (1 - M) R_a P_o$$

$$P_{max} = R_a P_o$$

∴ for low modulation depth (see later)

$$V = M\left(1 - \frac{R_{cavity}}{R_a}\right)$$

For 40m, M ~ 0.9,

| **Arm cavity** | $R_{cavity}$ | **Ta** | **V** |
|---|---|---|---|
| 40m recombined | 0.019 | 280 ppm | 0.88 |
| 40m recycled | 0.86 | 5750 ppm | 0.12 |

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# Phase Modulation and Signal for Locking Fabry Perot Cavity

# Simple Example

Frequencies: $\omega_0$ , $\omega_0 + \Omega$

Field incident on F-P cavity

$$E = E_0 \, e^{i\omega_0 t} \, (1 + a \, e^{i\Omega t})$$

Field after the reflection

$$E = E_0 \, e^{i\omega_0 t} \, ( \, \rho(x) + a \, e^{i\Omega t} \, )$$

Intensity at the photodiode

$$|E|^2 = E_0^2 \, ( \, 2a \, \mathrm{Im}\{\rho(x)\} \, \sin \Omega t \ldots)$$

Demodulation picks up $1\Omega$-component:

$$V \sim \mathrm{Im}\{\rho(x)\}.$$

Can lock on the sideband: less signal, wrong sign

# Phase Modulation Sidebands

Frequencies: $\omega_0 + \Omega$, $\omega_0$, $\omega_0 - \Omega$

Field after Pockel Cell

$$E = E_0 \, e^{i\omega_0 t} \, e^{i\Gamma \sin \Omega t}.$$

Expansion

$$1 + i\Gamma \sin \Omega t \quad = \quad 1 \; + \; \frac{\Gamma}{2} \, e^{i\Omega t} \quad - \quad \frac{\Gamma}{2} \, e^{-i\Omega t}$$

Same result

$$V \sim \mathrm{Im}\{\rho(x)\}.$$

# Phase Modulation Parameters

Modulation frequency

$$\Omega = 2\pi f_{mod}$$

Modulation wavelength

$$\lambda_{mod} = \frac{c}{f_{mod}}.$$

## 40m and LIGO Parameters

| | $f_{mod}$(MHz) | $\lambda_{mod}$(m) |
|---|---|---|
| Recombined 40m | 12.3 | 24.4 |
| Recycled 40m | 32.7 | 9.18 |
| LIGO | 24.5 | 12.2 |

# Bessel Functions

Fourier expansion

$$e^{i\Gamma \sin \Omega t} = \sum_{n=-\infty}^{\infty} J_n(\Gamma) e^{in\Omega t}$$

where $J_n(\Gamma)$ are Bessel functions

$\Gamma$ - Modulation Index

There is infinite number of sidebands

Frequencies

$$\omega_n = \omega_0 + n\,\Omega$$

The amplitudes

$$E_n = J_n(\Gamma)\,E$$

# Sideband Initial Phases

Bessel functions with negative index flip sign

$$J_{-n}(\Gamma) = (-1)^n \; J_n(\Gamma)$$

General rule:

Upper sidebands are all positive

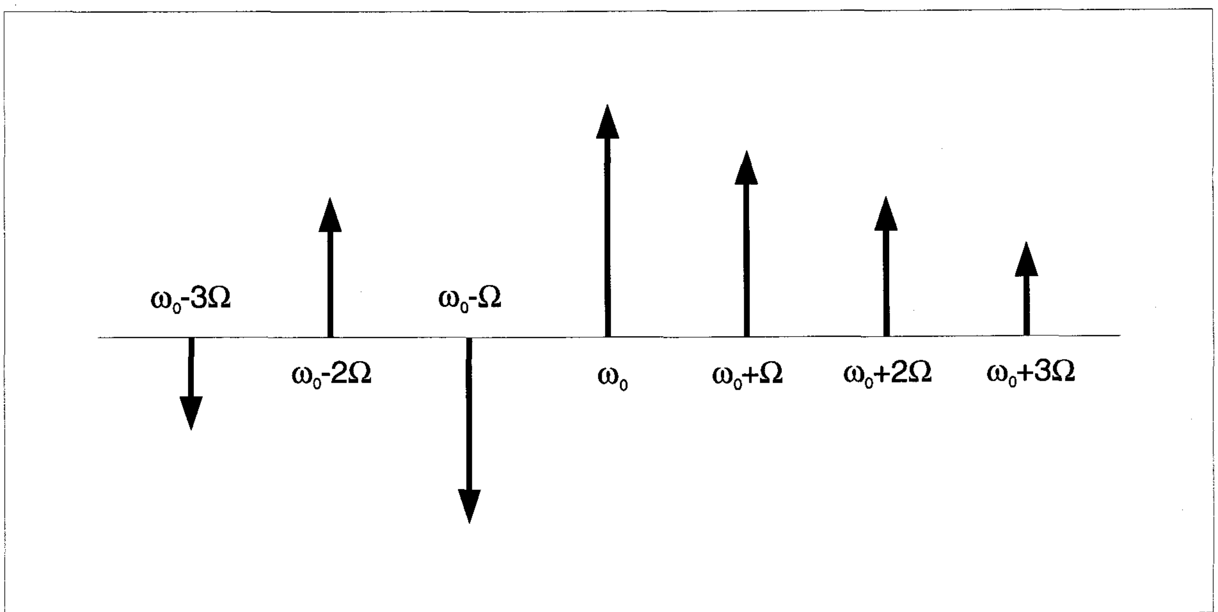
Lower even number sidebands are positive

Lower odd number sidebands are negative

Example:

$$\begin{aligned} E_{-1} &= -E_{+1}, \\ E_{-2} &= +E_{+2} \end{aligned}$$

Figure 1: Sideband Initial Phases



# Sideband Propagation

*Picture 1*

Sidebands have different frequencies.

For propagation in positive $z$-direction

$$\begin{aligned} E_0 &= E\ J_0\ e^{i\omega_0(t-\frac{z}{c})}, \\ E_n &= E\ J_n\ e^{i\omega_n(t-\frac{z}{c})} \end{aligned}$$

*Picture 2*

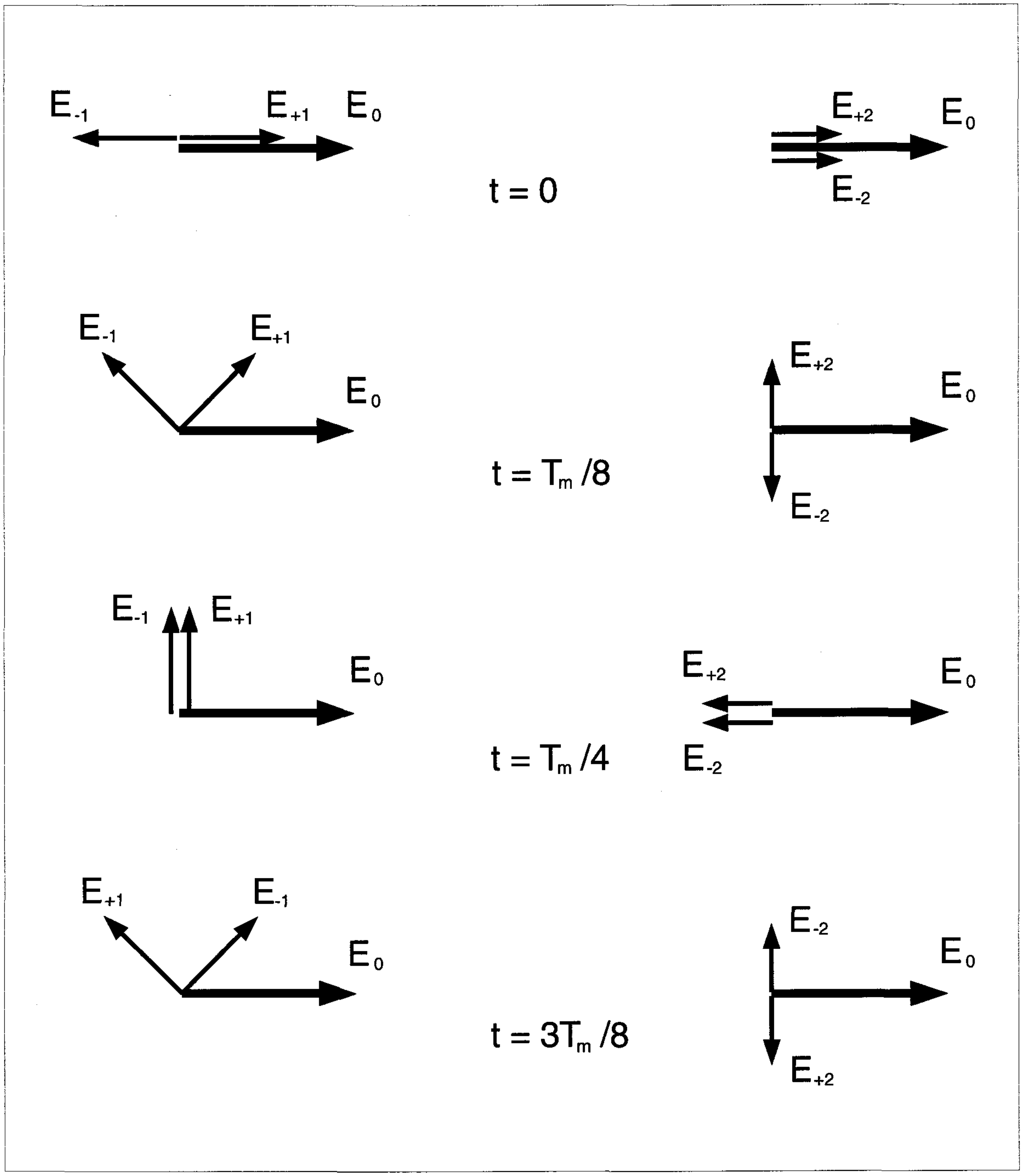
Sidebands have the same frequency as the carrier, but their phases ***slowly*** change with time

$$E_n = E\ J_n\ e^{i\omega_0(t-\frac{z}{c})}e^{iU_n}$$

$U_n(z,t)$ is the phase of the $n$th-sideband relative to carrier

$$U_n = n\,\Omega\,(t - \frac{z}{c}).$$

Figure 1: Phasor Diagram for 1st and 2nd Order Sidebands*



# Power in the Sidebands

Modulation index, $\Gamma$, can be found from measurements of sideband power.

The power in $n$th-sideband

$$P_n = P\ J_n^2(\Gamma)$$

The measurements are done using optical spectrum analyzer.

Resolving sidebands is necessary for accurate measurements of contrast defect.

Fig. Spectrum of Light (Gamma = 1.0)

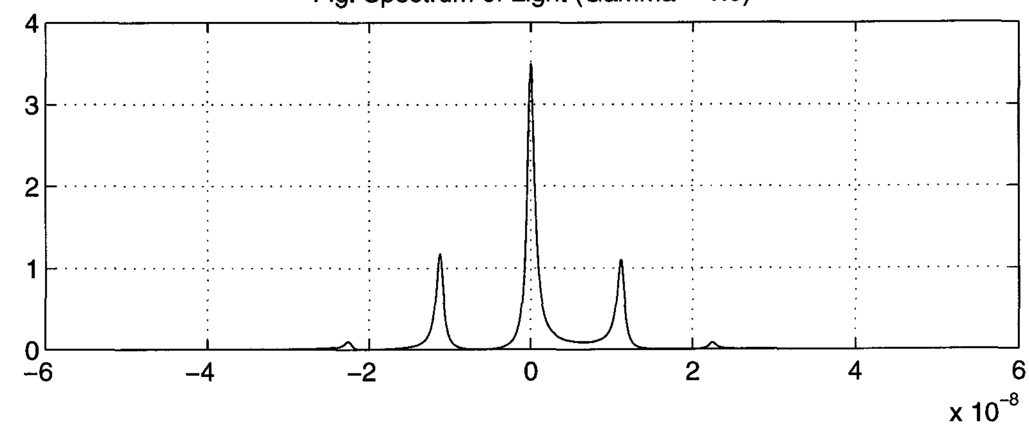
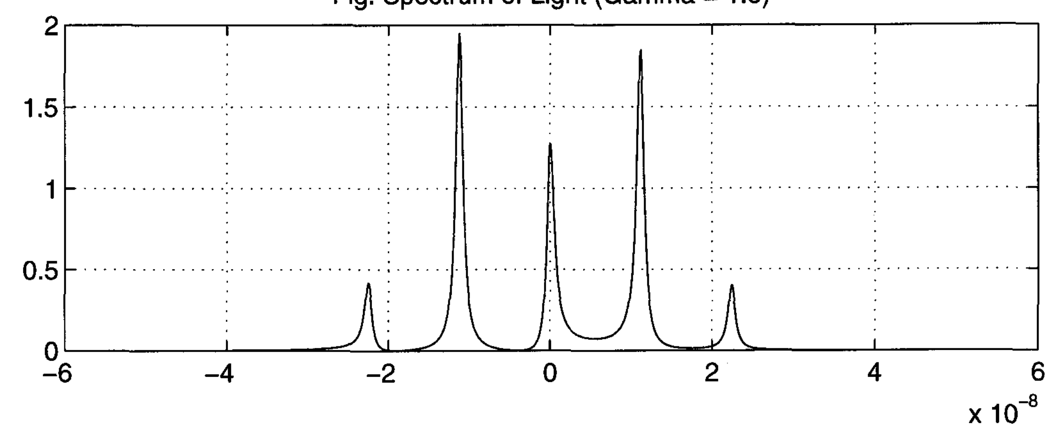



Fig. Spectrum of Light (Gamma = 1.6)



# Intensity at the Photodiode

If cavity is on resonance intensity is constant in the first order.

If cavity deviates from resonance the intensity is modulated

$$P = A_0 + 2A_1 \cos \Omega t + 2B_1 \sin \Omega t.$$

Either $A_1$ or $B_1$ is proportional to deviation of cavity from resonance.

The coefficients are

$$\begin{aligned} A_0 &= |E_0|^2 + |E_{+1}|^2 + |E_{-1}|^2, \\ A_1 &= \mathrm{Re}\left\{E_0^*(E_{+1} + E_{-1})\right\}, \\ B_1 &= \mathrm{Im}\left\{E_0^*(E_{+1} - E_{-1})\right\}. \end{aligned}$$

# Photodiode Responsivity

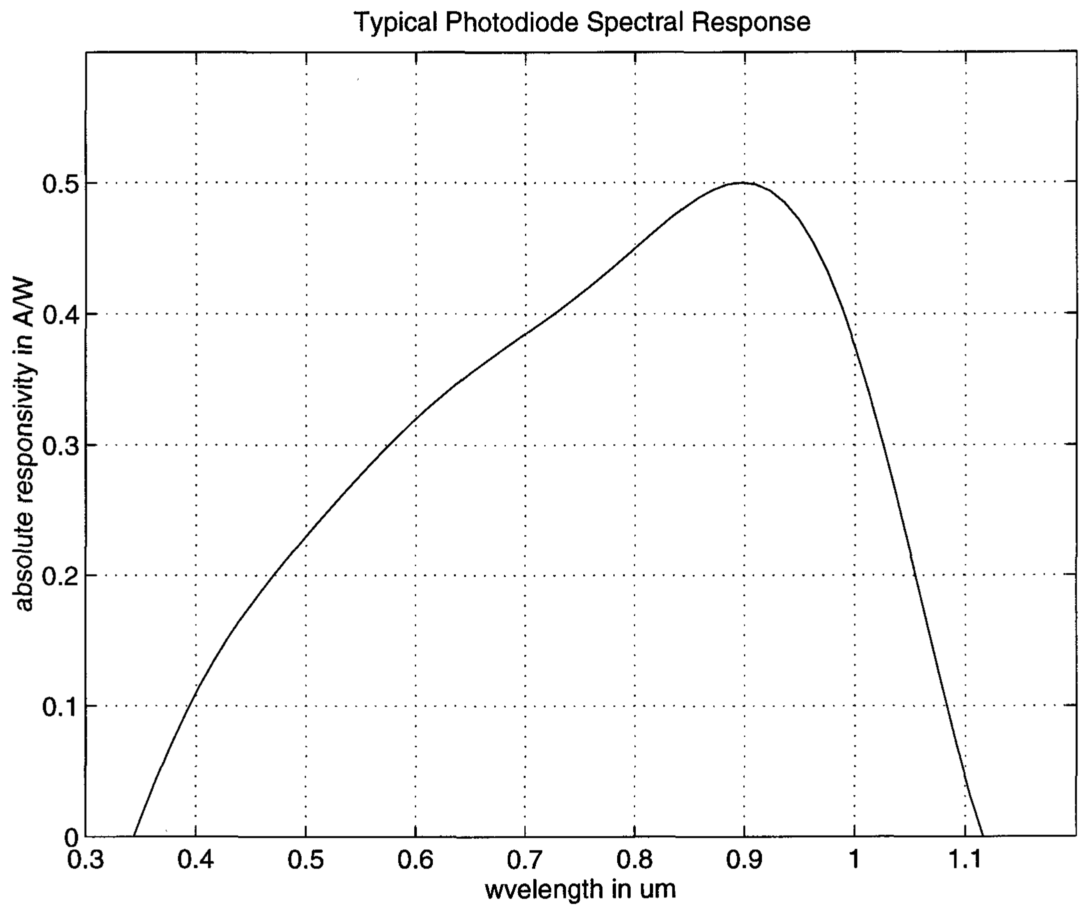
At the photodiode the intensity of light is converted to the photocurrent

$$I(t) = \sigma P(t)$$

$\sigma$ is the responsivity the photodiode

For the green light $\lambda = 514.5$ nm the responsivity is

$$\sigma = 0.25 \ \ \mathrm{A/W}$$



Typical Photodiode Spectral Response
absolute responsivity in A/W
0
0.1
0.2
0.3
0.4
0.5
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
wvelength in um

# Photodiode Quantum Efficiency

Each photon carries energy $h\nu$.

Intensity of light = total energy carried by photons in 1 sec

$$P = h\nu N_{ph}$$

Photocurrent = total charge carried across the photodiode in 1 sec

$$I = eN_e$$

Quantum efficiency of the photodiode $\kappa < 1$

$$N_e = \kappa N_{ph}$$

Photodiode responsivity

$$\sigma = \frac{\kappa e}{h\nu}$$

The choice above corresponds to $\kappa = 0.6$

# Photodiode Output

Photodiode has resonant circuit

$$\tilde{Z}(\omega) = \frac{R}{1 + i\omega RC(1 - \frac{\omega_r^2}{\omega^2})}$$

The resonance frequency $\omega_r = 1/\sqrt{LC}$ is tuned to coinside with the modulation frequency $\Omega$.

$C = 70$ pF (under bias),

$L$ is tuned inductance,

$R \approx 600$ Ohm

Photodiode output

$$V_{pd}(t) = R\ I(t)$$

# Demodulation

Demodulation signal $D(t)$ is a square wave.

$\beta$ adjustable phase

Approximation:

leave only the first term of the Fourier series

$$D(t) = \frac{4}{\pi} \sin(\Omega t + \beta).$$

Coefficient $\frac{4}{\pi}$ is important for comparison with calibration.

Mixer output

$$V = V_{pd}(t)\ D(t)$$

# Signal

In stationary approximation

$$V(x) = V_0 \frac{\sin 2kx}{1 + F \sin^2 kx}$$

40m recycling arm cavity $F = 4.48 \times 10^5$
The magnitude of the signal is defined by the constant

$$V_0 = \frac{4}{\pi} R \sigma P M J_0 J_1 C_0$$

$$2\ k\ V_0 = 160 \ \ \mathrm{V/nm}$$

Fig. Signal of Fabry–Perot Cavity

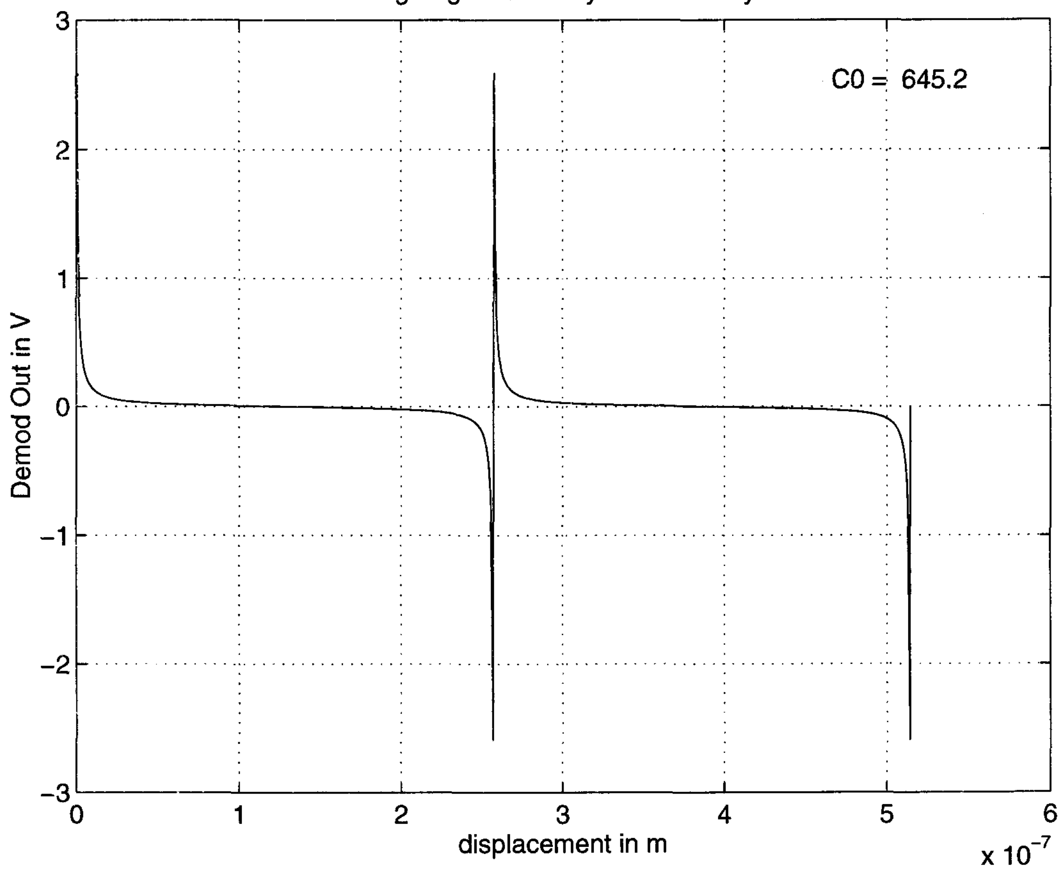
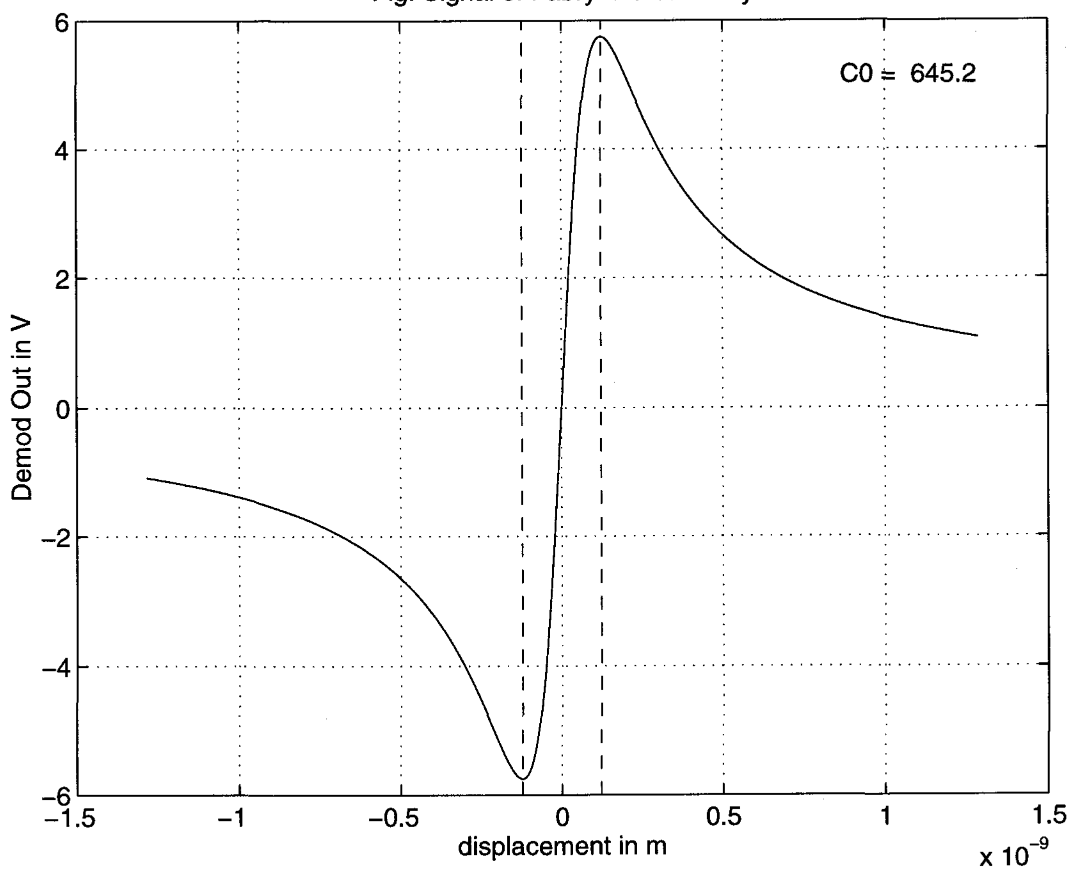


Fig. Signal of Fabry–Perot Cavity



# Maximum of Signal

Fine tuning

Signal as a function of modulation index

$$V_0 \sim J_0(\Gamma)\ J_1(\Gamma)$$

Maximum of the function is at

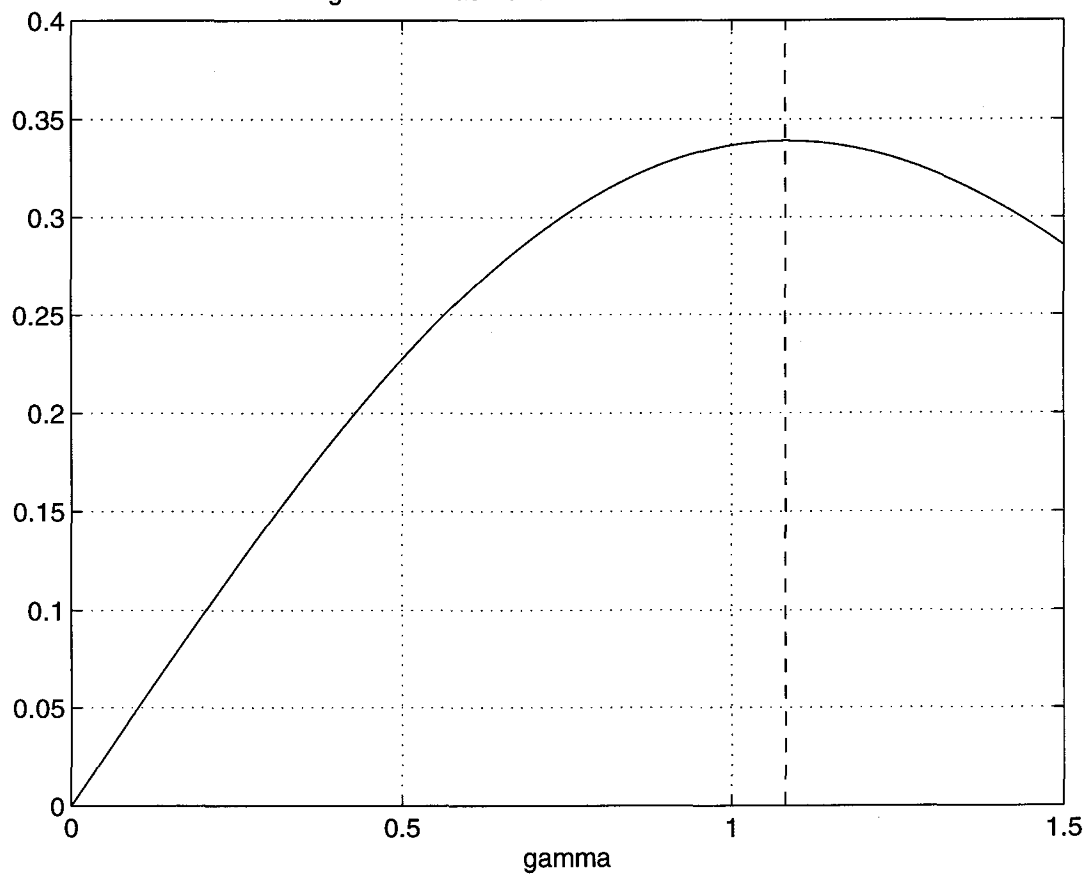
$$\Gamma_1 = 1.08$$

It corresponds to

$$\frac{J_1^2}{J_0^2} = 0.43$$

Beyond this point there is no increase in the signal

Fig. J0 * J1 as Function of Modulation Index



# Optimal Modulation

If we are shot noise limited then the important signal to noise ratio is

$$K(\Gamma) = \frac{J_0 \; J_1}{\sqrt{\rho^2 J_0^2 + 2J_1^2}}$$

For single Fabry-Perot cavity $\rho^2$ is a square of static cavity reflectivity

For recycled interferometer $\rho^2$ is contrast defect

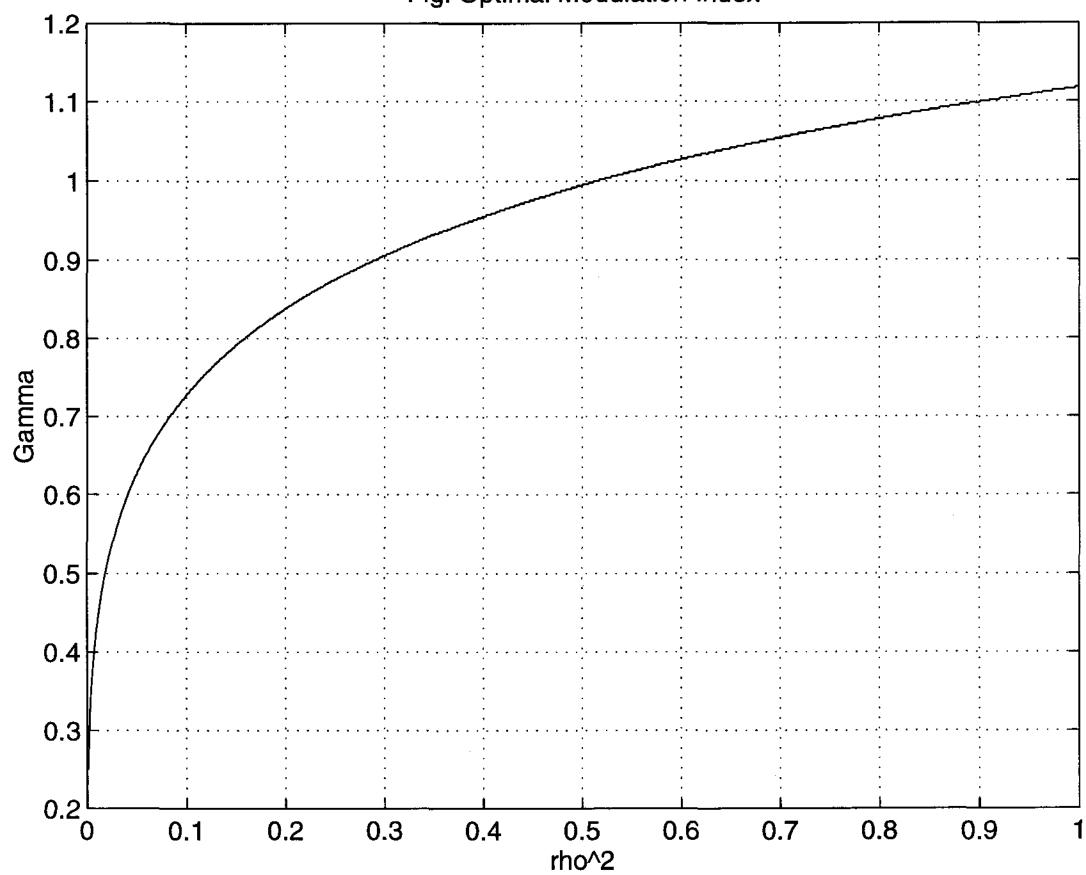
In general the range $\rho^2 < 1$

For any particular $\rho^2$ maximum can be found numerically. This will define optimal modulation index $\Gamma_{opt}$ .

$\Gamma_{opt}$ as a function of $\rho^2$ is found numerically

$$\Gamma_{opt} = \Gamma_{opt}(\rho^2)$$

Fig. Optimal Modulation Index



# Recycling II: Optical Design and Choice of Parameters

- the use of an asymmetric Michelson
- choice of modulation frequency
- choice of mirror reflectivites
- cavity dynamics
  - ringdowns
  - transfer functions

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# Frontal / Schnupp Modulation

- Have already seen the use of modulation to control a single Fabry Perot cavity

How do we apply this principle to a system with

several degrees of freedom?

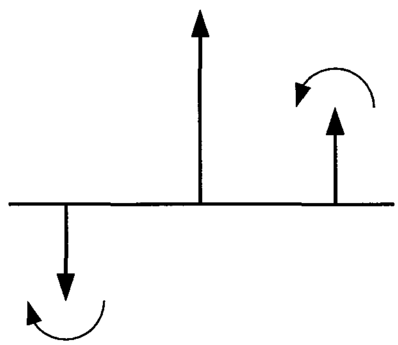
Modulate the light immediately before it enters the interferometer - called frontal or Schnupp modulation.

- Need an asymmetric Michelson.

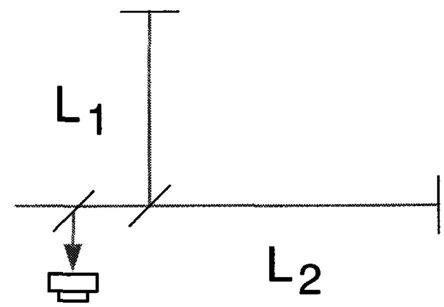
Why?

- Asymmetric Michelson is dispersive $\therefore$ whilst carrier light appears at only one port, sidebands are transmitted to both ports.

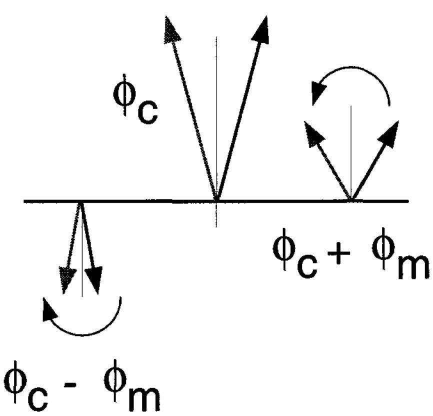
e.g. consider control of beamsplitter for simple Michelson:



Input light




Light from 1 has phase retarded
Light from 2 has phase advanced




If $\phi_c \neq 0$ then one sideband is larger than the other ∴ locking is equivalent to equalising the sideband heights.

# Optimal Asymmetry

The optimal asymmetry is calculated by considering the shot noise limited sensitivity of the GW. It is achieved for

$$\cos\left(\frac{2\pi\delta}{\lambda_{mod}}\right) = r_R T_P$$

For the 40m parameters:

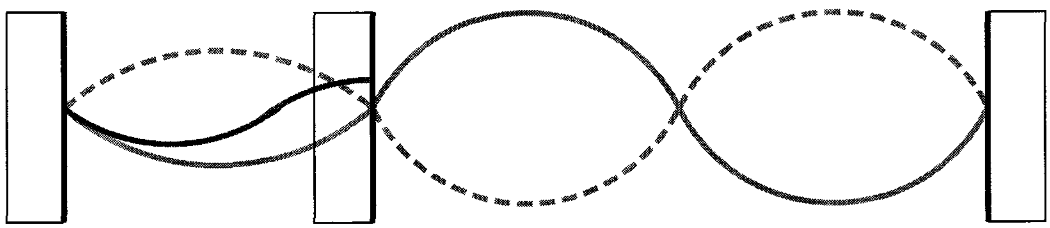
$$\delta_{opt} = 67 \text{ cm}$$

Due to space considerations

$$\delta = 54 \text{ cm}$$

# Choice of Modulation Frequency

- Sidebands + carrier are resonant in the recycling cavity, only carrier is resonant in the arms.



- Carrier has phase change of $\pi$ between light entering and leaving arm $\therefore$ for carrier resonant in the recycling cavity $\phi_c = 2k\pi$
- For sidebands to be resonant in the recycling cavity, we require a $\pi$ phase change i.e.

$$\phi_s = \phi_c + \phi_{\text{mod}} = 2n\pi + \pi$$

$$\phi_{\text{mod}} = \frac{4\pi l_R}{\lambda_{\text{mod}}} = 2m\pi + \pi$$

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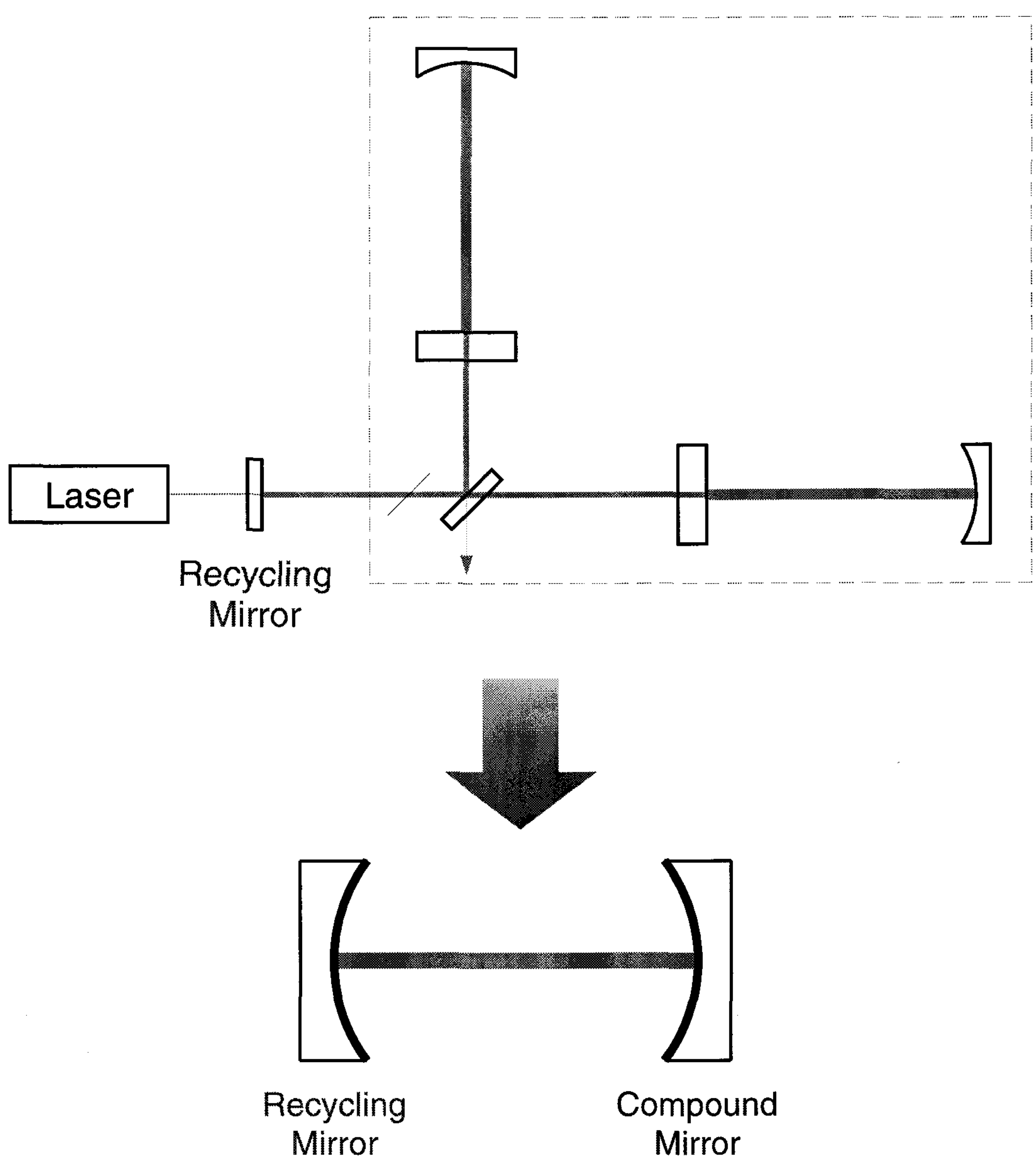
$$l_r = \frac{\lambda_{\text{mod}}}{2}\left(m + \frac{1}{2}\right)$$

From space considerations at 40m, $l_r = 2.3$m

=> $f_{\text{mod}} = 32.7$ MHz

# Recycling: Choice of Mirror Parameters



# Recombined Interferometer as a Compound Mirror

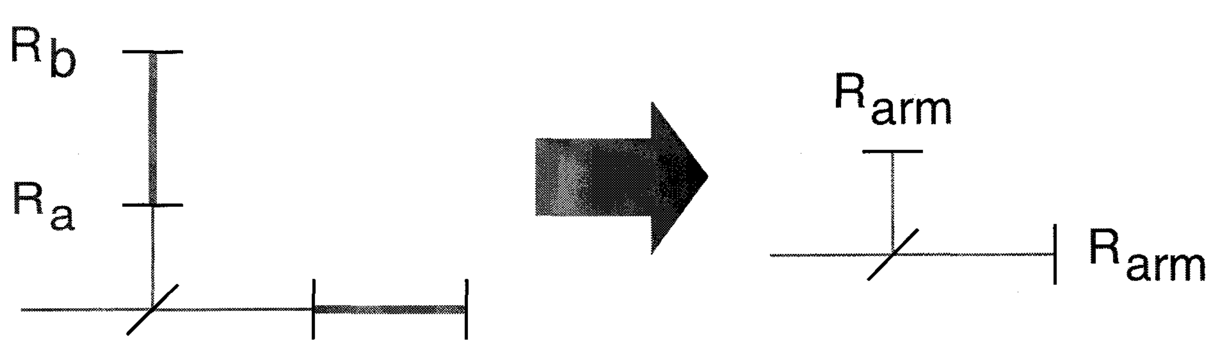
Recall, single FP cavity has a reflectivity

$$R_{\text{cavity}} = \left[\frac{r_a - r_b(1 - L_a)e^{i\phi}}{1 - r_a r_b e^{i\phi}}\right]^2$$

In reality only a fraction, M, of the input power is modematched to the cavity ∴ (1-M) of the light reflects directly off the front mirror. Define

$$R_{\text{arm}} = MR_{\text{cavity}} + (1 - M)R_a$$

Thus we may reduce the recombined topology to



# Contrast

The contrast of the interferometer is a measure of how perfectly light interferes at the beamsplitter:

$$C = \frac{P_B - P_D}{P_B + P_D}$$

$P_D$ is minimum carrier power at dark port with both arms in lock. $P_B$ is maximum carrier power at bright port with arms out of lock.

Some fraction, n, of carrier light recombining at the beamsplitter leaks out of the dark port