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A formalism for simulating the SEI/SUS system in the LIGO E2E model

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I. OVERVIEW

Starting from first principles, a formalism was developed for simluating the dynamics of a mechanical system such as the SEI/SUS system of LIGO which allows inclusion of all degrees of freedom including the internal modes of elastic elements. It is assumed that all displacements are small enough that the dynamics is governed by a system of *linear* differential/partial differential equations. In this report, we also assume that these equations are *time translation invariant*. This allows us to work in the Laplace or Fourier domain.

Note : While completing the present report I finally found a reference [1] (which I always suspected existed somewhere) that provides a general treatment of distributed systems along the same lines as developed here for a mechanical system*. A lot of details have to be filled in, however, when applying the contents of these references to the system we are considering here. Therefore, I continue with the formalism as developed independently in my earlier notes.

The basic idea behind the formalism is the computation of action-reaction (or, simply, reaction) forces between the elements constituting a mechanical system such as the SEI/SUS. The motion of each element (say, a wire, a spring or a rigid body) is solved, in a formal sense, in terms of the forces acting on the element. Some of these forces would be reactions and the rest would be external to the system (or, simply external forces). One then computes, again formally, the acceleration of each point at which the element is connected to other elements in the system. The same excercise is carried out for each element and finally the acceleration of the points of attachement are equated. This then allows us to solve for the reactions in terms of given external forces. As we shall show, it is easy to include externally specified displacements also in this framework. Once all the forces, reactions as well as external, are known, the motion of any part of the system can be computed.

When the dynamics of an element is described by *linear* differential/partial differential equations, the formal solutions can be obtained in terms of appropriate Green's functions. Further, for *time-invariant* linear systems, the Green's functions become transfer functions.

This formalism has the following properties which make it suitable for the e2e.

1. In this formalism each element of a mechanical system can be treated as a separate module. Each module is specified by a set of transfer functions that map external forces into accelerations of attachement points. This allows an arbitrary mechanical system to be constructed by joining modules representing different types of elements (wires, rigid bodies etc.).

2. The formalism was designed from the start to take into account back-action between elements.

3. For time invariant systems, the motion of any element is finally obtained in terms of transfer functions acting on externally specified forces and displacements. This makes a time domain implementation fairly straightforward.

4. It should be possible to incorporate experimentally measured force-to-displacement and displacement-to-displacement transfer functions easily.

5. The Green's functions can be tailored to retain some modes while dropping others. This allows modelling the same element in different ways.

In this report we explain the formalism via its application to some simple systems (sections II A and II B). We present frequency domain transfer functions for these systems. As far as possible the parameters used in these systems have been those in G. Cella's working note released in Nov, 1998. In Section III, we describe one way in which this formalism can be implemented for an arbitrary mechanical system. In Sections III A and III B, we give formal solutions for a few standard elements that would be required in putting together the LIGO SEI/SUS system. Section IV) contains an application of this formalism to a toy model of the suspension of large optics in LIGO. In Appendix D, plots of transfer functions from externally specified displacements of suspension points to translational and rotational motions of the mirror are presented for some simple types of displacements.

Conversion of transfer functions into time domain kernels is still a remaining task.

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*Subsequently, I went back to my reference table for Green's functions, compiled by the same author, and found a short account of the subject at the end!

II. SIMPLE EXAMPLES

A. Simple pendulum

Consider a simple pendulum as shown in Fig. 1.

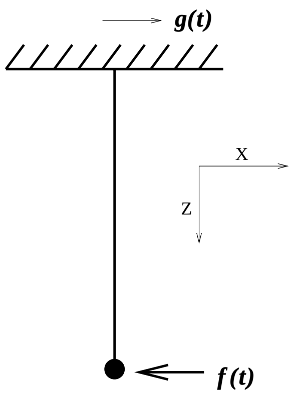

FIG. 1. The length of the string is l and the mass of the bob is m. The mass per unit length of the string is ρ.

We will assume that the transverse displacement of the string, denoted by $u(z,t)$, is confined to the XZ plane. We do not include the longitudinal mode of the string for the present. The driving force $f(t)$ acts on the bob alone. A transverse force $g(t)$ also acts at the suspension point of the string. The damping of the wire is assumed to be viscous (specified by γ). The tension in the string (assumed constant along the string) is T.

The equation of motion for the string and the appropriate boundary conditions are,

$$\frac{\partial^2 u(z,t)}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 u(z,t)}{\partial z^2} + \frac{\gamma}{\rho}\frac{\partial u(z,t)}{\partial t} = 0 \,, \tag{1}$$

$$-T\frac{\partial u(0,t)}{\partial z} = g(t) \; ; \; T\frac{\partial u(l,t)}{\partial z} = r(t) \,. \tag{2}$$

Here $r(t)$ is the force exerted *by* the bob *on* the string and is the unknown force that needs to be computed in order to solve for the bob's motion. Without loss of generality, assume the inital position and velocity of the string to be zero at all points :

$$u(z,0) = 0 \; ; \; \dot{u}(z,0) = 0 \,. \tag{3}$$

The velocity, a, of transverse waves is given by,

$$a = \sqrt{\frac{T}{\rho}} \,. \tag{4}$$

We can formally obtain a solution for $u(z,t)$ by using the Green's function $G(z,\xi,t,\tau)$ and the standardising function $w(z,t)$ for the above system of equations (see Appendix A),

$$u(z,t) = \int_0^t d\tau \int_0^l d\xi \, G(z,\xi,t,\tau) w(\xi,\tau) \,, \tag{5}$$

$$w(z,t) = -a^2\delta(z)\frac{(-g(t))}{T} + a^2\delta(z-l)\frac{r(t)}{T} \,. \tag{6}$$

Thus, we obtain,

$$u(z,t) = \frac{a^2}{T}\int_0^t d\tau \, G(z,0,t,\tau) g(\tau) + \frac{a^2}{T}\int_0^t d\tau G(z,l,t,\tau) r(\tau) \,. \tag{7}$$

Now, the wave equation is time translation invariant. Therefore, $G(z,\xi,t,\tau) = G(z,\xi,t-\tau)$. Thus it makes sense to talk about the Laplace transform of the Green's function here.

Let the Laplace transforms of $G(z,\xi,t-\tau)$ with respect to the time argument be $W(z,\xi,s)$. Let the Laplace transform of $u(z,t)$, $g(t)$, $r(t)$ and $f(t)$ be $U(z,s)$, $G(s)$, $R(s)$ and $F(s)$ respectively. From Eq. (7), we obtain

$$U(z,s) = \frac{a^2}{T}W(z,0,s)G(s) + \frac{a^2}{T}W(z,l,s)R(s) \ . \tag{8}$$

Under the initial conditions in Eq. (3), the Laplace transform of $\ddot{u}(z,t)$ is given by $s^2U(z,s)$.

Now comes the crucial step in the formalism. Since the bob and string are attached at all times, the bob and the end of the string attached to it *must* have the same acceleration. Let the transverse displacement of the bob be $x_b(t)$. Then,

$$m\ddot{x}_b = f(t) - r(t) \ . \tag{9}$$

Equating the acceleration of the bob and the string at $z = l$, we obtain in the Laplace domain,

$$s^2\left[\frac{a^2}{T}W(l,0,s)G(s) + \frac{a^2}{T}W(l,l,s)R(s)\right] = \frac{F(s) - R(s)}{m} \ . \tag{10}$$

Eq. (10) can be solved for $R(s)$ yielding,

$$R(s) = \frac{F(s)/m - s^2a^2W(l,0,s)G(s)/T}{1/m + s^2a^2W(l,l,s)/T} \ . \tag{11}$$

It is possible to get closed form expressions for $W(l,0,s)$ and $W(l,l,s)$. From a standard table [2], we get

$$W(l,0,s) = \frac{1}{a^2s'\sinh(s'l)} \ , \tag{12}$$

$$W(l,l,s) = \frac{1}{a^2s'\tanh(s'l)} \ , \tag{13}$$

$$s' = \frac{1}{a}\sqrt{s^2 + 2s\gamma/\rho} \ . \tag{14}$$

$$\tag{15}$$

One would expect that the transfer functions obtained above would include the pendulum mode. However, this is not so because that solution is obtained for a different set of boundary conditions where the displacement of the string at $z = 0$ is forced to be zero (or a specified function of time).

Let the motion of the string at the suspension point be a given function $x_s(t)$. Then the force $g(t)$ must be such that the displacement $u(0,t) = x_s(t)$. This implies that (using Eq. (8)),

$$G(s) = \frac{TX_s(s)/a^2 - W(0,l,s)R(s)}{W(0,0,s)} \ , \tag{16}$$

where $X_s(s)$ is the Laplace transform of $z_s(t)$. From the symmetry of Green's function,

$$W(0,l,s) = W(l,0,s) \ , \tag{17}$$

and it can be shown that, in the present case,

$$W(l,l,s) = W(0,0,s) \ . \tag{18}$$

Substituting for $G(s)$ in Eq. (10) and using the above relations, we get,

$$R(s) = \frac{F(s) - ms^2X_s(s)W(l,0,s)/W(0,0,s)}{1 + ms^2a^2\left[W(l,l,s) - W(l,0,s)W(0,l,s)/W(0,0,s)\right]/T} \ , \tag{19}$$

$$= \frac{F(s) - ms^2X_s(s)/\cosh(s'l)}{1 + ms^2\tanh(ls')/(Ts')} \ , \tag{20}$$

$$X_b(s) = F(s)\frac{\tanh(ls')/(Ts')}{1 + ms^2\tanh(ls')/(Ts')} + X_s(s)\frac{1/\cosh(s'l)}{1 + ms^2\tanh(ls')/(Ts')} \ , \tag{21}$$

where $X_b(s)$ is the Laplace transform of the bob's displacement.

As a check[†] on the above analysis, let $F(t) = 0$ and $\gamma = 0$ ($\Rightarrow s' = s/a$). In the Fourier domain, the transfer function becomes[‡] for $a \to \infty$,

$$\tilde{x}_b(\omega) = \frac{1}{1 - \omega^2 l/g} \tilde{x}_s(\omega) \tag{22}$$

The transfer function above has a pole at $\sqrt{g/l}$ which is precisely the pendulum frequency.

1. Summary of steps involved in the above calculation

Step 1: Obtain the equations of motion of each element of the mechanical system. These are Eqs. (1), (2) and (9). Impose appropriate boundary and initial conditions.

Step 2: Formally solve the equations in the Laplace domain. This was done for the string in Eq. (8) and is a trivial expression for the pendulum bob which is treated as a point mass here.

Step 3: In the Laplace domain, equate the acceleration of the string and the bob at their attachment point. This was done in Eq. (10).

Step 4: Solve for the internal force.

The internal force was then used to obtain the motion of the bob. In the rest of the report we follow the same basic steps for every system that we consider.

2. Results

In Fig. 2, we have plotted the modulus and phase of the transfer function which takes $X_s(s)$ to $X_b(s)$ (see Eq. (21)). The modulus and phase of the transfer function that takes $F(s)$ to $X_b(s)$ is plotted in Fig. 3. In the latter, the violin mode resonances appear highly suppressed. A zoomed in view near the first mode is shown in Fig 4. The parameters used for the wire and the bob are the same as used in Cella's working note[§],

$$\begin{aligned}
&m = 223 \text{ Kg} , \\
&\rho = \mu\pi r^2 \ , \mu \text{ is the density of wire material and } r \text{ is the wire radius.} \\
&\mu = 8.0 \times 10^3 \ \ \text{Kg/m}^3 , \\
&r = 2.0 \times 10^{-3} \text{ m} , \\
&l = 1.0 \text{ m} .
\end{aligned}$$

[†]**NOTE :** In the present example, it is the wire which is damped *not* the mass. So, a direct comparison with the usual case of a massless, undamped wire and damped mass is possible only when the damping is set to zero.

[‡]The transfer function, being causal in the time domain, is zero for $t < 0$. Hence, the transfer function in the Fourier domain is obtained by $s \to i\omega$.

[§]Received around the middle of November 1998. See Section 3.3 of that note.

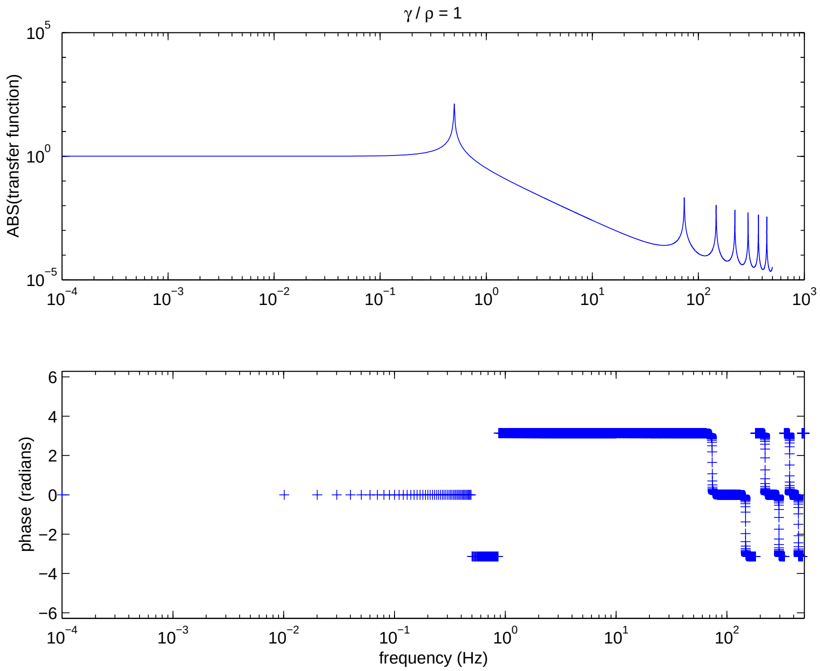

FIG. 2. The modulus of the $X_s(s)$ to $X_b(s)$ transfer function for $\gamma/\rho = 1.0$.

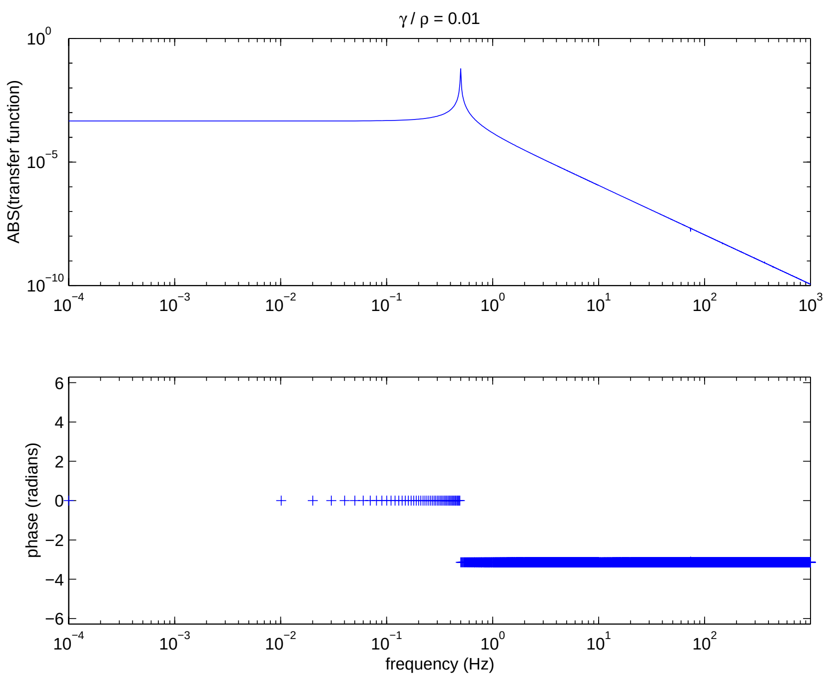

FIG. 3. The modulus of the $F(s)$ to $X_b(s)$ transfer function for $\gamma/\rho = 0.01$.

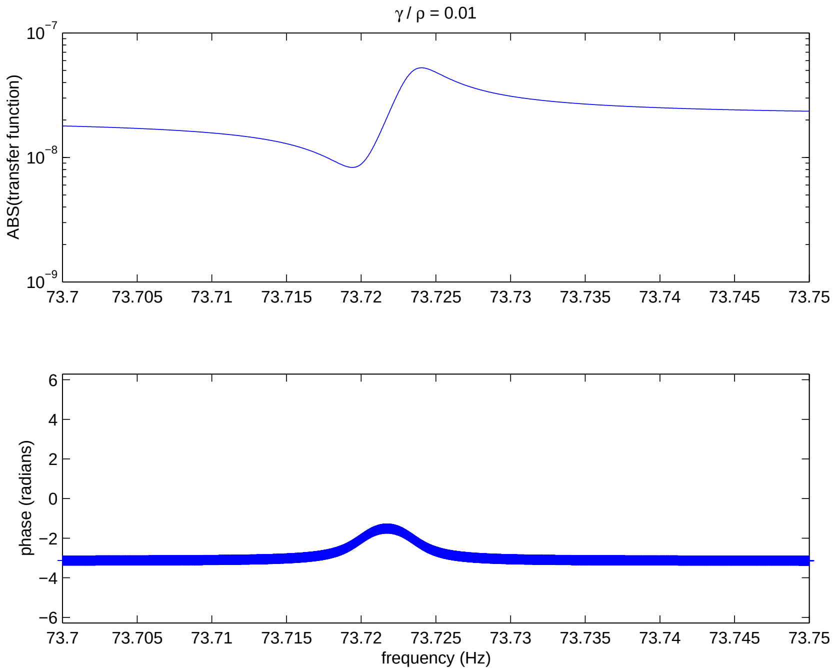

FIG. 4. The modulus of the $F(s)$ to $X_b(s)$ transfer function around the first violin mode for $\gamma/\rho = 0.01$.

B. Simple pendulums in series

We will now consider a more complicated system which will illustrate the method that will be followed for the SEI/SUS model. Consider the system shown in Fig. 5. We follow the same basic steps as summarised in Section II A 1.

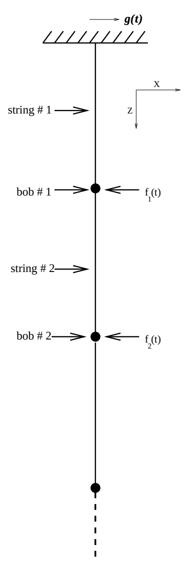

FIG. 5. Simple pendulums in series.

Step 1 : Obtaining equations of motion, boundary and initial conditions for the elements in the system.

The equation of motion of the j^{th} string and its boundary conditions are given by,

$$\frac{\partial^2 u_j(z,t)}{\partial t^2} - \frac{T_j}{\rho_j}\frac{\partial^2 u_j(z,t)}{\partial z^2} + \frac{\gamma_j}{\rho_j}\frac{\partial u_j(z,t)}{\partial t} = 0 \ , \tag{23}$$

$$-T_j\frac{\partial u_j(t_j,t)}{\partial z} = r_{j-1,j}(t) \ ; \ T_j\frac{\partial u(b_j,t)}{\partial z} = r_{j,j}(t) \ , \tag{24}$$

where $r_{i,j}(t)$ is the reaction *of* the i^{th} bob *on* the j^{th} string, $r_{01}(t)$ is understood to be the same as $g(t)$ and t_j denotes the z coordinate of the top of the j^{th} string while b_j denotes the same for the bottom. The equation of motion of the k^{th} bob is,

$$m\ddot{x}_k = f_k(t) - r_{k,k}(t) - r_{k,k+1}(t) \ . \tag{25}$$

The initial positions and velocities are assumed to be zero.

Step 2: Formal solution of the equations of motions in the Laplace domain.

As in the case of the simple pendulum considered earlier, $u_j(z,t)$ can be formally solved in terms of the reaction forces. For the j^{th} string we get,

$$U_j(z,s) = \frac{a_j^2}{T_j}W_j(z-t_j,0,s)R_{j-1,j}(s) + \frac{a_j^2}{T_j}W_j(z-t_j,l_j,s)R_{j,j}(s) \ , \tag{26}$$

where $W_j(z,\xi,s)$ is the same as $W(z,\xi,s)$ defined in the case of the simple pendulum earlier but with the sound speed a and γ/ρ for the j^{th} wire.

Step 3: Equating the accelerations of different elements at their attachment points

Equate the acceleration of the k^{th} bob with those of the end points of each of the two strings attached to it. Thus, we get two acceleration balance equations for the k^{th} bob,

$$s^2U_k(b_k,s) = \frac{F_k}{m_k} - \frac{R_{k,k}+R_{k,k+1}}{m_k} \ , \tag{27}$$

$$s^2U_{k+1}(t_{k+1},s) = \frac{F_k}{m_k} - \frac{R_{k,k}+R_{k,k+1}}{m_k} \ . \tag{28}$$

which yield,

$$\frac{s^2a_k^2}{T_k}W_k(l_k,0,s)R_{k-1,k}(s) + \left[\frac{s^2a_k^2}{T_k}W_k(l_k,l_k,s) + \frac{1}{m_k}\right]R_{k,k}(s) + \frac{R_{k,k+1}(s)}{m_k} = \frac{F_k(s)}{m_k} \ , \tag{29}$$

$$\frac{R_{k,k}(s)}{m_k} + \left[\frac{s^2a_{k+1}^2}{T_{k+1}}W_{k+1}(l_{k+1},l_{k+1},s) + \frac{1}{m_k}\right]R_{k,k+1}(s) + \frac{s^2a_{k+1}^2}{T_{k+1}}W_{k+1}(l_{k+1},l_{k+1},s)R_{k+1,k+1}(s) = \frac{F_k(s)}{m_k} \ . \tag{30}$$

For $k=N$, we have only the first of the above equations with $R_{k,k+1}=0$.

For $k=1$, Eq. (29) needs to be modified. For an externally specified force $G(s)$, the equation becomes,

$$\left[\frac{s^2a_k^2}{T_k}W_k(l_k,l_k,s) + \frac{1}{m_k}\right]R_{k,k}(s) + \frac{R_{k,k+1}(s)}{m_k} = \frac{F_k}{m_k} - \frac{s^2a_k^2}{T_k}W_k(l_k,0,s)G(s) \ . \tag{31}$$

While for an externally specified displacement, $X_s(s)$, of the topmost suspension point the equation becomes,

$$\left[\frac{s^2a_k^2}{T_k}\left(W_k(l_k,l_k,s) - \frac{W_k^2(l_k,0,s)}{W_k(l_k,l_k,s)}\right) + \frac{1}{m_k}\right]R_{k,k}(s) + \frac{R_{k,k+1}(s)}{m_k} = \frac{F_k}{m_k} - s^2\frac{W_k(l_k,0,s)}{W_k(l_k,l_k,s)}X_s(s) \ . \tag{32}$$

Step 4 : Solve for the internal forces

Collecting all the acceleration balance equations together gives a matrix equation of the form,

$$\mathbf{T}\mathbf{R} = \mathbf{A}_{d+f}\mathbf{X} \ , \tag{33}$$

$$\mathbf{R} = [R_{1,1}(s),\ \ldots,\ R_{j,j-1}(s),\ R_{j,j}(s),\ R_{j,j+1}(s),\ \ldots,\ R_{N,N}(s)]^T \ , \tag{34}$$

$$\mathbf{X} = [X_s(s), F_1(s), F_2(s),\ F_3(s),\ \ldots, F_N(s)]^T \ , \tag{35}$$

with $X_s(s)$ replaced by $G(s)$ when appropriate. The matrix $\mathbf{A}_{d+f}$ is given by,

$$\mathbf{A}_{d+f} = \left.\overbrace{\begin{pmatrix} -s^2\frac{W_1(l_1,0,s)}{W_1(l_1,l_1,s)} & \frac{1}{m_1} & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{m_1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{m_2} & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{m_2} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \frac{1}{m_3} & \ldots & 0 \\ 0 & 0 & 0 & \frac{1}{m_3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{m_{N-1}} & 0 \\ 0 & 0 & 0 & \ldots & \frac{1}{m_{N-1}} & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{m_N} \end{pmatrix}}^{N+1}\right\} 2N-1 \ . \tag{36}$$

$\mathbf{T}$ has dimensions $(2N-1)\times(2N-1)$ with the first row (in the case of externally specified displacement) given by,

$$\left[\frac{s^2 a_1^2}{T_1}\left(W_1(l_1,l_1,s) - \frac{W_1^2(l_1,0,s)}{W_1(l_1,l_1,s)}\right) + \frac{1}{m_1}, \frac{1}{m_1}, 0, 0, \ldots, 0\right] \ , \tag{37}$$

row $2k$, $k = 1, \ldots, N-1$, given by

$$\left[\overbrace{0,0,\ldots,0}^{2k-2}, \frac{1}{m_k}, \frac{s^2 a_{k+1}^2}{T_{k+1}} W_{k+1}(l_{k+1}, l_{k+1}, s) + \frac{1}{m_k}, \frac{s^2 a_{k+1}^2}{T_{k+1}} W_{k+1}(l_{k+1}, 0, s), 0, 0, \ldots, 0\right] \ , \tag{38}$$

row $2k-1$, $k = 2, \ldots, N-1$, given by

$$\left[\overbrace{0,0,\ldots,0}^{2k-3}, \frac{s^2 a_k}{T_k} W_k(l_k, 0, s), \frac{s^2 a_k^2}{T_k} W_k(l_k, l_k, s) + \frac{1}{m_k}, \frac{1}{m_k}, 0, 0, \ldots, 0\right] \ , \tag{39}$$

and, finally, the last row is given by

$$\left[\overbrace{0,0,\ldots,0}^{2N-3}, \frac{s^2 a_N}{T_N} W_N(l_N, 0, s), \frac{s^2 a_N^2}{T_N} W_k(l_N, l_N, s) + \frac{1}{m_N}\right] \ . \tag{40}$$

Given $\mathbf{X}$, one can solve for $\mathbf{R}$,

$$\mathbf{R} = \mathbf{T}^{-1}\mathbf{A}_{d+f}\mathbf{X} \ . \tag{41}$$

Using the reaction forces, one can then solve for the motion of each bob in the Laplace domain.

In Fig. (6) to Fig. (10) we show the transfer functions from $x_s(t)$, the motion of the top suspension point, to the motion of bob# 1 and # 2 in a double pendulum. The parameters used are :
$m_1 = 142.0$ Kg, $m_2 = 898.0$ Kg, $l_1 = l_2 = 1.0$ m, radius of wires $= 1.75 \times 10^{-3}$ m, wire density $= 8 \times 10^3$ kg/m^3. The value of γ/ρ used and the bob number is indicated at the top of each figure. "Transfer function" is abbreviated

as "t.f.". Note that for $\gamma = 0$, the transfer functions should be real. We have plotted the ratio of the imaginary to real parts of the transfer functions to show the error incurred in practice.

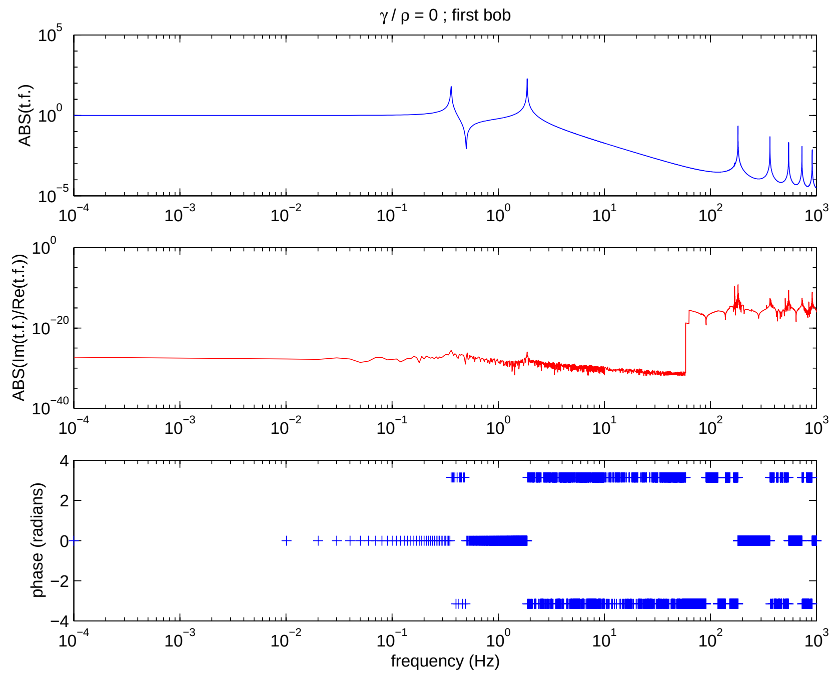

FIG. 6. Transfer function for two stage pendulum.

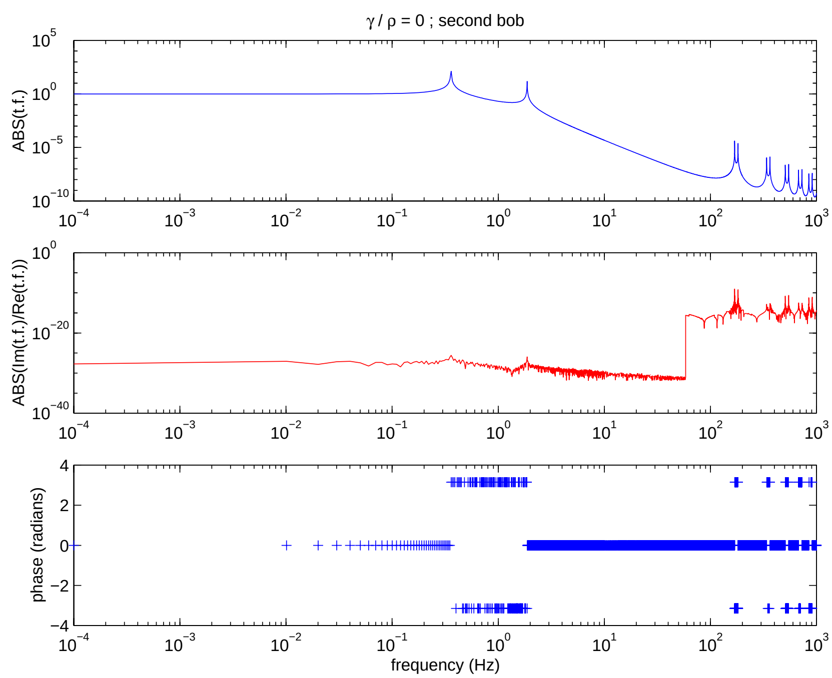

FIG. 7. Transfer function for two stage pendulum.

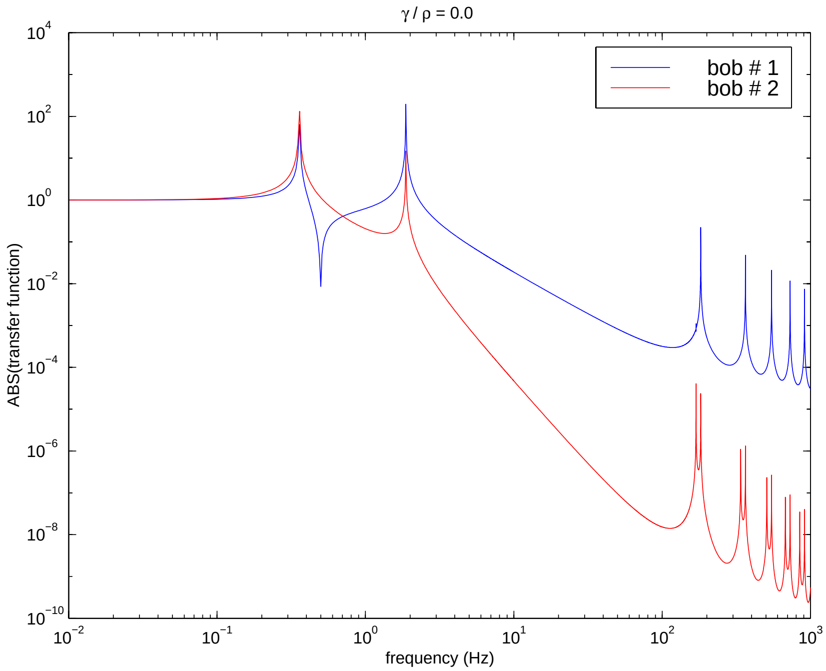

FIG. 8. Transfer functions for two stage pendulum.

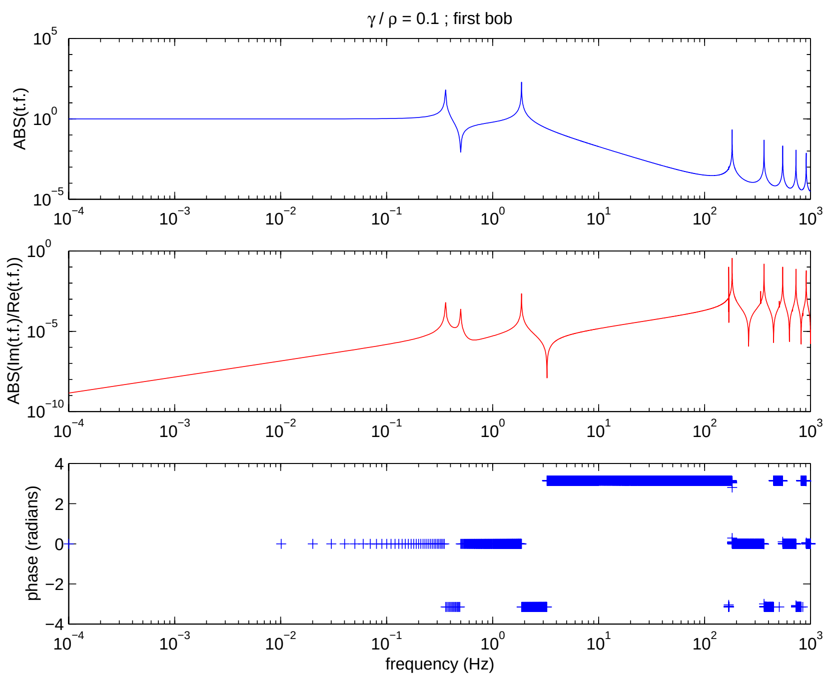

FIG. 9. Transfer function for first stage of a two stage chain.

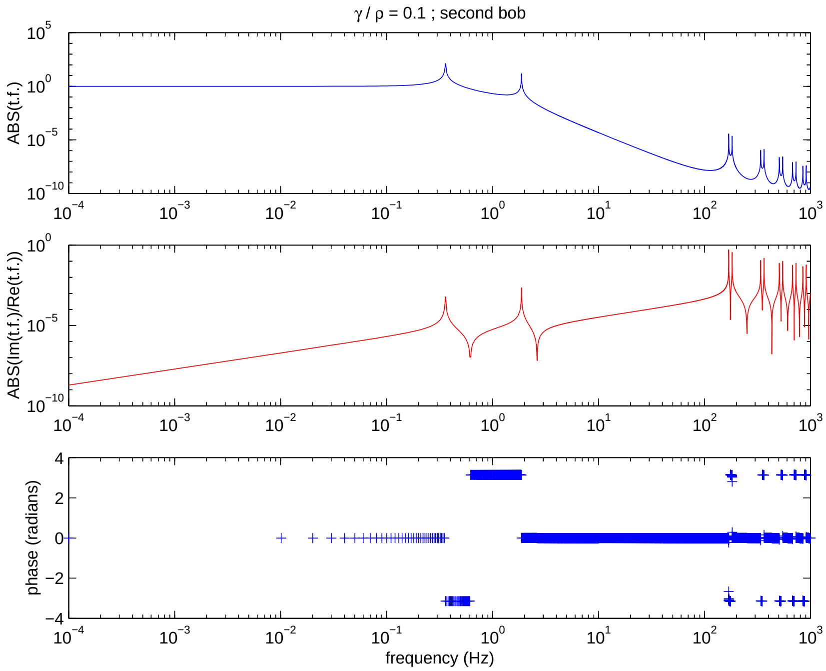

FIG. 10. Transfer function for second stage of a two stage chain.

Fig. (11) to Fig. (18) show transfer function modulii and phase for all the bobs in a seven stage chain. The parameters used are listed below** (wire density $= 8 \times 10^{-3}$ Kg/m^3).

stage #	M (kg)	L (m)	$d = 2r$ (mm)
1	142	1	3.5
2	164	1	3.5
3	135	1	3
4	132	1	3
5	128	1	3
6	116	1	2
7	163	1	2
8	30	1	1
9	30	1	.2

The transfer sunction is negative in sign in the segments shown in cyan. **Note :** There seem to be discrepancies between the high frequency region of the seventh stage (see Fig 18), and the plot given in G. Cella's working note.

**Taken from G. Cella's working note.

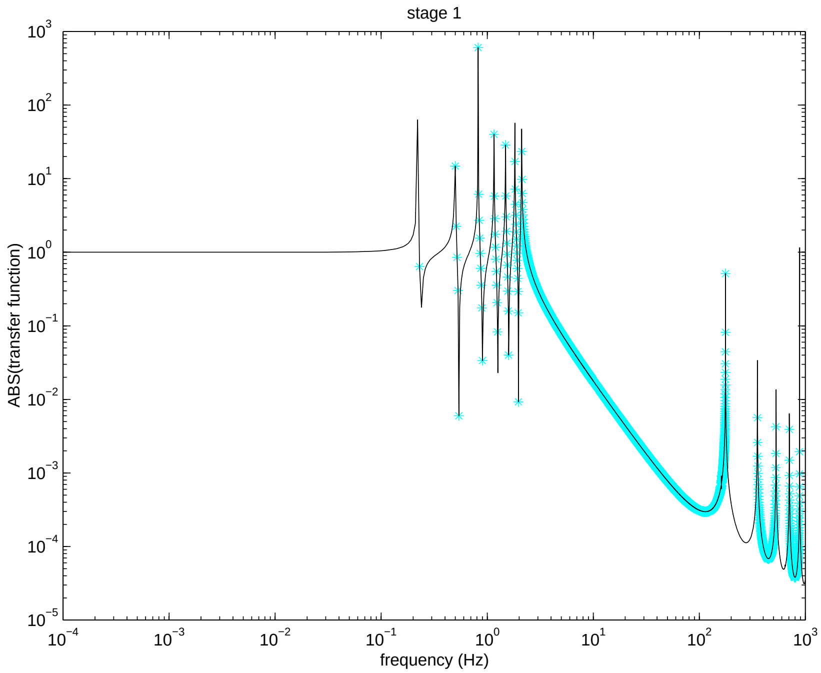
stage 1
ABS(transfer function)
frequency (Hz)

FIG. 11.

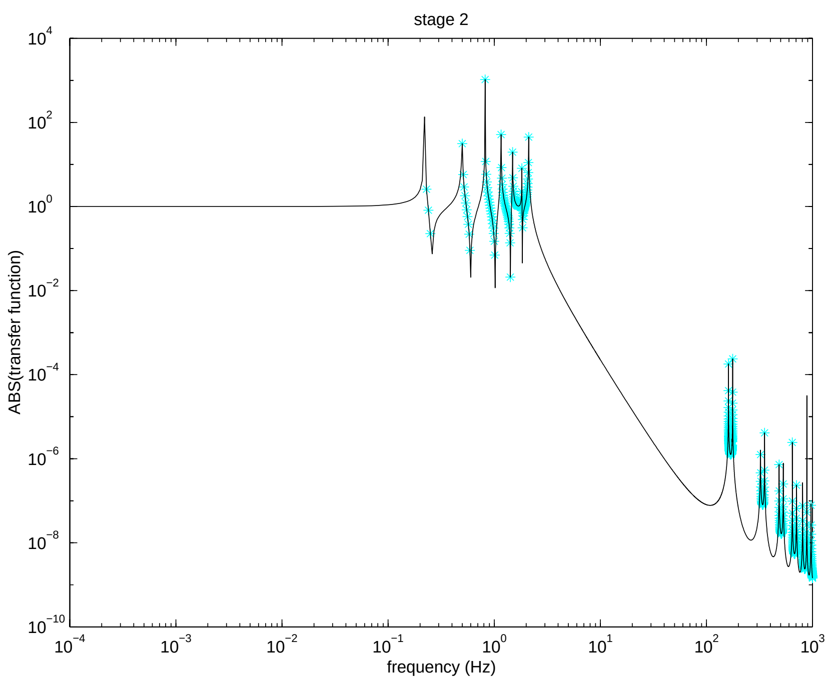
stage 2
ABS(transfer function)
frequency (Hz)

FIG. 12.

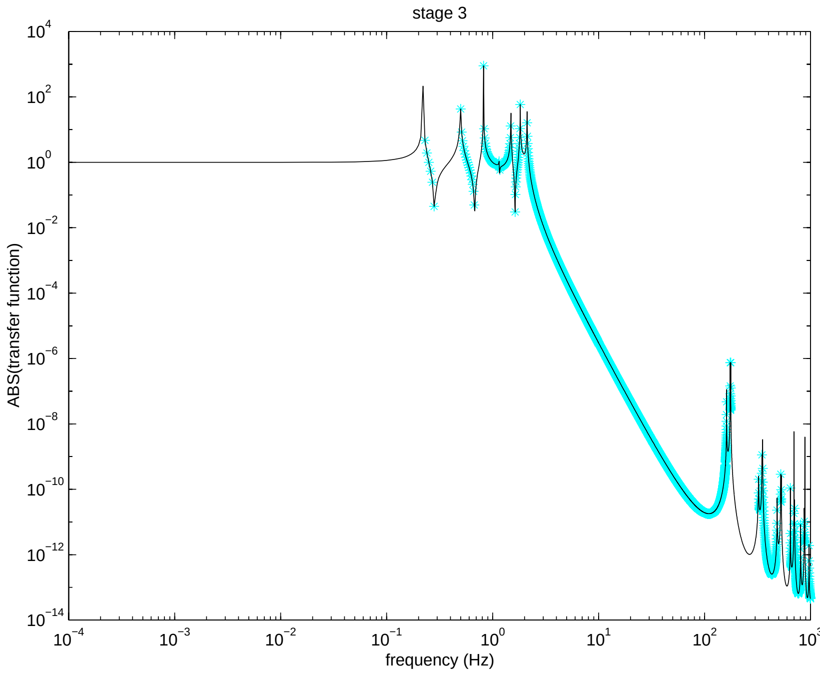
stage 3
ABS(transfer function)
frequency (Hz)
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

FIG. 13.

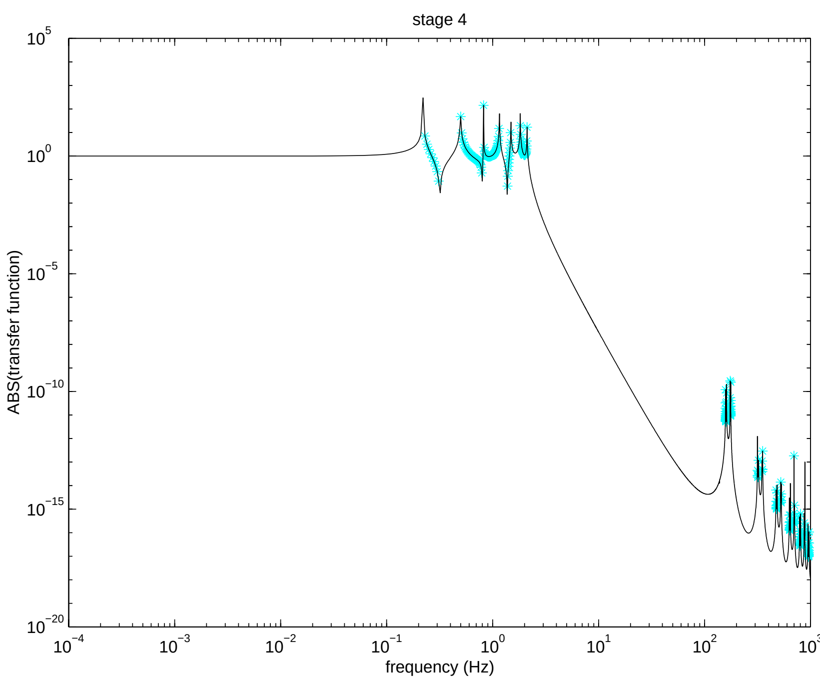
stage 4
ABS(transfer function)
frequency (Hz)
10^{5}
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

FIG. 14.

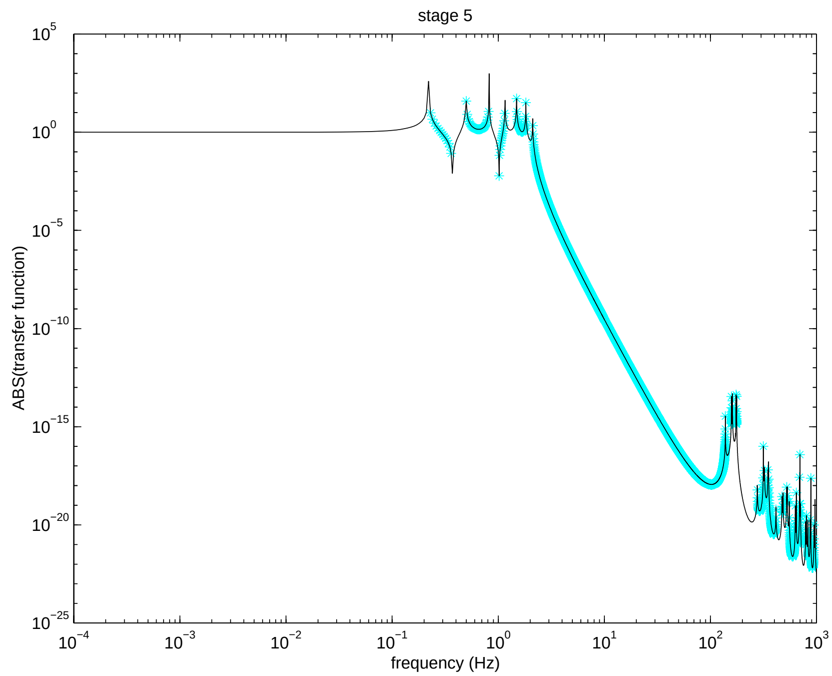
stage 5
ABS(transfer function)
frequency (Hz)

FIG. 15.

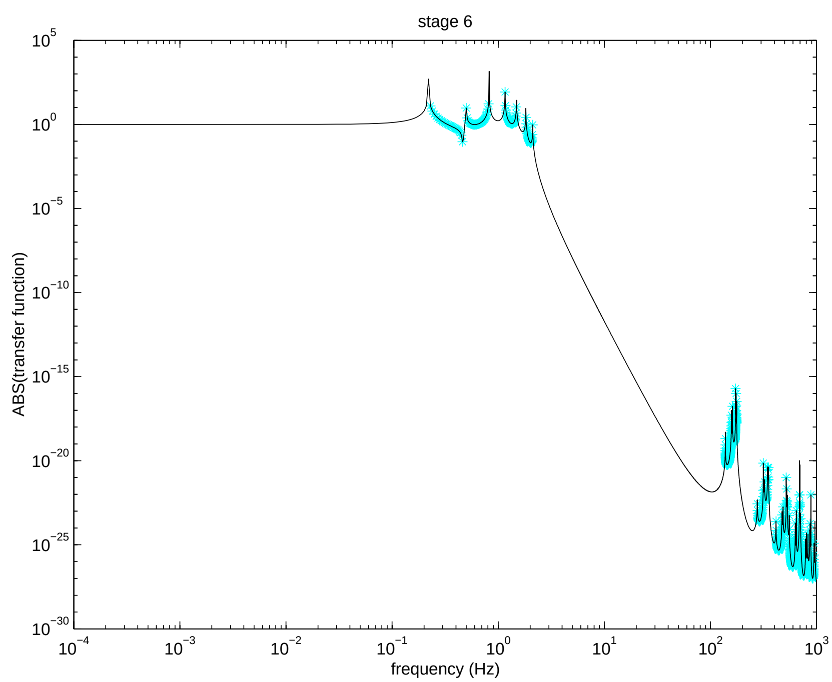
stage 6
ABS(transfer function)
frequency (Hz)

FIG. 16.

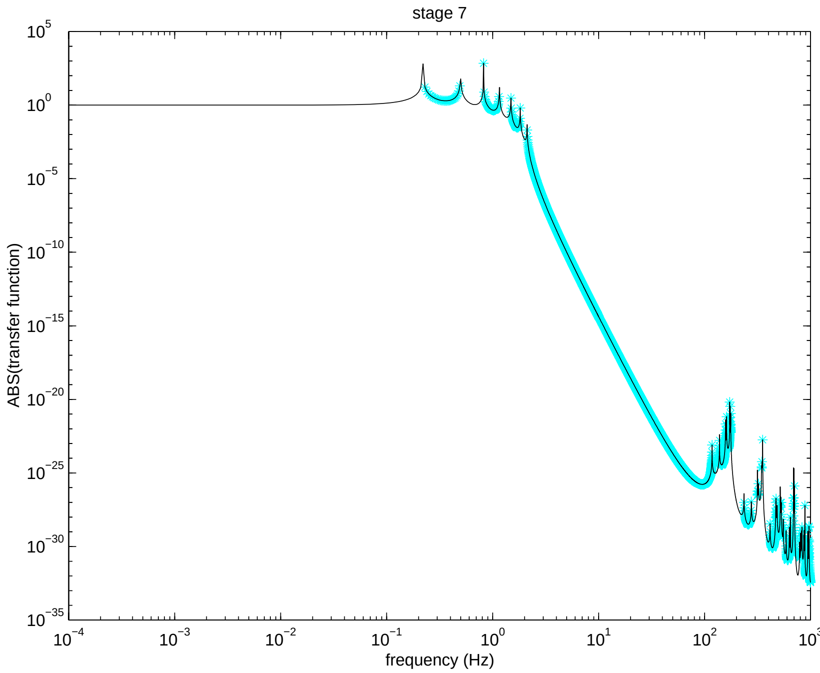

FIG. 17.

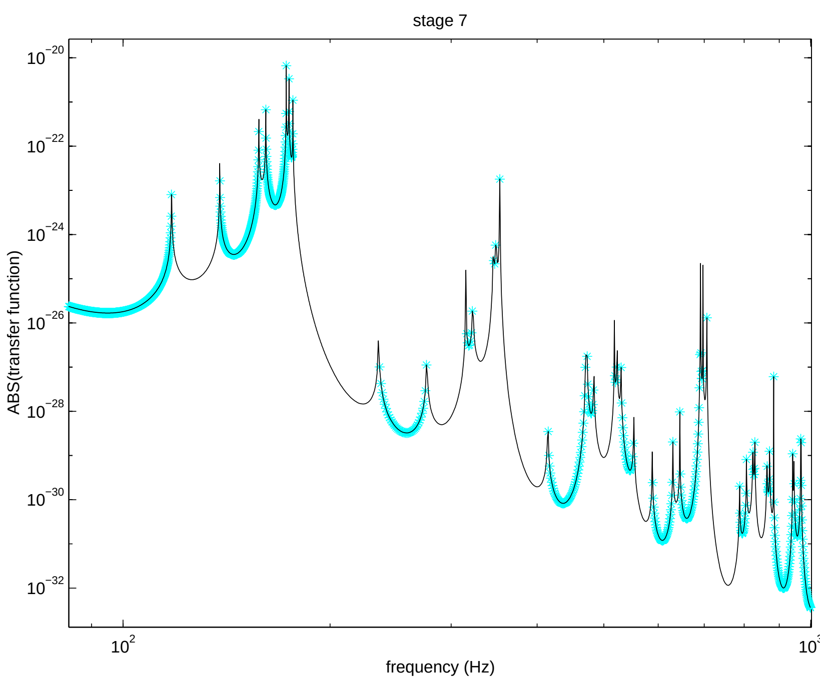

FIG. 18. Zoomed in view of the last stage transfer function.

1. The structure of $\mathbf{T}$

Pictorially, the matrix $\mathbf{T}$ would look somewhat like the schematic in Fig. 19. As one would expect, most of the matrix is zero.

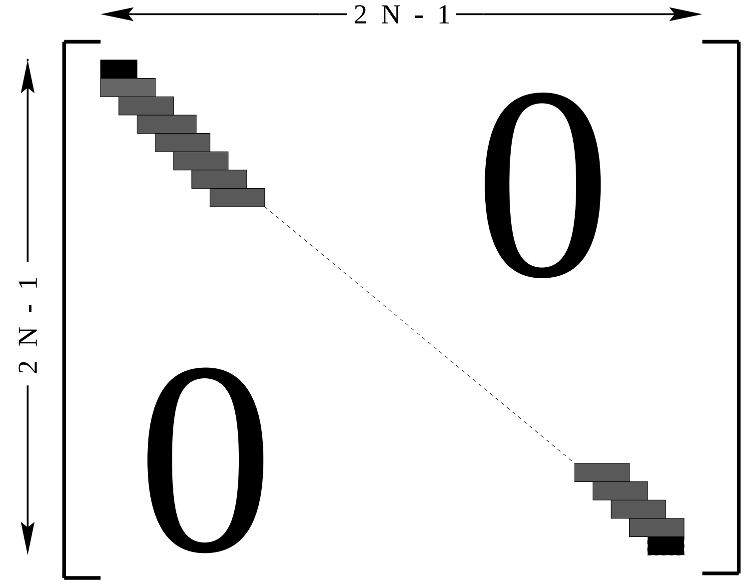

FIG. 19. The structure of $\mathbf{T}$. Each gray box is a 1×3 matrix. A black box has dimensions 1×2.

III. AN IMPLEMENTATION SCHEME FOR THE FORMALISM

For a general mechanical system, essentially the same kind of expression as in Eq. (33) would be obtained. the matrix $\mathbf{T}$ would, in general, be a sparse matrix. This is because the acceleration balance equations for any one point of attachement would involve only a *proper* subset of the full set $\mathcal{R}$ of action-reaction forces. Hence, the row of $\mathbf{T}$ that corresponds to this point of attachement would have all elements zero apart from the ones that pick up the proper subset of $\mathcal{R}$ that is involved. However, it is difficult to say whether this sparse matrix would have a nice pattern as in the previous example. If it does have a special pattern, there exist special techniques for inversion of such matrices that are quite efficient.

For any form of $\mathbf{T}$, $\mathbf{T}$ can be inverted either using a symbolic algebra program or numerically for each frequency that is required. Each row of $\mathbf{T}^{-1}$ would be a collection of transfer functions which can be converted into digital filters. Thus, all the action-reaction forces would finally be obtained as digital filters acting on the given external forces or displacements in the time domain.

It should be noted that for linear systems all that is required to describe an element is the set of Green's functions associated with its equation of motion and boundary conditions. These can be computed before hand, either experimentally or theoretically, and stored as separate modules. Once a system design has been specified, these modules can be connected together to yield $\mathbf{T}$.

We now describe an algorithm that can be used for setting up the matrix $\mathbf{T}$ in Eq. (33). It should be noted that many alternatives to this algorithm are possible. The steps are as follows.

(1) Fix a global cartesian reference frame (GRF). This frame is global in the sense that it describes an entire SEI/SUS system at any one location in the interferometer. There may exist a frame that describes the entire interferometer itself. Such a frame would be related to the GRF frame via a fixed rotation matrix.

(2) Consider the state of mechanical equilibrium for the entire system. In this state record the locations of all the points of attachments between elements and also, for each element, the rotational transformation needed to go from the GRF frame to the *preferred frame* (see below) of that element.

(3) Each element has a preferred frame (cartesian) in which its Green's functions are expressible in the simplest manner. For instance, for a wire the preferred frame is one where the wire lies along one of the axes. (Say, the Z axis as in the previous examples.) Let the rotational transformation from the GRF frame to the preferred frame for the i^{th} element be denoted by $\mathcal{R}_i$.

For every element, transform the GRF frame components of the external forces acting on it into its preferred frame. The preferred frame Green's functions would be computed and stored beforehand. Formally solve for the acceleration of all the points of attachments of the element. Transform the acceleration components back into the GRF frame.

(4) So now we have the GRF frame components of the acceleration of each point of attachment. Apply the equality of acceleration condition for each point of attachment and transfer terms containing the action-reaction force terms to the LHS and terms containing driving forces to the RHS.
(5) The set of algebraic equations so obtained then furnishes the matrix **T**.

We will now compute the preferred frame formal solutions (**Step 1** and **Step 2** of Section II B) for accelerations of points of attachments for several elements out of which a model of the LIGO SEI/SUS system may be built.

A. A straight wire

1. One transverse polarization, no longitudinal mode, viscous damping

This case has already been discussed (see Section II A). Here the relevant expression is the one given in Eq. (8) which can be used to obtain the accelerations at the two points of attachements.

2. One transverse polarization, no longitudinal mode, internal damping

The equation of motion of the wire [3] is,

$$\frac{\partial^2 u(z,t)}{\partial t^2} - a^2 \frac{\partial^2 u(z,t)}{\partial z^2} + \frac{EI}{T}\frac{\partial^4 u(z,t)}{\partial z^4} = 0 \ , \tag{42}$$

where E is the Young's modulus and I is the area moment of inertia. with boundary conditions,

$$EI\frac{\partial^3 u(0,t)}{\partial z^3} - T\frac{\partial u(0,t)}{\partial z} = g(t) \ , \tag{43}$$

$$EI\frac{\partial^3 u(l,t)}{\partial z^3} - T\frac{\partial u(l,t)}{\partial z} = -r(t) \ . \tag{44}$$

We assume the initial conditions $u(z,0) = \dot{u}(z,0) = 0$. To introduce damping in the wire, E will be made complex in the final transfer function.

3. Two transverse polarizations, no longitudinal mode and viscous damping

$$\frac{\partial^2 u_x}{\partial t^2} - a_x^2 \frac{\partial^2 u_x}{\partial z^2} + \frac{\gamma_x}{\rho}\frac{\partial u_x}{\partial t} = 0 \ , \tag{45}$$

$$-T\frac{\partial u_x(0,t)}{\partial z} = g_x(t) \ ; \ T\frac{\partial u_x(l,t)}{\partial z} = r_x(t) \ , \tag{46}$$

$$\frac{\partial^2 u_y}{\partial t^2} - a_y^2 \frac{\partial^2 u_y}{\partial z^2} + \frac{\gamma_y}{\rho}\frac{\partial u_y}{\partial t} = 0 \tag{47}$$

$$-T\frac{\partial u_y(0,t)}{\partial z} = g_y(t) \ ; \ T\frac{\partial u_y(l,t)}{\partial z} = r_y(t) \ , \tag{48}$$

The required values of the Green's function for each degree of freedom have already been provided in Eq. (12) and Eq. (13). However, γ/ρ should be replaced by γ_x/ρ or γ_y/ρ and a should be replaced by a_x or a_y as appropriate. We attach the appropriate subscript, 'x' or 'y', to distinguish between the two Green's functions and their Laplace transforms. Thus, from Eq. (8), we get,

$$U_x(z,s) = \frac{a_x^2}{T}W_x(z,0,s)G_x(s) + \frac{a_x^2}{T}W_x(z,l,s)R_x(s) \ , \tag{49}$$

$$U_y(z,s) = \frac{a_y^2}{T}W_y(z,0,s)G_y(s) + \frac{a_y^2}{T}W_y(z,l,s)R_y(s) \ . \tag{50}$$

Using the above expressions we can write down the acceleration at the end points.

4. Two transverse polarizations, longitudinal mode and viscous damping

As discussed above, in the small deformation approximation, the three degrees of freedom can be decoupled. Hence, the equations of motion for the transverse polarizations are the same as before (see Eq. (49) and Eq. (50)). For the longitudinal mode, we have

$$\frac{\partial^2 u_z}{\partial t^2} - a_z^2 \frac{\partial^2 u_z}{\partial z^2} + \frac{\gamma_z}{\rho} \frac{\partial u_z}{\partial t} = 0 \, , \tag{51}$$

$$-Y \frac{\partial u_z(0,t)}{\partial z} = g_z(t) \; ; \; Y \frac{\partial u_z(l,t)}{\partial z} = r_z(t) \tag{52}$$

where $Y = \mathcal{Y} \pi r_{\rm wire}^2$, $r_{\rm wire}$ being the radius of the wire and $\mathcal{Y}$ is its Young's modulus. In the Laplace domain,

$$U_z(z,s) = \frac{a_z^2}{Y} W_z(z,0,s) G_z(s) + \frac{a_z^2}{Y} W_z(z,l,s) R_z(s) \, . \tag{53}$$

5. Two transverse polarizations, longitudinal mode and internal damping

B. A rigid body

We consider[††] the case of small angular motions for which Euler's equations for the motion of a rigid body can be linearised. Euler's equations evolve the angles that describe the rotation from a fixed orientation frame attached to the body's centre of mass, say S, to the body axes that rotate with the body (say, frame B). The angles used here are defined in Appendix B. We begin with a description of the various reference frames that will be needed here apart from the GRF.

Let the body frame B be attached to the centre of mass (CM) of the body and oriented along its principal axes. Let I_1, I_2 and I_3 be the principal moments of inertia. Determine the orientation of B when the system is in mechanical equilibrium. We have already mentioned that S is attached to the CM. However, it has a fixed orientation. We orient our space axes to match the body axes *at equilibrium*. Let the rotational transform *from* GRF *to* S be denoted by $\mathcal{R}$ (which is a constant matrix).

Let the forces on the body be $(\overline{T}_1 + \overline{r}_1, \overline{T}_2 + \overline{r}_2, \ldots, \overline{T}_n + \overline{r}_n)$, the set of action-reaction forces, and $(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m)$, the set of external forces (such as control forces). Here $\overline{T}_i$ are constants, that is the forces at equilibrium. We assume that the f forces are of the same order as r forces and that they are small, $r_i(t) \ll T_i$. For instance, in the case of a rigid body hung from wires, $\overline{T}_i$ are the tension forces at equilibrium and $r_i(t)$, $f_i(t)$ are the time dependent forces that are small if all displacements are small.

The equation of motion for the centre of mass in the GRF is

$$M \ddot{\overline{\mathbf{x}}}_{\rm cm} = \sum_{i=1}^{m} \overline{f}_i + \sum_{i=1}^{n} \overline{r}_i \, , \tag{54}$$

where M is the body's mass and $\overline{\mathbf{x}}_{\rm cm}$ is the position of the centre of mass.

The rotational transformation from B to S at any instant is denoted here by R_{b2s},

$$R_{b2s} = \mathbf{1} + \epsilon \, , \tag{55}$$

$$\epsilon = \begin{pmatrix} 0 & -\phi & \eta \\ \phi & 0 & -\theta \\ -\eta & \theta & 0 \end{pmatrix} \, . \tag{56}$$

For the derivation and definition of the angles used here, see Appendix B. For any vector $\overline{X}$,

$$\left[\overline{X}\right]_B = R_{b2s}^T \left[\overline{X}\right]_S \, . \tag{57}$$

[††]**Note :** This section has been revised completely.

The components of a vector in the S frame are obtained from the components in the GRF frame as,

$$[\overline{X}]_S = \mathcal{R}\,[\overline{X}]_{\mathrm{GRF}} \; . \tag{58}$$

Note that since ϵ is an antisymmetric matrix,

$$\epsilon^T\,\overline{X} \equiv \overline{e}\times\overline{X} \; , \tag{59}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} := -\begin{pmatrix} \theta \\ \eta \\ \phi \end{pmatrix} \; . \tag{60}$$

Let the r-forces act at points $\overline{p}_1,\ldots,\overline{p}_n$ respectively and let the f-forces act at points $\overline{P}_1,\ldots,\overline{P}_m$, where these position vectors are measured from the CM. If we assume that the points at which forces are applied are fixed with respect to B, the above position vectors would have constant components in B.

Let the sum of all external (both r and f forces) torques be $\overline{N}$. Using Eq. (55), Eq. (57) and Eq. (59) we get

$$\begin{aligned}
[\overline{N}]_B &= \sum_{i=1}^{n}[\overline{p}_i]_B\times\left[R_{b2s}^T\left([\overline{r}_i]_S+[\overline{T}_i]_S\right)\right]+\sum_{i=1}^{m}[\overline{P}_i]_B\times\left(R_{b2s}^T\,[\overline{f}_i]_S\right) \; , \\
&= \mathbf{D}\begin{pmatrix} \theta \\ \eta \\ \phi \end{pmatrix}+\sum_{i=1}^{n}[\overline{p}_i]_B\times[\overline{r}_i]_S+\sum_{i=1}^{m}[\overline{P}_i]_B\times[\overline{f}_i]_S-\mathbf{A}\begin{pmatrix} \theta \\ \eta \\ \phi \end{pmatrix} \; ,
\end{aligned} \tag{61}$$

$$\mathbf{D} = \sum_{i=1}^{n}\begin{pmatrix} 0 & -[\overline{p}_i]_{B,3} & [\overline{p}_i]_{B,2} \\ [\overline{p}_i]_{B,3} & 0 & -[\overline{p}_i]_{B,1} \\ -[\overline{p}_i]_{B,2} & [\overline{p}_i]_{B,1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -[\overline{T}_i]_{S,3} & [\overline{T}_i]_{S,2} \\ [\overline{T}_i]_{S,3} & 0 & -[\overline{T}_i]_{S,1} \\ -[\overline{T}_i]_{S,2} & [\overline{T}_i]_{S,1} & 0 \end{pmatrix} \; , \tag{62}$$

$$\mathbf{A}_{kl} = \sum_{i=1}^{n}\left[\left([\overline{p}_i]_B\cdot[\overline{r}_i]_S\right)\delta_{ij}-[\overline{p}_i]_{B,k}\,[\overline{r}_i]_{S,l}\right]+\sum_{i=1}^{m}\left[\left([\overline{P}_i]_B\cdot[\overline{f}_i]_S\right)\delta_{ij}-[\overline{P}_i]_{B,k}\,[\overline{f}_i]_{S,l}\right] \; , \tag{63}$$

where $[\overline{X}]_{Y,j}$ denotes the j^{th} component of $\overline{X}$ in the frame Y.

We can neglect the last term in Eq. (61) since it is at a lower order in the angles with respect to the other terms. Thus, we get for Euler's equations (see Eq. (B16) – Eq. (B18))

$$\begin{pmatrix} I_1\ddot{\theta} \\ I_2\ddot{\eta} \\ I_3\ddot{\phi} \end{pmatrix}-\mathbf{D}\begin{pmatrix} \tilde{\theta} \\ \tilde{\eta} \\ \tilde{\phi} \end{pmatrix} = \sum_{i=1}^{n}[\overline{p}_i]_B\times[\overline{r}_i]_S+\sum_{i=1}^{m}[\overline{P}_i]_B\times[\overline{f}_i]_S \; . \tag{64}$$

The above equations can be solved in the Laplace domain. Let the Laplace transforms of the angles be denoted by placing a '$\sim$' on top of the corresponding time domain symbol. Then[‡‡],

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\eta} \\ \tilde{\phi} \end{pmatrix} = \mathbf{J}^{-1}(s)\left[\sum_{i=1}^{n}[\overline{p}_i]_B\times[\overline{R}_i(s)]_S+\sum_{i=1}^{m}[\overline{P}_i]_B\times[\overline{F}_i(s)]_S\right] \; . \tag{65}$$

$$\mathbf{J}(s) = s^2\begin{pmatrix} I_1^{-1} & 0 & 0 \\ 0 & I_2^{-1} & 0 \\ 0 & 0 & I_3^{-1} \end{pmatrix}-\mathbf{D} \; . \tag{66}$$

Now, we need to obtain the acceleration of each of the points $\overline{p}_j$. The GRF components of $\overline{p}_j$ are given by,

$$\begin{aligned}
[\overline{p}_j]_{\mathrm{GRF}} &= \mathcal{R}^T R_{b2s}[\overline{p}_j]_B+\overline{x}_{\mathrm{cm}} \; , \\
&= \mathcal{R}^T[\overline{p}_j]_B-\mathcal{R}^T\left[\overline{e}\times[\overline{p}_j]_B\right]+\overline{\mathbf{x}}_{\mathrm{cm}} \; .
\end{aligned} \tag{67}$$

[‡‡]Assuming as before that the initial values of the angles and angular velocities are zero.

In terms of Laplace transforms,

$$\left[\,\overline{p}_j(s)\,\right]_{\text{GRF}} = \frac{1}{s}\mathcal{R}^T[\,\overline{p}_j\,]_B + \mathcal{R}^T\left[\begin{pmatrix}\tilde{\theta}\\ \tilde{\eta}\\ \tilde{\phi}\end{pmatrix}\times[\,\overline{p}_j\,]_B\right] + \overline{X}_{\text{CM}}(s)\ , \tag{68}$$

In the Laplace domain, the acceleration is given by (using Eq. (54) and Eq. (65)),

$$s^2\left[\,\overline{p}_j(s)\,\right]_{\text{GRF}} - s\left[\mathcal{R}^T[\,\overline{p}_j\,]_B \ + \overline{\mathbf{x}}_{\text{cm}}(0)\right] = \sum_{i=1}^{n}\mathcal{R}^T\mathbf{H}_1(s;\,i,j)\left[\,\overline{R}_i(s)\,\right]_S + \sum_{i=1}^{m}\mathcal{R}^T\mathbf{H}_2(s;\,i,j)\left[\,\overline{F}_i(s)\,\right]_S + \frac{1}{M}\left[\sum_{i=1}^{n}\overline{R}_i(s) + \sum_{i=1}^{m}\overline{F}_i(s)\right]\ , \tag{69}$$

$$\mathbf{H}_1(s;\,i,j) = -s^2\begin{pmatrix}0 & -p_{j,3} & p_{j,2}\\ p_{j,3} & 0 & -p_{j,1}\\ -p_{j,2} & p_{j,1} & 0\end{pmatrix}\mathbf{J}^{-1}(s)\begin{pmatrix}0 & -p_{i,3} & p_{i,2}\\ p_{i,3} & 0 & -p_{i,1}\\ -p_{i,2} & p_{i,1} & 0\end{pmatrix}\ , \tag{70}$$

$$\mathbf{H}_2(i,j) = -s^2\begin{pmatrix}0 & -p_{j,3} & p_{j,2}\\ p_{j,3} & 0 & -p_{j,1}\\ -p_{j,2} & p_{j,1} & 0\end{pmatrix}\mathbf{J}^{-1}(s)\begin{pmatrix}0 & -P_{i,3} & P_{i,2}\\ P_{i,3} & 0 & -P_{i,1}\\ -P_{i,2} & P_{i,1} & 0\end{pmatrix}\ , \tag{71}$$

$$\tag{72}$$

where $p_{i,m}$ stands for the m^{th} component of $[\,\overline{p}_i\,]_B$ and similarly for $p_{j,n}$. **Eq. (69) is the key equation for this module.**

IV. LOS MODEL : VISCOUSLY DAMPED SUSPENSION WIRES AND CYLINDRICAL MIRROR

A. The Global reference frame and Equilibrium values

Consider the model shown in Fig. 20 for the LIGO end mirror suspension. Note that instead of a single loop of wire we have modelled the suspension as having two independent wires. The GRF is shown as the set of axes XYZ.

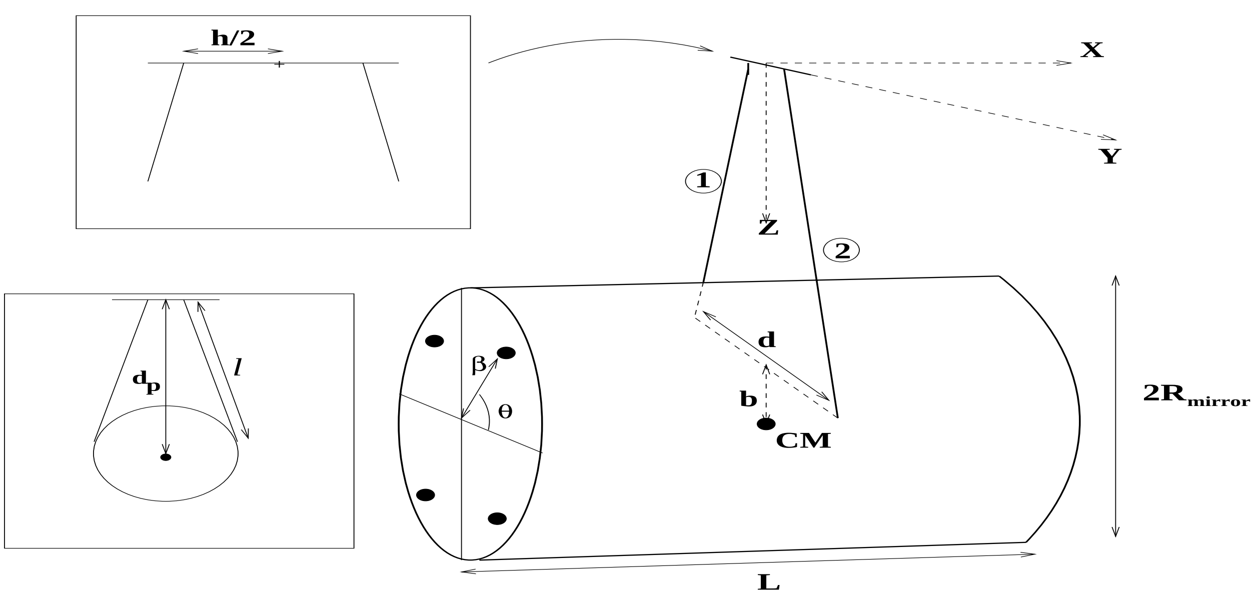

FIG. 20. A mirror suspension model. The mass of the mirror is M and the centre of mass (CM) lies directly on the Z axis. The length, l, of each wire is the length at mechanical equilibrium. The black circles on one of the faces show the location of control forces. Each location is at an angle θ to the horizontal and at a distance β from the centre of the face.

The mirror :

We choose the space frame axes of the mirror to lie along XYZ also. This implies that $\mathcal{R}$, the rotation matrix from the GRF to the mirror space frame S, is the identity matrix $\mathbf{1}$. We assume that at equilibrium the body axes B lie along the axes of S. The origins of both the B and S frame are attached to the CM of the mirror.

The wires :

In our model, we assume that the wires are attached on the mirror in such a way that each wire is tangent to the mirror cross-section at its respective attachment point§§. Thus,

$$b = R_{\text{mirror}} \cos \left[\tan^{-1}\left(\frac{h/2}{d_p}\right) + \cos^{-1}\left(\frac{R_{\text{mirror}}}{\sqrt{d_p^2 + (h/2)^2}}\right) \right] , \tag{73}$$

$$l = \left[d_p^2 + (h/2)^2 - R_{\text{mirror}}^2\right]^{1/2} , \tag{74}$$

$$d = 2\left[R_{\text{mirror}}^2 - b^2\right]^{1/2} . \tag{75}$$

The rotation matrix, say $\mathcal{R}_{\text{wire}}^{(j)}$, *from* the GRF *to* the preferred frame of wire # j is given by***,

$$\mathcal{R}_{\text{wire}}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} , \tag{76}$$

$$\mathcal{R}_{\text{wire}}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} , \tag{77}$$

$$\sin\phi = \frac{d-h}{2l} . \tag{78}$$

The equilibrium value of the tension†††, T, is,

$$T = \frac{Mg}{2\cos\phi} , \tag{79}$$

in each wire.

B. Step 1 and 2 : Equations of motion and their formal solution

The mirror :

The list of $\mathbf{H}_1(i,j)$ and $\mathbf{H}_2(i,j)$ (see Eq. (69) and Eq. (70)) matrices are given in Appendix C. Since $\mathcal{R} = \mathbf{1}$ here, the acceleration of the attachment point of the j^{th} wire is given by

$$\sum_{i=1}^{2} \mathbf{H}_1(s;\, i,j)\overline{R}_i(s) + \sum_{i=1}^{4} \mathbf{H}_2(s;\, i,j)\overline{F}_i(s) + \frac{1}{M}\left[\sum_{i=1}^{2}\overline{R}_i(s) + \sum_{i=1}^{4}\overline{F}_i(s)\right] . \tag{80}$$

The wires :

Let the preferred frame of the j^{th} wire be denoted by PW_j. The acceleration of the wire–mirror attachment point of the j^{th} wire, in terms of the forces acting on the wire, can be obtained from Eq. (49), (50) and Eq. (53). The acceleration components in the GRF are,

---

§§In the actual design [5], the wire takeoff points will be fixed arbitrarily by using wire standoffs.

***A sign error in these expressions has been rectified.

†††The factor of 2 was missing earlier.

$$s^2\,\mathcal{R}_{\text{wire}}^{(j)T}\begin{pmatrix} \frac{a_x^2}{T}W_x(l,0,s)\left[\,\overline{G}_j\,\right]_{PW_j,1}+\frac{a_x^2}{T}W_x(l,l,s)\left[\,-\overline{R}_j\right]_{PW_j,1} \\ \frac{a_y^2}{T}W_y(l,0,s)\left[\,\overline{G}_j\,\right]_{PW_j,2}+\frac{a_y^2}{T}W_y(l,l,s)\left[\,-\overline{R}_j\right]_{PW_j,2} \\ \frac{a_z^2}{Y}W_z(l,0,s)\left[\,\overline{G}_j\,\right]_{PW_j,3}+\frac{a_z^2}{Y}W_z(l,l,s)\left[\,-\overline{R}_j\right]_{PW_j,3} \end{pmatrix}\,. \tag{81}$$

In the above expression we have used the fact that the force on the wire because of the mirror is opposite in direction to $\overline{R}_j$ which was the force acting *on the mirror*.

We can re-express the acceleration as,

$$s^2\,\mathcal{R}_{\text{wire}}^{(j)T}\left[\mathbf{W}_0\mathcal{R}_{\text{wire}}^{(j)}\overline{G}_j\;+\mathbf{W}_1\mathcal{R}_{\text{wire}}^{(j)}(-\overline{R}_j)\right]\,,$$

$$\mathbf{W_0}=\text{diag}\left[\frac{a_x^2}{T}W_x(l,0,s),\frac{a_y^2}{T}W_y(l,0,s),\frac{a_z^2}{Y}W_z(l,0,s)\right]\;;\;\mathbf{W}_1=\text{diag}\left[\frac{a_x^2}{T}W_x(l,l,s),\frac{a_y^2}{T}W_y(l,l,s),\frac{a_z^2}{Y}W_z(l,l,s)\right]\,. \tag{82}$$

Let $\mathcal{R}_{\text{wire}}^{(j)T}\mathbf{W}_k\mathcal{R}_{\text{wire}}^{(j)}$ be denoted by $\widetilde{\mathbf{W}}_k^{(j)}$, for $k=0,\,1$. Then the acceleration reduces to,

$$s^2\left[\widetilde{\mathbf{W}}_0^{(j)}\overline{G}_j+\widetilde{\mathbf{W}}_1^{(j)}(-\overline{R}_j)\right]\,, \tag{83}$$

where the explicit expressions for $\widetilde{\mathbf{W}}_0^{(j)}$ and $\widetilde{\mathbf{W}}_1^{(j)}$ are given in Appendix C.

C. Step 3 : Equating accelerations

We have to now equate the acceleration components given by Eq. (80) and Eq. (83). We get,

$$\sum_{i=1}^{2}\mathbf{H}_1(s;\,i,1)\overline{R}_i(s)+\sum_{i=1}^{4}\mathbf{H}_2(s;\,i,1)\overline{F}_i(s)+\frac{1}{M}\left[\sum_{i=1}^{2}\overline{R}_i(s)+\sum_{i=1}^{4}\overline{F}_i(s)\right]=s^2\left[\widetilde{\mathbf{W}}_0^{(1)}\overline{G}_1(s)+\widetilde{\mathbf{W}}_1^{(1)}(-\overline{R}_1(s))\right]\,, \tag{84}$$

$$\sum_{i=1}^{2}\mathbf{H}_1(s;\,i,2)\overline{R}_i(s)+\sum_{i=1}^{4}\mathbf{H}_2(s;\,i,2)\overline{F}_i(s)+\frac{1}{M}\left[\sum_{i=1}^{2}\overline{R}_i(s)+\sum_{i=1}^{4}\overline{F}_i(s)\right]=s^2\left[\widetilde{\mathbf{W}}_0^{(2)}\overline{G}_2(s)+\widetilde{\mathbf{W}}_1^{(2)}(-\overline{R}_2(s))\right]\,. \tag{85}$$

Rewriting the above expressions, we get

$$\left[\mathbf{H}_1(s;\,1,1)+s^2\widetilde{\mathbf{W}}_1^{(1)}+\frac{1}{M}\right]\overline{R}_1(s)+\left[\mathbf{H}_1(s;\,2,1)+\frac{1}{M}\right]\overline{R}_2(s)=s^2\widetilde{\mathbf{W}}_0^{(1)}\overline{G}_1(s)-\sum_{i=1}^{4}\left(\mathbf{H}_2(s;\,i,1)+\frac{1}{M}\right)\overline{F}_i(s) \tag{86}$$

$$\left[\mathbf{H}_1(s;\,1,2)+\frac{1}{M}\right]\overline{R}_1(s)+\left[\mathbf{H}_1(s;\,2,2)+\frac{1}{M}+s^2\widetilde{\mathbf{W}}_1^{(2)}\right]\overline{R}_2(s)=s^2\widetilde{\mathbf{W}}_0^{(2)}\overline{G}_2(s)-\sum_{i=1}^{4}\left(\mathbf{H}_2(s;\,i,2)+\frac{1}{M}\right)\overline{F}_i(s)\,. \tag{87}$$

Finally, we can write the above expressions compactly as,

$$\mathbf{T}_{r2f}\mathbf{R}=\mathbf{A}_f\mathbf{F}\,, \tag{88}$$

where

$$\mathbf{R}=\begin{pmatrix}\overline{R}_1(s)\\ \overline{R}_2(s)\end{pmatrix}\,, \tag{89}$$

$$\mathbf{F}=\begin{pmatrix}\overline{G}_1(s)\\ \overline{G}_2(s)\\ \overline{F}_1(s)\\ \vdots\\ \overline{F}_4(s)\end{pmatrix}\,. \tag{90}$$

and the explicit expressions for $\mathbf{T}_{r2f}$ and $\mathbf{A}_f$ are given in Appendix C. We call $\mathbf{T}_{r2f}$ the reaction–to–force transfer function.

D. Step 4: The solution

1. The reaction–to–displacement+force transfer function

As was done in Section II A (see Eq. (16)), we can express the forces at the suspension points, i,e., $\overline{G}_j(t)$, in terms of given displacements of the suspension points. Let $\overline{X}_j(t)$ be the displacement of the suspension point of the j^{th} wire in the GRF. Then the force–to–displacement+force (because, we now have a mixture of externally specified displacements and forces) transfer function can be obtained by replacing $\left[\overline{G}_j\right]_{PW_j,i}$ in the expression (81) by

$$\left[\overline{G}_j\right]_{PW_j,i} \to \frac{1}{W_i(0,0,s)}\left(\frac{T}{a_i^2}\left[\overline{X}_j\right]_{PW_j,i} - W_i(0,l,s)\left[-\overline{R}_j\right]_{PW_j,i}\right) \text{ ;for } i = 1,\, 2, \tag{91}$$

$$\left[\overline{G}_j\right]_{PW_j,i} \to \frac{1}{W_i(0,0,s)}\left(\frac{Y}{a_i^2}\left[\overline{X}_j\right]_{PW_j,i} - W_i(0,l,s)\left[-\overline{R}_j\right]_{PW_j,i}\right) \text{ ,for } i = 3\,. \tag{92}$$

After making the above replacements, the acceleration of the attachment point of the j^{th} wire is reexpressed as,

$$s^2\mathcal{R}^{(j)T}_{\text{wire}}\left[\mathbf{W}_0\mathcal{R}^{(j)}_{\text{wire}}\overline{X}_j + \mathbf{W}_1\mathcal{R}^{(j)}_{\text{wire}}(-\overline{R}_j)\right]\ , \tag{93}$$

where, now, $\mathbf{W}_0$ and $\mathbf{W}_1$ are redefined as[‡‡‡],

$$\mathbf{W}_0 = \text{diag}\left[\frac{W_x(l,0,s)}{W_x(l,l,s)}, \frac{W_y(l,0,s)}{W_y(l,l,s)}, \frac{W_z(l,0,s)}{W_z(l,l,s)}\right]\ , \tag{94}$$

$$\mathbf{W}_1 = \text{diag}\left[\frac{a_x^2}{T}\left(W_x(l,l,s) - \frac{W_x^2(l,0,s)}{W_x(l,l,s)}\right), \frac{a_y^2}{T}\left(W_y(l,l,s) - \frac{W_y^2(l,0,s)}{W_y(l,l,s)}\right), \frac{a_z^2}{Y}\left(W_z(l,l,s) - \frac{W_z^2(l,0,s)}{W_z(l,l,s)}\right)\right]\ . \tag{95}$$

The rest of the analysis goes through as before and we get,

$$\mathbf{T}_{r2d+f}\mathbf{R} = \mathbf{A}_{d+f}\mathbf{X}\ , \tag{96}$$

where $\mathbf{T}_{r2d+f}$ stands for the reaction–to–displacement+force transfer function and

$$\mathbf{X} = \begin{pmatrix} \overline{X}_1(s) \\ \overline{X}_2(s) \\ \overline{F}_1(s) \\ \vdots \\ \overline{F}_4(s) \end{pmatrix}\ . \tag{97}$$

The matrix $\mathbf{A}_{d+f}$ has the same forms as for $\mathbf{A}_f$ respectively as given in Appendix C apart from the redefinition of the $\mathbf{W}$ matrices given above. We do not give the explicit forms for either $\mathbf{T}_{r2d+f}$ or $\mathbf{A}_{d+f}$ in this report because as far as the code is concerned, they are obtained by redefining the $\mathbf{W}$ matrices and going through the same construction as for $\mathbf{T}_{r2f}$.

To obtain $\mathbf{R}$ from given external forces, we simply need to invert $\mathbf{T}_{r2d+f}$.

2. The displacement+force–to–displacement tranfer function

We will now derive the transfer function from externally specified displacements of the suspension points and externally specified control forces to the displacment of the mirror centre of mass.

Recall that in the present case we have chosen the GRF to be the same as the space frame S. Thus $\left[\overline{R}_i(s)\right]_S$ and $\left[\overline{F}_i(s)\right]_S$ in Section III B are the same as $\overline{R}_i$ and $\overline{F}_i$ respectively that are used in the above. This was mentioned at the beginning of the section itself but is again stated here as a reminder.

[‡‡‡]Recall that $W(0,l,s) = W(l,0,s)$ and $W(0,0,s) = W(l,l,s)$.

The motion of the centre of mass is governed by Eq. (54), which in the Laplace domain, can be expressed as

$$\overline{X}_{\rm cm}(s) = \frac{1}{Ms^2}\left[\sum_{i=1}^{i=2}\overline{R}_i(s) + \sum_{i=1}^{i=4}\overline{F}_i(s)\right] , \tag{98}$$

$$= \frac{1}{Ms^2}\left[\omega_1\mathbf{R} + \omega_{\mathbf{2}}\mathbf{X}\right] + \frac{\overline{\mathbf{x}}_{\rm cm}(0)}{s} , \tag{99}$$

$$\omega_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} , \tag{100}$$

$$\omega_{\mathbf{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} . \tag{101}$$

W have assumed the initial velocity $\dot{\mathbf{x}}_{\rm cm}(0) = 0$ and

$$\mathbf{x}_{\rm cm}(0) = \begin{pmatrix} 0 \\ 0 \\ b + l\cos\phi \end{pmatrix} . \tag{102}$$

Now, we can substitute for $\mathbf{R}$ from Eq. (96) to get,

$$\mathbf{X}_{\rm cm} = \frac{1}{Ms^2}\left[\omega_1\mathbf{T}^{-1}_{r2d+f}\mathbf{A}_{d+f} + \omega_{\mathbf{2}}\right]\mathbf{X} + \frac{\overline{\mathbf{x}}_{\rm cm}(0)}{s} . \tag{103}$$

If, instead of $\overline{\mathbf{x}}_{\rm cm}$ which is the position vector from the origin of the GRF, we look at the displacement of the CM from its initial position, $\overline{\mathbf{x}}_{\rm cm}(0)$, at equilibrium then the last term in Eq. (103) disappears. We call the coefficient of $\mathbf{X}$ as the force+displacement to displacement transfer function.

3. The displacement+force–to–angular displacement tranfer function

We will now derive the transfer function from externally specified displacements of the suspension points and externally specified control forces to the angular displacments of the mirror about its centre of mass.

Let $\mathcal{A}(\overline{x})$, where $\overline{x}$ is an arbitrary vector be the antisymmetric matrix,

$$\mathcal{A}(\overline{x}) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} . \tag{104}$$

Thus,

$$\overline{x} \times \overline{y} = \mathcal{A}(\overline{x})\overline{y} . \tag{105}$$

Using Eq. (65) and Eq. (96), we get

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\eta} \\ \tilde{\phi} \end{pmatrix} = \mathbf{J}^{-1}(s)\left(\epsilon_p\mathbf{T}^{-1}_{r2d+f}\mathbf{A}_{d+f} + \epsilon_P\right)\mathbf{X} , \tag{106}$$

$$\epsilon_p = \begin{pmatrix} \mathcal{A}\left([\overline{p}_1]_B\right) & \mathcal{A}\left([\overline{p}_2]_B\right) \end{pmatrix} , \tag{107}$$

$$\epsilon_P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}\left([\overline{P}_1]_B\right) & \mathcal{A}\left([\overline{P}_2]_B\right) & \mathcal{A}\left([\overline{P}_3]_B\right) & \mathcal{A}\left([\overline{P}_4]_B\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & \end{pmatrix} . \tag{108}$$

We call the coefficient of $\mathbf{X}$, the displacement+force–to–angular displacement transfer function.

In Appendix D, we plot some sample displacement+force–to–displacement/angular displacement transfer functions.

[1] Anatoliy G. Butkovskiy, *Structural theory of distributed systems*, 1983 (John Wiley, New York).
[2] Anatoliy G. Butkovskiy, *Green's Functions and transfer functions handbook*, 1982 (John Wiley, New York).
[3] Gabriela I. Gonzalez, Peter R. Saulson, J. Accoust. Soc. Am. **96** (1), 207 (1994).
[4] Herbert Goldstein, *Classical Mechanics*, Seventh edition, 1965 (Addison-Wesley, Massachusetts).
[5] Seiji Kawamura, Janeen Hazel, Mark Barton, LIGO-T970158-06.

APPENDIX A: GREEN'S FUNCTION : A BRIEF OUTLINE.

Given[§§§] (i) a linear partial differential equation that describes a process in an open spatial set $\mathcal{D}$,

$$L(u(x,t)) = f(x,t)\ ,\ x \in \mathcal{D}\ ,\ t > t_0\ , \tag{A1}$$

where L is the partial differential operator, (ii) associated boundary conditions

$$\Gamma(u(x,t)) = g(x,t)\ ,\ x \in \partial\mathcal{D}\ , \tag{A2}$$

$\partial\mathcal{D}$ being the boundary of $\mathcal{D}$ and Γ being a linear partial differential operator on the boundary, and (iii) initial conditions,

$$N(u(x,t)) = u_0(x)\ ,\ x \in \mathcal{D}\ ,\ t = t_0\ , \tag{A3}$$

it can be shown that there is a *generalized* function $w(x,t)$ such that solving Eq. (A1), (A2) and Eq. (A3) is equivalent to solving the following system with homogenous boundary and initial conditions,

$$L(u(x,t)) = w(x,t)\ ,\ x \in \mathcal{D}\ ,\ t > t_0\ , \tag{A4}$$
$$\Gamma(u(x,t)) = 0\ ,\ x \in \partial\mathcal{D}\ ,\ t > t_0\ , \tag{A5}$$
$$N(u(x,t)) = 0\ ,\ x \in \mathcal{D}\ ,\ t = t_0\ . \tag{A6}$$

$w(x,t)$, called the *standardising function* is a linear combination of $f(x,t)$, $g(x,t)$ and $u_0(x)$.

Now, the system of equations, Eq. (A4), (A5) and Eq. (A6), can be solved completely in terms of the Green's function $G(x,\xi,t,\tau)$ which satisfies the equations,

$$L(G(x,\xi,t,\tau)) = \delta(x-\xi)\delta(t-\tau)\ ,\ x \in \mathcal{D}\ ,\ t > t_0\ , \tag{A7}$$
$$\Gamma(G(x,\xi,t,\tau)) = 0\ ,\ x \in \partial\mathcal{D}\ ,\ t > t_0\ , \tag{A8}$$
$$N(G(x,\xi,t,\tau)) = 0\ ,\ x \in \mathcal{D}\ ,\ t = t_0\ . \tag{A9}$$

Knowing $G(x,\xi,t,\tau)$ and $w(x,t)$, one can solve for $u(x,t)$,

$$u(x,t) = \int_{t_0}^{t}\int_{\mathcal{D}} G(x,\xi,t,\tau)w(\xi,\tau)d\xi d\tau\ . \tag{A10}$$

APPENDIX B: EULER'S EQUATIONS

The angular motion of a rigid body can be obtained in terms of Euler's equations. We present a brief derivation of these equations now. In the space frame S attached to the centre of mass, the equation of motion is,

$$\frac{d\overline{\mathbf{L}}}{dt} = \overline{\mathbf{N}}\ , \tag{B1}$$

[§§§]See a standard reference on Green's functions. A concise treatment is given in [2].
That is, it includes Dirac delta functions or *distributions*.

where $\overline{\mathbf{L}}$ is the body's angular momentum and $\overline{\mathbf{N}}$ is the total external torque as measured by an observer at rest in S. Let R_{b2s} be the rotation matrix from the body frame B to the S. Then, $R_{s2b} = R_{b2s}^{-1} = R_{b2s}^{T}$.

Eq. (B1) can be rexpressed in the body frame as,

$$\frac{d}{dt}(R_{b2s} \left[\overline{\mathbf{L}}\right]_B) = R_{b2s} \left[\overline{\mathbf{N}}\right]_B \ , \tag{B2}$$

where the subscript 'B' indicates the vector components in the body frame. We know that, for any vector $\overline{\mathbf{X}}$,

$$\frac{d\,R_{b2s}}{dt} \left[\overline{\mathbf{X}}\right]_B = \overline{\mathbf{\Omega}}\times(R_{b2s} \left[\overline{\mathbf{X}}\right]_B) \ , \tag{B3}$$

where $\overline{\mathbf{\Omega}}$ is the angular velocity in the space frame. Applying this to Eq. (B2), we get

$$\begin{aligned} (R_{b2s} \left[\overline{\mathbf{\Omega}}\right]_B)\times(R_{b2s} \left[\overline{\mathbf{L}}\right]_B) + R_{b2s} \left[\dot{\overline{\mathbf{L}}}\right]_B &= R_{b2s} \left[\overline{\mathbf{N}}\right]_B \ , \\ \left[\overline{\mathbf{\Omega}}\right]_B \times \left[\overline{\mathbf{L}}\right]_B + \left[\dot{\overline{\mathbf{L}}}\right]_B &= \left[\overline{\mathbf{N}}\right]_B \ . \end{aligned} \tag{B4}$$

Now,

$$\begin{aligned} \left[\dot{\overline{\mathbf{L}}}\right]_B &= \sum_i m_i \left[\overline{\mathbf{r}}_i\right]_B \times(\left[\overline{\mathbf{\Omega}}\right]_B \times \left[\overline{\mathbf{r}}_i\right]_B) \ , \\ &= \sum_i m_i \left(\left[\overline{\mathbf{r}}_i\right]_B^2 \left[\overline{\mathbf{\Omega}}\right]_B - \left(\overline{\mathbf{r}}_i.\overline{\mathbf{\Omega}}\right) \left[\overline{\mathbf{r}}_i\right]_B\right) \ , \\ &= \mathbf{I} \left[\overline{\mathbf{\Omega}}\right]_B \ , \end{aligned} \tag{B5}$$

where $\mathbf{I}$ is the inertia tensor which is a constant in frame B,

$$\mathbf{I}_{mn} = \sum_i m_i \left(\overline{\mathbf{r}}_i^2 \delta_{mn} - \left[\overline{\mathbf{r}}_i\right]_{B,m} \left[\overline{\mathbf{r}}_i\right]_{B,n}\right) \ , \tag{B6}$$

where the subscript 'B, m' denotes the m^{th} body frame component. For simplicity in the following, we will assume that the B frame axes are oriented along the principal axes of the body. By the definition of principal axes, this implies that the inertia tensor is diagonal, $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$. Substituting Eq. (B5) in Eq. (B4), we get

$$\left[\overline{\mathbf{\Omega}}\right]_B \times(\mathbf{I} \left[\overline{\mathbf{\Omega}}\right]_B) + \mathbf{I} \left[\dot{\overline{\mathbf{\Omega}}}\right]_B = \left[\overline{\mathbf{N}}\right]_B \tag{B7}$$

or,

$$I_1 \dot{\Omega}_1 - (I_2 - I_3)\Omega_2\Omega_3 = N_1 \ , \tag{B8}$$
$$I_2 \dot{\Omega}_2 - (I_3 - I_1)\Omega_3\Omega_1 = N_2 \ , \tag{B9}$$
$$I_1 \dot{\Omega}_3 - (I_1 - I_2)\Omega_1\Omega_2 = N_3 \ , \tag{B10}$$

where we have dropped the subscript 'B' for convenience.

In order to get the equations motion for the angles, we must express $\overline{\mathbf{\Omega}}$ in term of the rotational angles. It can be shown that if one chooses these angles to be the Euler angles, linearisation of Eq. (B8)–Eq. (B10) yields a non-dynamical equation of motion (i.e., the lowest order terms do not have the second derivatives of the angles). Thus one has to choose a different set of angles. Euler angles are given in Fig. (21) while the new set of angles called Kardan angles is given in Fig. (22).

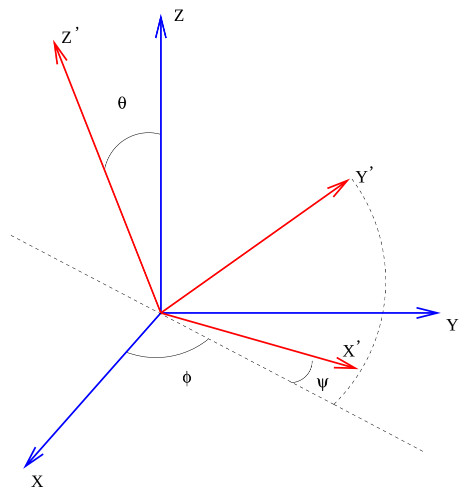

FIG. 21. The Euler angles (θ, ϕ, ψ). The red axes (primed) are the body axes while the space axes (unprimed) are in blue. The angle ψ lies in the $X'Y'$ plane.

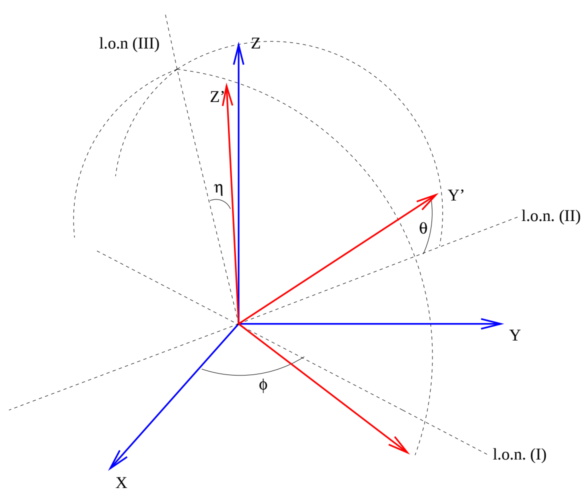

FIG. 22. The Kardan angles (θ, ϕ, η). The red axes are the body axes while the space axes are in blue. The first rotation is around Z by angle ϕ giving the line of nodes (l.o.n.) **l.o.n. (I)**. The second rotation is around the new X axis by an angle θ. (as measured from **l.o.n. (II)**.) The third rotation is around the new Y axis, Y', by an angle η. (measured from **l.o.n. (III)**.)

We can obtain $\overline{\mathbf{\Omega}}$ from the definition given in Eq. (B3). In terms of Euler angles, R_{b2s} is given in [4]. For Kardan angles,

$$R_{b2s} = [\mathtt{B}\,\mathtt{C}\,\mathtt{D}]^T \ ,$$

$$\mathtt{B} = \begin{pmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{pmatrix} \ , \tag{B11}$$

$$\mathtt{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \ , \tag{B12}$$

$$\mathtt{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{B13}$$

To first order in θ, ϕ and η, R_{b2s} is obtained as

$$R_{b2s} = \begin{pmatrix} 1 & -\phi & \eta \\ \phi & 1 & -\theta \\ -\eta & \theta & 1 \end{pmatrix} . \tag{B14}$$

Finally $\dot{R}_{b2s} R_{b2s}^T$ gives the components of $\overline{\mathbf{\Omega}}$,

$$\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{\phi} & \dot{\eta} \\ \dot{\phi} & 0 & -\dot{\theta} \\ -\dot{\eta} & \dot{\theta} & 0 \end{pmatrix} \tag{B15}$$

To lowest order, Eq. (B8)–Eq. (B10) then become,

$$I_1 \ddot{\theta} = \left[\overline{N}\right]_{B,1} , \tag{B16}$$
$$I_2 \ddot{\eta} = \left[\overline{N}\right]_{B,2} , \tag{B17}$$
$$I_3 \ddot{\phi} = \left[\overline{N}\right]_{B,3} , \tag{B18}$$

APPENDIX C: SOME QUANTITIES REQUIRED FOR THE MIRROR SUSPENSION MODEL

The components, in frame B, of the points at which the wires are attached to the mirror are :

$$\overline{p}_1 = (0, -d/2, -b) , \tag{C1}$$
$$\overline{p}_2 = (0, d/2, -b) , \tag{C2}$$

for wire # 1 and #2 respectively. The B frame components for the points of application of control forces are :

$$\overline{P}_1 = (-L/2, \beta\cos\theta, -\beta\sin\theta) , \tag{C3}$$
$$\overline{P}_2 = (-L/2, -\beta\cos\theta, -\beta\sin\theta) , \tag{C4}$$
$$\overline{P}_3 = (-L/2, -\beta\cos\theta, \beta\sin\theta) , \tag{C5}$$
$$\overline{P}_4 = (-L/2, \beta\cos\theta, \beta\sin\theta) . \tag{C6}$$

The matrix $\mathbf{T}_{r2f}$ is presented below. Here, $W_x \equiv W_x(l,l,s)$, $W_y \equiv W_y(l,l,s)$ and $W_z \equiv W_z(l,l,s)$. The expression for ϕ is provided in Eq. (78).

$$\mathbf{T}_{r2f} = \left(\begin{array}{c|c} \mathbf{T}_1 & \mathbf{T}_2 \\ \hline \mathbf{T}_3 & \mathbf{T}_4 \end{array} \right) , \tag{C7}$$
$$\mathbf{T}_1 = \tag{C8}$$

The matrix $\mathbf{A}_f$ is now presented,

$$\mathbf{A}_f = -\begin{pmatrix} -s^2\widetilde{\mathbf{W}}_0^{(1)}, & \mathbf{0}, & \mathbf{H}_2(s;\,1,1)+\frac{1}{M}, & \mathbf{H}_2(s;\,2,1)+\frac{1}{M}, & \mathbf{H}_2(s;\,3,1)+\frac{1}{M}, & \mathbf{H}_2(s;\,4,1)+\frac{1}{M} \\ \mathbf{0}, & -s^2\widetilde{\mathbf{W}}_0^{(2)}, & \mathbf{H}_2(s;\,1,2)+\frac{1}{M}, & \mathbf{H}_2(s;\,2,2)+\frac{1}{M}, & \mathbf{H}_2(s;\,3,2)+\frac{1}{M}, & \mathbf{H}_2(s;\,4,2)+\frac{1}{M} \end{pmatrix} \tag{C9}$$

APPENDIX D: DISPLACEMENT+FORCE–TO–DISPLACEMENT/ANGULAR DISPLACEMENT TRANSFER FUNCTIONS : FIGURES

The parameters used to produce the following figures are chosen to match corresponding ones in [5].

Wires :	
Density	7.8×10^3 Kg/m^3
Diameter	0.31 mm
Young's Modulus ($\mathcal{Y}$)	2.068×10^{11} N/m^2
$\gamma_x = \gamma_y = \gamma_z$	0.0 Kg/m sec
Mirror :	
Mass (M)	10.7 Kg
Diameter ($2R_{\rm mirror}$)	25.0 cm
Length (L)	10.0 cm
Suspension :	
d_p	45.0 cm
h	3.3 cm

The derived suspension parameters are :

wire length (l)	43.26 cm
b	3.02 cm
d	24.26 cm
ϕ	14.0°

Some derived dynamical parameters:

mass per unit length (ρ)	0.6 gm/m
tension per wire	54.04 N
sound velocity ($a_x = a_y$)	302.97 m/sec
longitudinal stress (Y)	1.6×10^4 N
sound velocity (a_z)	5.15×10^3 m/sec

For comparision, the following frequencies from [5] are helpful,

Pendulum frequency	0.74 Hz
Pitch frequency	0.6 Hz
Yaw frequency	0.497 Hz
Violin frequency	341 Hz
vertical frequency	12.63 Hz

Note, however, that without wire-standoffs, our geometry is different than the one used in [5]. For instance, $b \equiv d_{\rm pitch} = 0.724$ cm in [5]) which implies a lower pitch frequency than obtained here.

When we talk of the motion of the mirror centre of mass here, it should be understood as the displacement of the centre of mass around its position at equilibrium. Hence the last term on the RHS of Eq. (103) can be dropped. We denote the matrix

$$\frac{1}{Ms^2}\left[\omega_1 \mathbf{T}^{-1}_{r2d+f}\mathbf{A}_{f+d} + \omega_{\mathbf{2}}\right] ,$$

in Eq. (103) by $\mathcal{T}_{\rm CM}$. The matrix,

$$\mathbf{J}^{-1}(s)\left(\epsilon_p \mathbf{T}^{-1}_{r2d+f}\mathbf{A}_{d+f} + \epsilon_P\right) ,$$

in Eq. (106) is denoted by $\mathcal{T}_{\rm rot}$ in the following. The $(i,j)^{\rm th}$ element of these matrices will be denoted by $\mathcal{T}_{\rm CM}(i,j)$ and $\mathcal{T}_{\rm rot}(i,j)$.

Fig. 23 : X motion of mirror centre of mass when suspension points are moved, *in phase* and with equal amplitudes, in the X direction alone. The transfer function plotted here is given by $\mathcal{T}_{\rm CM}(1,1) + \mathcal{T}_{\rm CM}(1,4)$. As expected, Y, Z motion of mirror centre of mass ($\mathcal{T}_{\rm CM}(2,1) + \mathcal{T}_{\rm CM}(2,4)$ and $\mathcal{T}_{\rm CM}(3,1) + \mathcal{T}_{\rm CM}(3,4)$ respectively) for the same suspension point displacements as in Fig. 23 are zero.

Fig. 24 : Pitch (η) motion ($\mathcal{T}_{\rm rot}(2,1) + \mathcal{T}_{\rm rot}(2,4)$) for the same suspension point displacements as in Fig. 23.

Fig. 25 : yaw (ϕ) and roll (θ) motion ($\mathcal{T}_{\rm rot}(3,1) + \mathcal{T}_{\rm rot}(3,4)$ and $\mathcal{T}_{\rm rot}(1,1) + \mathcal{T}_{\rm rot}(1,4)$ respectively) for the same suspension point displacements as in Fig. 23. Ideally there should be no coupling to θ and ϕ motions.

Fig. 26 : X motion of mirror centre of mass when suspension points are moved *out of phase* and with a relative amplitude of 1:2 in the X direction alone. The transfer function plotted here is given by $\mathcal{T}_{\rm CM}(1,1) - 2\mathcal{T}_{\rm CM}(1,4)$. Note the change in level for low frequencies. Pitch motion ($\mathcal{T}_{\rm rot}(2,1) - 2\mathcal{T}_{\rm rot}(2,4)$) and yaw motion ($\mathcal{T}_{\rm rot}(3,1) - 2\mathcal{T}_{\rm rot}(3,4)$) for the same suspension point displacements.

Fig. 27 : Z motion of mirror centre of mass when the suspension points are moved *in phase* and with equal amplitude along the Z direction only ($\mathcal{T}_{\rm CM}(3,3) + \mathcal{T}_{\rm CM}(3,6)$).

Fig. 28 : Y motion of centre of mass when the suspension points move *out of phase* and with equal amplitudes in the Z direction only ($\mathcal{T}_{\rm CM}(2,3) - \mathcal{T}_{\rm CM}(2,6)$).

Fig. 29 : θ (roll) for the same motion of suspension points as in Fig. 28 ($\mathcal{T}_{\rm rot}(1,3) - \mathcal{T}_{\rm rot}(1,6)$).

Fig. 30 : Z motion of the centre of mass when the suspension points are moved *out of phase* and with equal amplitudes in the Y direction alone ($\mathcal{T}_{\rm CM}(3,2) - \mathcal{T}_{\rm CM}(3,5)$). All other motions are zero.

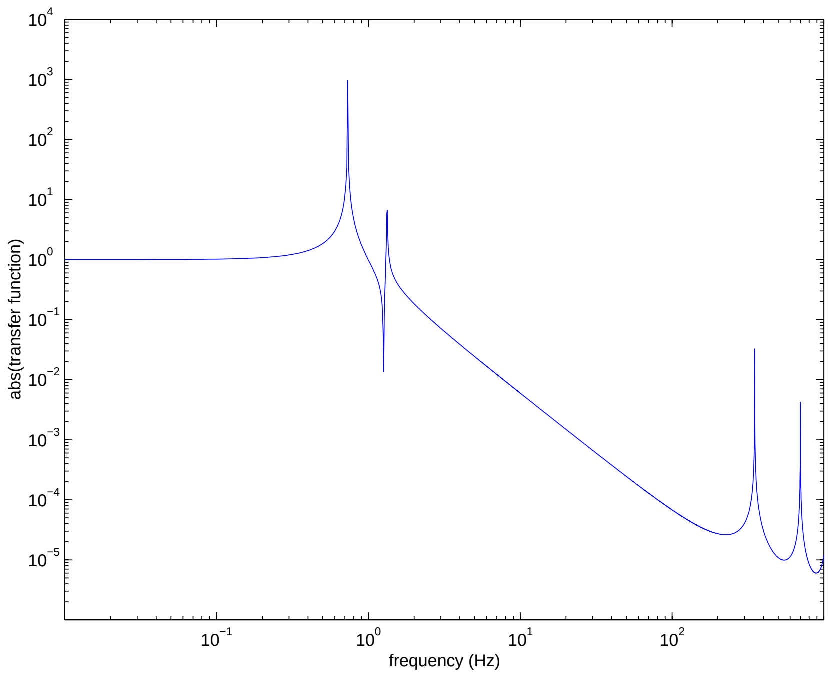

FIG. 23.

Note : What are the two resonances at high frequencies? Is there a coupling because the wires are at an angle to the vertical?

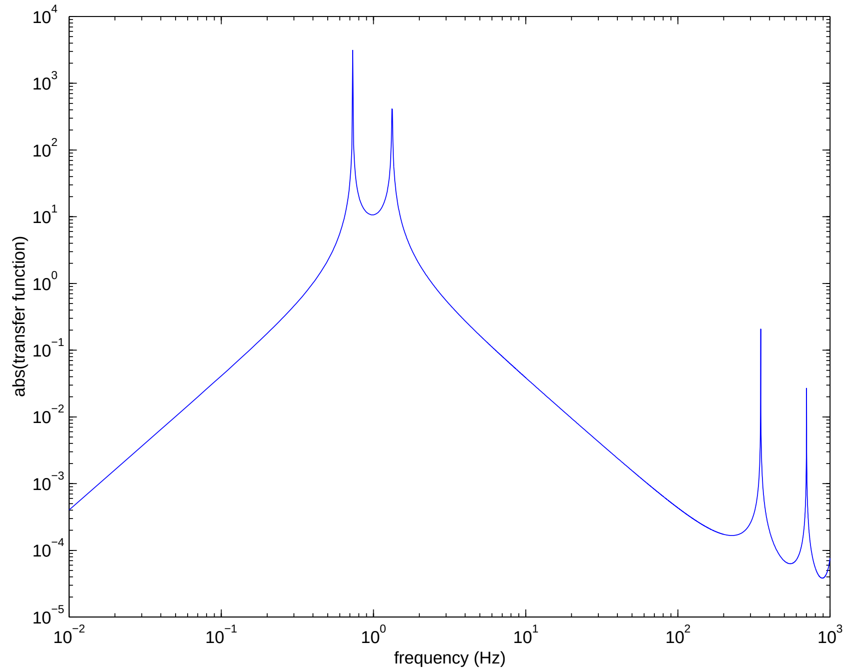
abs(transfer function)
frequency (Hz)

FIG. 24.

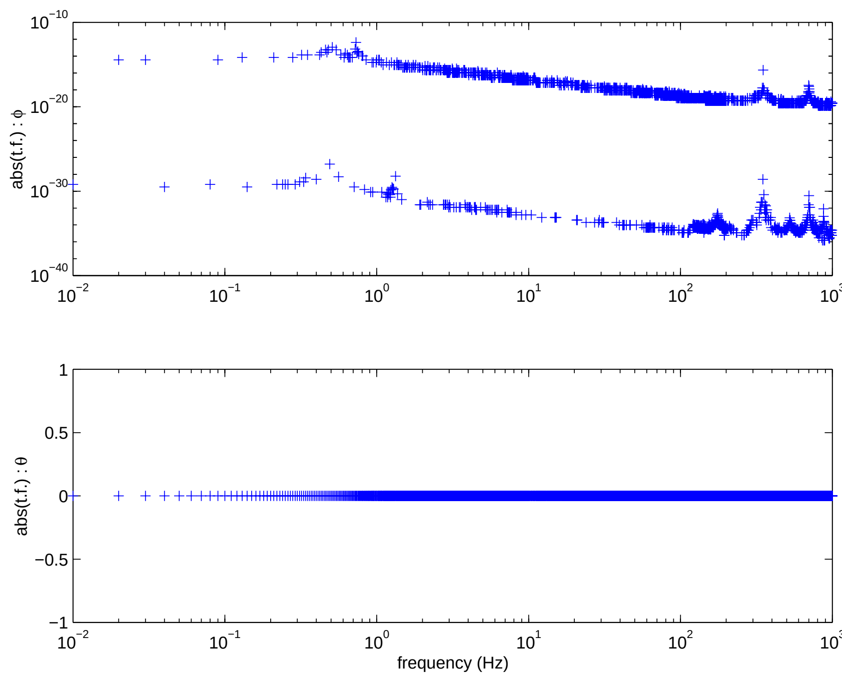
abs(t.f.) : φ
abs(t.f.) : θ
frequency (Hz)

FIG. 25.

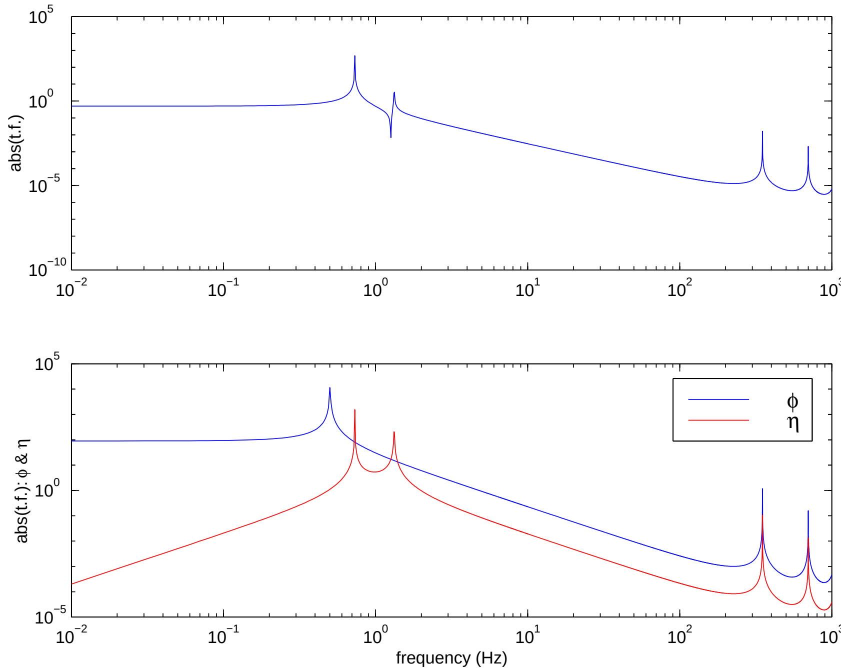
abs(t.f.)
abs(t.f.): ϕ & η
ϕ
η
frequency (Hz)

FIG. 26.

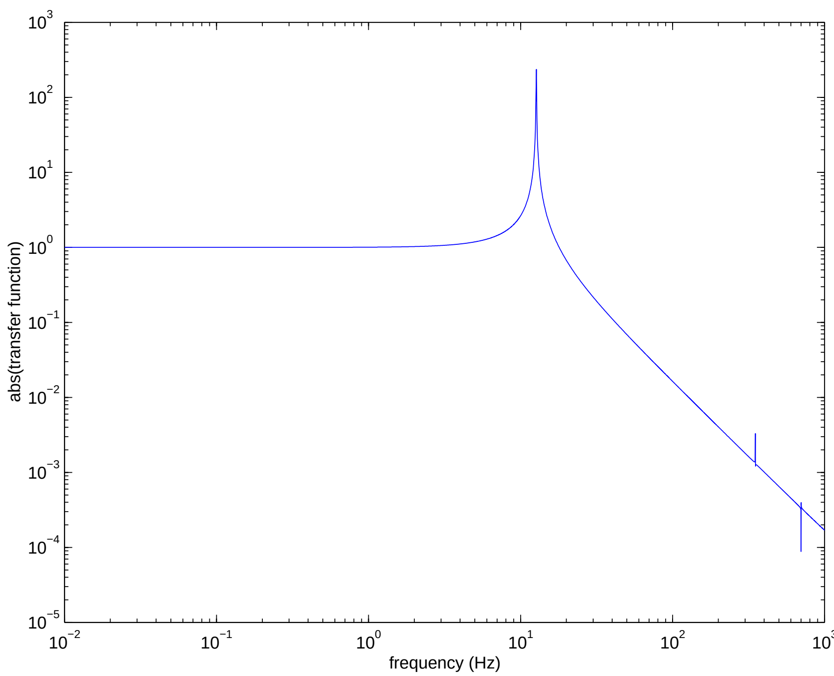
abs(transfer function)
frequency (Hz)

FIG. 27.

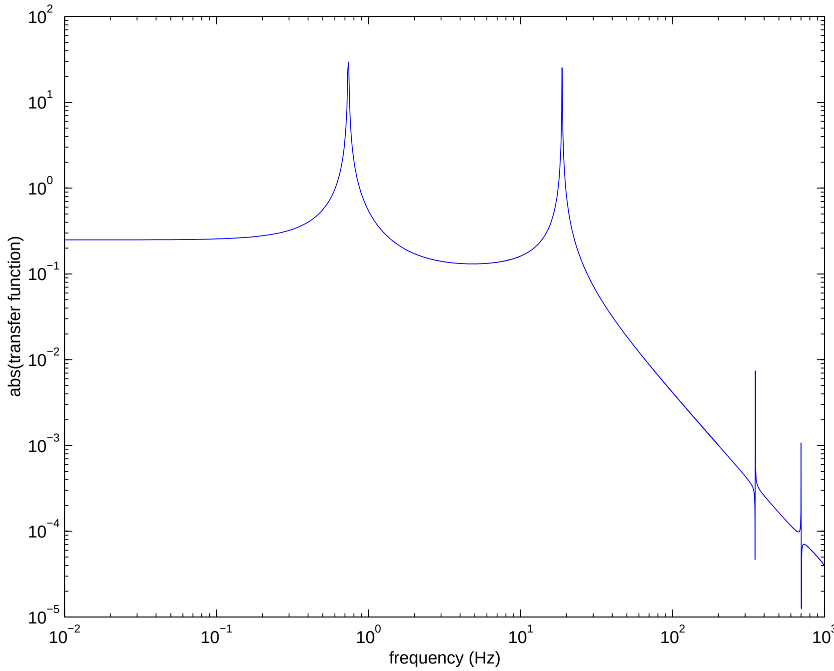
abs(transfer function)
frequency (Hz)
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

FIG. 28.

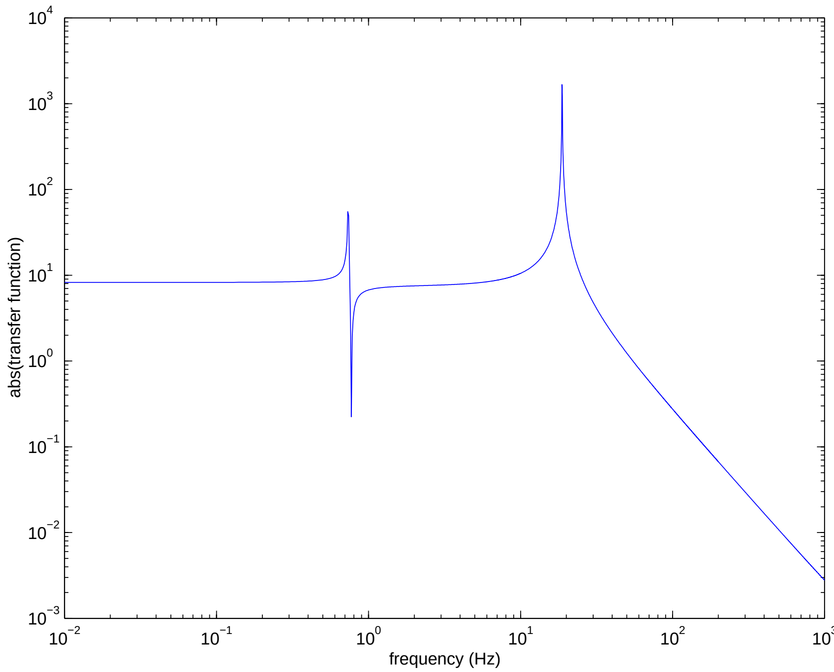
abs(transfer function)
frequency (Hz)
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

FIG. 29.

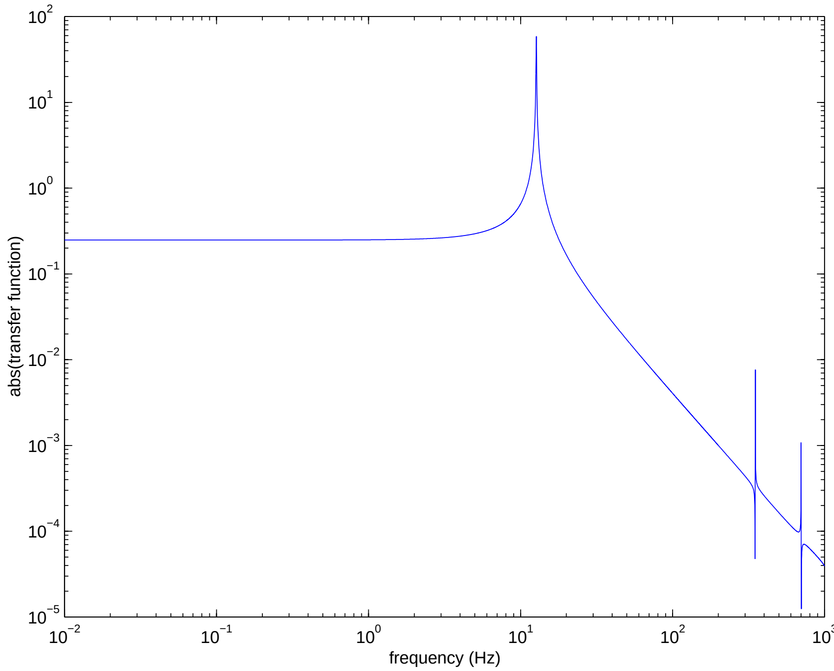
abs(transfer function)
frequency (Hz)
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

FIG. 30.