Singular Value Decomposition Applied to Binary Black Hole Waveforms

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Abstract

Singular Value Decomposition (SVD) has been shown to eliminate redundancy inherent in template banks of post-Newtonian waveforms used to search for gravitational-wave (GW) signals from binary neutron stars. Here we present results from a similar analysis applied to binary black hole (BBH) GW waveforms from systems with total mass in the range $M \in [30M_\odot, 200M_\odot)$.

Binary Black Hole Waveforms

Breakthroughs in numerical relativity during the last decade have led to the generation of accurate GW waveforms models from coalescing BBHs. However, numerical relativity simulations are computationally expensive and individual simulations can only explore tracks through the parameter space. With this in mind, phenomenological models of these GW waveforms have been constructed to fill in the gaps.

One particularly useful phenomenological model can be used to generate GW waveforms for systems whose objects' spins are (anti-)aligned with the angular momentum of the system[1]. This model computes GW waveforms in the frequency-domain given the masses of the two objects m_1, m_2, and a parameter characterizing the objects' spins χ,

$$h(f) = A(f)\mathrm{e}^{-\mathrm{i}\Psi(f)},$$

where the is

$$\Psi(f) \equiv 2\pi f t_0 + \varphi_0 + \sum_{k=0}^{7} \psi_k f^{(k-5)/3},$$

where ψ_k are functions of (m_1, m_2, χ), and the amplitude $A(f)$ is split into three pieces, which describe the separate phases of coalescence: inspiral, merger, and ringdown,

$$A(f) \equiv C \begin{cases} \sum_{k=0}^{3} \alpha_k f^{(2k-7)/6} & \text{if } f < f_{\mathrm{merger}} \\ \sum_{k=0}^{2} \epsilon_k f^{(k-2)/3} & \text{if } f_{\mathrm{merger}} \le f < f_{\mathrm{ringdown}} \\ \mathcal{L}(f, m_1, m_2, \chi) & \text{if } f_{\mathrm{ringdown}} \le f < f_{\mathrm{cutoff}} \\ 0 & \text{if } f > f_{\mathrm{cutoff}}, \end{cases}$$

where C is an overall scaling factor, α_k and ϵ_k are functions of (m_1, m_2, χ), $\mathcal{L}(f, m_1, m_2, \chi)$ is a Lorentzian whose shape depends on (m_1, m_2, χ), and f_{merger}, f_{ringdown}, and f_{cutoff} are the frequencies where the transitions between the phases of coalescence occur and are functions of (m_1, m_2, χ).

Parameter Space

The parameter space in which we perform our investigations is $\{M, q, \chi | M \in [30M_\odot, 200M_\odot), q \in [1, 10), \chi = 0\}$, where $M \equiv m_1 + m_2$ is the total mass of the system, $q \equiv m_1/m_2$ is the mass ratio, $\chi \equiv (1+\delta)(S_1/m_1^2)/2 + (1-\delta)(S_2/m_2^2)/2$ is a dimensionless effective spin parameter, $\delta \equiv (m_1 - m_2)/M$ is the mass difference, and S_i is the spin angular momentum of the ith black hole. We also compare template banks of different dimensionality using $\chi \in [-0.85\delta, 0.85\delta)$.

Stochastic Template Banks

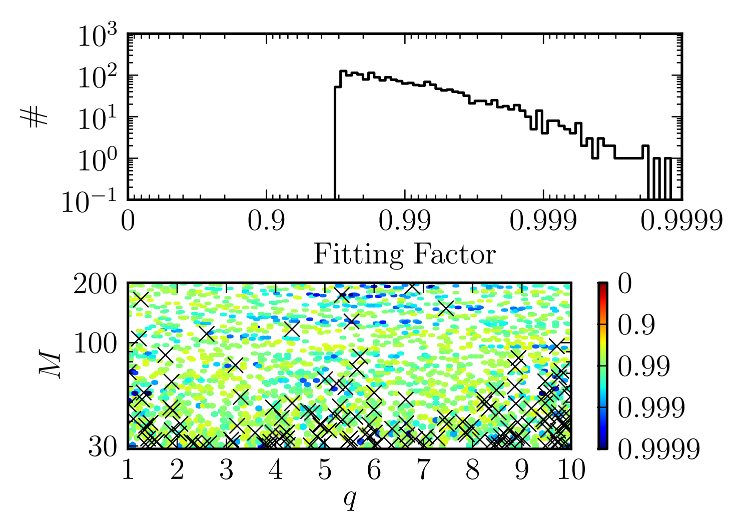
The parameter space of associated GW signals from BBH coalescences that ground-based GW detectors are sensitive to can be parameterized by a handful of parameters. Some of these parameters can be efficiently searched over analytically or numerically (called intrinsic parameters). The remaining extrinsic parameters (given in the Parameter Space section) must be searched over by brute force. Discrete points in this parameter space must be chosen in order to match filter the GW data against a finite set of waveform models. We choose these points by constructing a stochastic template bank [2] in the region of parameter space we are interested in. The algorithm is as follows:

1. Choose a proposed point randomly in that region of parameter space.
2. Compute the fitting factor between the waveform from that point and all other previously accepted points.
3. If the maximum fitting factor is below the minimal match of the template bank, accept the point as a template and add it to the list of the accepted points.
4. Repeat with step 1 until the termination condition is met.

where the fitting factor is given by the match between two waveforms maximized over phase- and time-offsets, and the termination condition is "stop when the number of proposed points exceeds 100 times the number of accepted points".

Template Bank Coverage

Below we depict a template bank generated in this way where the minimal match of the template bank was chosen to be 0.97.



We choose 2000 points randomly from within this space and find the templates cover the space to the degree specified. From this we conclude that these stochastic template banks are able to uniformly cover the parameter space.

Singular Value Decomposition

SVD is a type of principal component analysis that decomposes an arbitrary matrix $\mathbf{S}$ into unique unitary matrices,

$$\mathbf{S} = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^{\mathrm{T}},$$

where $\mathbf{V}$ and $\mathbf{U}$ are unitary matrices and $\mathbf{\Sigma}$ is a diagonal matrix of the singular values of $\mathbf{S}$. Truncating the SVD of a matrix of time-domain GW templates has been shown to eliminate redundancy in inspiral GW template banks [3]. The expected fractional SNR loss from SVD truncation is

$$\left\langle \frac{\delta\rho}{\rho} \right\rangle = \frac{1}{2N} \sum_{\mu=N'+1}^{N} \sigma_\mu^2,$$

where N is the total number of real-valued filters in $\mathbf{S}$, N' is the last basis vector of $\mathbf{U}$ that is retained in the reconstruction of $\mathbf{S}$, and σ_μ is the μth element of $\mathbf{\Sigma}$. Truncating the SVD such that $\langle \delta\rho/\rho \rangle = 10^{-4}$, we measure the actual fractional loss of SNR for every template,

$$\left(\frac{\delta\rho}{\rho} \right)_\alpha = \frac{1}{4} \sum_{\mu=N'+1}^{N} \left(v_{(2\alpha-1)\mu}^2 + v_{(2\alpha)\mu}^2 \right) \sigma_\mu^2,$$

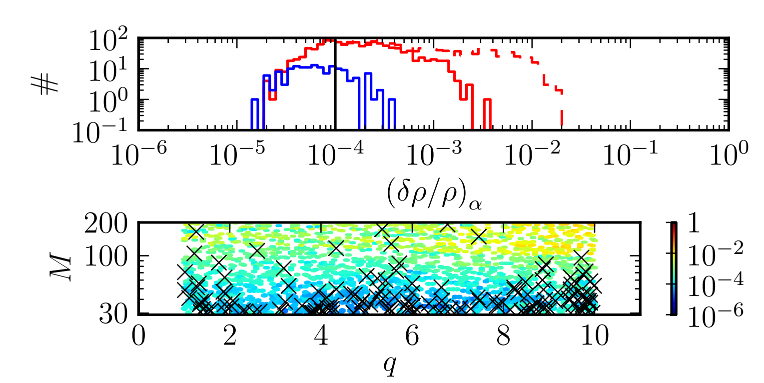
where $v_{(2\alpha-1)\mu}$ and $v_{(2\alpha)\mu}$ are the reconstruction coefficients of $\mathbf{V}$ associated with the μth basis vector and αth template.

Enclosing The Signal Manifold

As in [4], we test to see how well these basis vectors enclose the signal manifold by generating waveforms from random points in the parameter space, projecting them onto the basis vectors, and then computing $(\delta\rho/\rho)_\alpha$ for those points. The reconstruction coefficients $v_{(2\alpha-1)\mu}$ and $v_{(2\alpha)\mu}$ for the αth waveform are

$$v'_{(2\alpha-1)\mu} + \mathrm{i}v'_{(2\alpha)\mu} = \frac{1}{\sigma_\mu} \sum_j h_{\alpha j} u_{\mu j},$$

where $h_{\alpha j}$ is the jth time-sample of the αth waveform and $u_{\mu j}$ is the jth time-sample of the μth basis vector from $\mathbf{U}$.



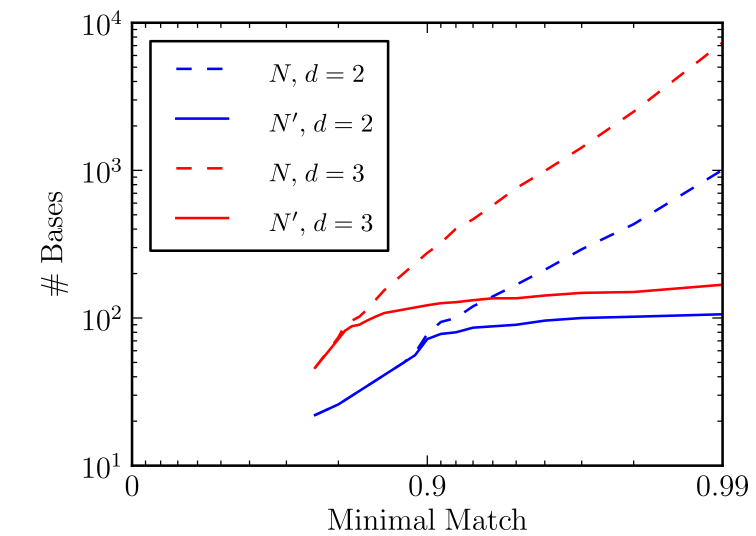
We see that basis vectors from the SVD of a 97% minimal match bank reconstruct all of the template waveforms (solid blue) and test-point waveforms below $100M_\odot$ (solid red) to high accuracy. The dashed red line shows a histogram of the results from all the test-points.

Independent Basis Vectors And Parameter Space Dimensionality

A rough number of templates needed to cover a given region of parameter space is

$$N \propto (1-m)^{-d/2} \int \sqrt{\det g}\, \mathrm{d}\lambda^d,$$

where m is minimal match of the template bank, d is the dimensionality of the parameter space, $\det g$ is the determinant of the metric of the parameter space parametrized by $\{\lambda\}$, and λ are the parameters we use to parametrize the space.



The number of basis vectors required to reconstruct a given region of parameter space depends on the density of the input template banks and the required accuracy. However, for a fixed accuracy, above a certain density, the number of basis vectors stabilizes. We find the point at which this stabilization occurs is at smaller minimal matches for the higher dimensional template bank that allows non-zero χ (i.e., $d = 3$).

Conclusions

In this investigation, we have found the stochastic template placement algorithm can effectively be used for constructing template banks to search for (non-)spinning BBH GWs. We also find that SVD of these template banks can enclose the signal manifold of BBH GW waveforms. Interestingly, the point at which the number of basis vectors stabilizes decreases for increasing dimensionality of the template bank in the simulations we have performed.

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