EVOLUTION OF ACCRETING NEUTRON STARS NEAR GRAVITATIONAL RADIATION EQUILIBRIUM

Robert V. Wagoner & Joseph F. Hennawi

Dept. of Physics, Stanford University

wagoner@stanford.edu

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EVOLUTION OF ACCRETING NEUTRON STARS

Consider a fluid displacement

$$\vec{\xi} = \vec{f}(r,\theta)e^{i(m\phi+\sigma t)} \sim \alpha R\ , \qquad m \neq 0\ , \qquad \alpha \ll 1\ .$$

It can induce other ($m = 0, 1, \ldots$) perturbations of order α^2. Friedman & Schutz (1978a) showed that any mass and entropy conserving perturbation of a Newtonian equilibrium star with angular momentum J_* produces a total angular momentum $J = J_* + \int \rho \Delta v_\phi dV + J_c + \mathcal{O}(\alpha^3)$. Thus we shall adopt the decomposition

$$J = J_*(M,\Omega) + (1-K_j)J_c\ , \qquad J_c = -K_c\alpha^2 J_*\ . \tag{1}$$

The canonical angular momentum obeys the relation

$$dJ_c/dt = 2J_c[(F_g(M,\Omega) - F_v(M,\Omega,T)]\ , \tag{2}$$

where F_g is the gravitational radiation growth rate and F_v is the viscous damping rate.

Conservation of angular momentum then requires that

$$dJ/dt = 2J_cF_g + \dot{J}_a(t)\ , \tag{3}$$

where $\dot{J}_a = j_a\dot{M}$ is the rate of accretion of angular momentum.

Combining these equations then gives

$$\frac{1}{\alpha}\frac{d\alpha}{dt} = F_g - F_v + [K_jF_g + (1-K_j)F_v]K_c\alpha^2 - \left(\frac{j_a}{2J_*}\right)\dot{M}(t)\ , \tag{A}$$

$$\left(\frac{I_*}{J_*}\right)\frac{d\Omega}{dt} = -2[K_jF_g + (1-K_j)F_v]K_c\alpha^2 + \left[\frac{(j_a - j_*)}{J_*}\right]\dot{M}(t)\ ; \tag{B}$$

where $I_*(M,\Omega) = \partial J_*/\partial\Omega$ and $j_*(M,\Omega) = \partial J_*/\partial M$.

Thermal energy conservation for the star gives

$$\int \frac{\partial T}{\partial t} c_v dV \equiv C_v(T)\frac{dT}{dt} = 2\tilde{E}_c F_v(T_v) + K_n\langle \dot{M}\rangle c^2 - L_\nu(T_\nu) - L_\gamma(T_s)\ , \quad (4)$$

where the rotating frame canonical energy $\tilde{E}_c = -(\sigma/m + \Omega)J_c = K_e \Omega J_* \alpha^2$. In what follows we shall assume that thermal conductivity timescales are short enough to give the relations $T_v(T)$ and $T_\nu(T)$. The nuclear heating rate involves an average mass accretion rate.

The mass accretion rate can be estimated from accretion energy conservation: $L_{\gamma,acc} \approx (3GM/4R)\dot{M}(t)$.

Since we are only considering conditions in which $\alpha^2 \ll 1$, we will have two slower varying functions (than α and T):

$$\Omega(t) = \Omega_0[1 + \zeta_\Omega(t)]\ ,\ |\zeta_\Omega| \ll 1\ ; \qquad M(t) = M_0[1 + \zeta_M(t)]\ ,\ |\zeta_M| \ll 1\ .$$

For any property Q_* of the equilibrium star, let $Q_0 \equiv Q_*(M_0, \Omega_0)$. Then to lowest order in ζ_Ω, equation (B) becomes

$$\frac{1}{\Omega}\frac{d\Omega}{dt} \cong \frac{d\zeta_\Omega}{dt} = -2[K_j F_g + (1 - K_j)F_v]K_c'\alpha^2 + F_a(t)\ , \qquad (B')$$

where $K_c' = (J_0/I_0\Omega_0)K_c$ and $F_a \equiv (j_a - j_0)(M_0/I_0\Omega_0)d\zeta_M/dt$.

TEMPERATURE INDEPENDENT VISCOSITY

We now consider conditions in which $\partial F_v/\partial T = 0$, and let

$$F_g(\Omega_0, M_0) = F_v(\Omega_0, M_0) \equiv F_0 \implies \Omega_0,\ M_0\ .$$

(a) Evolution of α

Averaging over the fluctuations in the mass accretion rate and neglecting the smaller contribution of $\partial(F_g - F_v)/\partial M$, equations (A) and (B$'$) give

$$\frac{d^2x}{dt^2} - \gamma(x)\frac{dx}{dt} + \frac{dV}{dx} = 0\ , \qquad x \equiv \ln\alpha\ , \tag{5}$$

where

$$\gamma(x) = 2K_c F_0 e^{2x}\ , \qquad V(x) = (p-n)F_0[K_c' F_0 e^{2x} - \langle F_a\rangle x]\ ,$$

with $F_g \propto \Omega^p$, $F_v \propto \Omega^n$. The sign of the damping term is opposite to that of Levin(1999). We assume a superfluid core, with mutual friction [$F_v = \tilde{\tau}_{MF}^{-1}(\Omega/\sqrt{\pi G\langle\rho\rangle})^5$, Lindblom & Mendell 2000] dominating the viscosity of $l = m = 2$ r-modes ($p = 6$). The electron scattering viscosity is less if

$$\frac{\Omega}{\Omega_{max}}\left(\frac{T}{10^9\ \mathrm{K}}\right)^{2/5} \gtrsim 0.2\left(\frac{\tilde{\tau}_{MF}}{4\times10^3\ \mathrm{sec}}\right)^{1/5}\ .$$

The governing radiation-viscous and accretion time scales are

$$\tau_0 \equiv \frac{1}{F_0} \sim 10^4\ \mathrm{sec}\ , \qquad \tau_a \equiv \frac{1}{\langle F_a\rangle} \sim 0.15\left(\frac{\Omega_0}{\Omega_{max}}\right)\frac{M_0}{\langle\dot{M}\rangle} \gtrsim 5\times10^6\ \mathrm{yr}\ ,$$

corresponding to the choice $\Omega_0 = 0.25\sqrt{\pi G\langle\rho\rangle} \approx 0.38\Omega_{max}$ ($\tilde{\tau}_{MF} \approx 13$ sec).

The minimum of the potential $V(\alpha)$ (Figure 1) gives an equilibrium at an amplitude $\alpha_0 \cong [\tau_0/(2K_c'\tau_a)]^{1/2} \lesssim 2\times10^{-5}$, but there is significant overshoot (Figure 2) to

$$(\alpha_{max}/\alpha_0)^2 \cong 2\ln\alpha_{min}^{-1}\ .$$

The seed amplitude is taken to be α_{min}. The period of the oscillation and the fraction during which $\alpha > \alpha_0$ are given by

$$P \cong \sqrt{8\tau_0\tau_a \ln \alpha_{min}^{-1}} \gtrsim 300 \text{ yr} , \qquad \frac{\Delta P}{P} \cong \frac{\ln(2\ln\alpha_{min}^{-1})}{4\ln\alpha_{min}^{-1}} .$$

We have employed a slowly increasing energy $E(t) \cong F_0\langle F_a\rangle \ln\alpha_{min}^{-1}$ by writing equation (5) in the form

$$\frac{dE}{dt} \equiv \frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] = \gamma(x)\left(\frac{dx}{dt}\right)^2 .$$

It is found that this energy increases by a fractional amount $\Delta E/E \cong (32K_c^2/9K_c')^{1/2}\alpha_{max}$ per cycle.

(b) Evolution of Ω

To lowest order in ζ_Ω, equation (B$'$) now becomes

$$\frac{1}{\Omega}\frac{d\Omega}{dt} \cong \frac{d\zeta_\Omega}{dt} \cong -2K_c'F_0\alpha^2 + F_a(t) .$$

Note the spin-up of Ω at the mass accretion rate and the more rapid gravitational radiation spin down when α becomes large (Figure 3). Averaging over the period of the oscillation, we find that $\langle\alpha^2\rangle \cong \alpha_0^2$. We see that this is also the value of α^2 when Ω reaches its maximim or minimum. Finally, integration during the spin-up phase shows that

$$\zeta_\Omega(\text{max}) \cong -\zeta_\Omega(\text{min}) \cong \tfrac{1}{2}\langle F_a\rangle P .$$

(c) Evolution of T

The initial temperature T_0 is determined from the balance of nuclear heating and total luminosity, just before α begins to grow. We consider mass accretion rates of the strongest X-ray sources, in which case crust neutrino bremsstrahlung should dominate the luminosity $[L_\nu = L_0(T/T_0)^6]$. Since $C_v = C_0 T/T_0$, equation (4) then gives

$$\tilde{T}\frac{d\tilde{T}}{dt} \cong K_r F_0 \alpha^2 - F_c(\tilde{T}^6 - 1) \ , \qquad \tilde{T} \equiv T/T_0 \ , \tag{6}$$

where $K_r = 2K_e\Omega_0 J_0/C_0T_0 \sim E_{rot}/E_{thermal} \sim 10^5$ and $F_c = L_0/C_0T_0 \sim (10^3\mathrm{yr})^{-1}$ is a cooling rate.

The evolution of $T(t)$ is shown in Figure (4a), and that of the heating and cooling terms of equation (6) in Figure (4b). Averaging over one period of oscillation of the temperature, equation (6) gives $\langle\tilde{T}^6\rangle = K_r(F_0/F_c)\langle\alpha^2\rangle + 1$.

If instead we integrate from $\tilde{T}_{min}$ to $\tilde{T}_{max}$ and assume that $\tilde{T}^2_{max} \gg \tilde{T}^2_{min}$, the heating dominates. We can then use the behavior

$$\alpha \cong \alpha_{max}/\cosh[(2K_c')^{1/2}\alpha_{max}(t - t_{max})/\tau_0]$$

near α_{max} to find that

$$\tilde{T}^2_{max} \approx (8/K_c')^{1/2} K_r \alpha_{max} \ .$$

Finally, we integrate from $\tilde{T}_{max}$ to $\tilde{T}_{min}$ and also assume that $\tilde{T}^6_{min} \gg 1$. With cooling dominating, we obtain

$$\tilde{T}^{-4}_{min} \approx 4F_cP \ .$$

We have also taken the time interval to be close to the period P of the oscillation, which is a good approximation since the heating interval can be seen to be much less than P.

BEHAVIOR NEAR EQUILIBRIUM

We define the equilibrium states X_0^i of our dynamical variables

$$X^i(t) = \{\alpha, \Omega, T\} = X_0^i[1 + \zeta^i(t)] , \qquad |\zeta^i| \ll 1 , \quad |\alpha_0| \ll 1$$

by the vanishing of the evolution equations (2), (3), (4). [Note that T_0 will be larger than that defined previously, and now $C_0 = C_v(T_0)$.] The function $M(t)$ is separately specified, and we employ the averaged accretion rate. The evolution equations are then

$$d\zeta^i/dt = A^{ij}\zeta^j , \qquad \zeta^i \propto \exp(\lambda t) , \quad ||A^{ij} - \lambda\delta^{ij}|| = 0 .$$

Assume now that $|\partial F_v/\partial T| \sim F_v/T_0$, etc.

The coefficients of the eigenvalue equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ are

$$\begin{aligned}
a_2 &\cong \frac{1}{C_0}\left[\left(\frac{\partial L}{\partial T}\right)_0 - 2(\tilde{E}_c)_0\left(\frac{\partial F_v}{\partial T}\right)_0\right] \sim K_r\alpha_0^2 F_0 , \\
a_1 &\cong \frac{4(\tilde{E}_c)_0 F_0}{C_0}\left(\frac{\partial F_v}{\partial T}\right)_0 \sim K_r\alpha_0^2 F_0^2 , \\
a_0 &\cong \frac{4K_c\Omega_0\alpha_0^2 F_0}{C_0}\left[\frac{\partial(F_g - F_v)}{\partial\Omega}\right]_0\left(\frac{\partial L}{\partial T}\right)_0 \\
&\quad - \frac{16K_c(\tilde{E}_c)_0\alpha_0^2 F_0^2}{C_0}\left(\frac{\partial F_v}{\partial T}\right)_0 \sim K_r\alpha_0^4 F_0^3 .
\end{aligned}$$

We have used the fact that the cooling rate $F_c \sim K_r\alpha_0^2 F_0$.

Now we also employ the inequalities $(\tilde{E}_c)_0(\text{mode energy})/E_{thermal} \sim K_r\alpha_0^2 \lesssim 10^{-4} \Rightarrow |a_1| \gg a_2^2$ and $E_{rot}/E_{thermal} \sim K_r \gg 1$ to obtain the eigenvalues

$$\lambda_{1,2} \cong -a_2/2 \pm \sqrt{-a_1} , \qquad \lambda_3 \cong -a_0/a_1 .$$

We have used the fact that $|\lambda_3| \sim \alpha_0^2 F_0 \ll |\lambda_1| \sim |\lambda_2|$.

We now examine the two relevant possibilities. For cases such as superfluid shear viscosity (produced by $e-e$ scattering), with $\tilde{E}_c > 0$,

$$a_1 \propto (\tilde{E}_c)_0 (\partial F_v/\partial T)_0 < 0 \Longrightarrow \lambda_{1,2} \cong \pm\sqrt{-a_1} \sim K_r^{1/2} \alpha_0 F_0 \ .$$

Thus this equilibrium is unstable, with a growth rate λ_1 that is of the same magnitude as found by Levin (1999).

The other possibility $a_1 > 0 \Longrightarrow \lambda_{1,2} \cong -a_2/2 \pm i\sqrt{a_1}$. Thus stability requires that $a_0 > 0$ and $a_2 > 0$. From their relations above, we see that this means that the variation in the cooling rate with temperature must be greater than both 1 and $2/(p-n)$ times the variation in the viscous heating rate with temperature.

THERMAL TIME SCALES

Define the thermal time scales τ_i from thermal energy conservation:

$$\frac{1}{\Delta V}\int \frac{1}{T}\frac{\partial T}{\partial t} dV = \frac{1}{\Delta V}\int \frac{1}{c_v T}\left[\nabla\cdot(K\nabla T) + \sum_a \dot{\varepsilon}_a\right] dV$$
$$= \pm\frac{1}{\tau_{cond}} + \frac{1}{\tau_{visc}} + \frac{1}{\tau_{nucl}} - \frac{1}{\tau_\nu}$$

for each region of the neutron star.

A) Core

$$\tau_{cond} \sim 0.4 \text{ yr}\ , \quad \tau_{visc} \sim 10^{-8}/\langle\alpha^2\rangle \text{ yr}\ .$$

B) Inner crust

$$\tau_{cond} \sim 60 \text{ yr}\ , \quad \tau_{visc} \sim 10^{-9}/\langle\alpha^2\rangle \text{ yr}\ , \quad \tau_{nucl} \sim 5 \text{ yr}\ , \quad \tau_\nu \sim 6 \text{ yr}\ .$$

C) Outer crust

$$\tau_{cond} \sim 10 \text{ yr}\ .$$

UNCERTAINTIES IN NEUTRON STAR PHYSICS

1) The neutrino luminosity $L_\nu(T)$ (especially in the crust).

2) The heat capacity $C_v(T)$ and equation of state.

3) The superfluid transition temperature $T_c(\sim 6 \times 10^{8-9}$ K).

4) Thermal conductivities.

5) The viscous damping rate $F_v(\Omega, T)$.

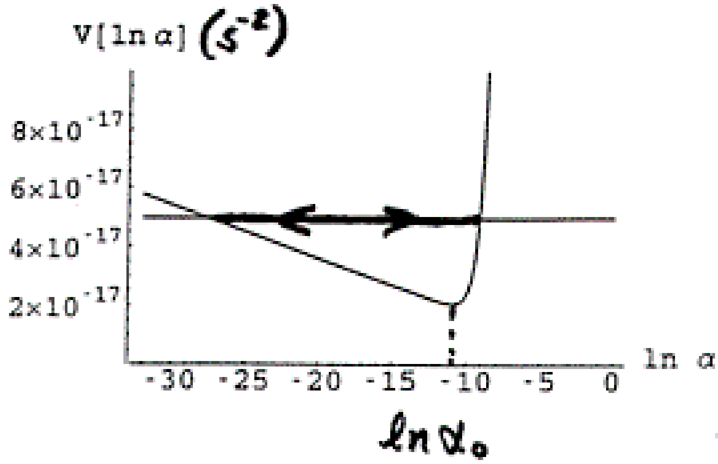
6) The fraction K_n of the accreted rest-mass energy that heats the crust.

7) The angular momentum accretion rate.

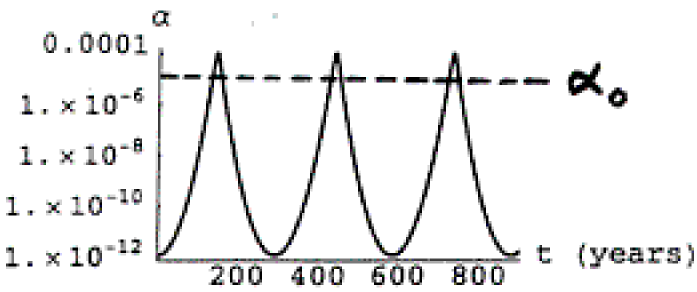
8) The relation between gravitational wave frequency and neutron star angular velocity (Ω/Ω_{max} and general relativistic corrections).

Approach: embed the range of uncertainties within a parameterized description.

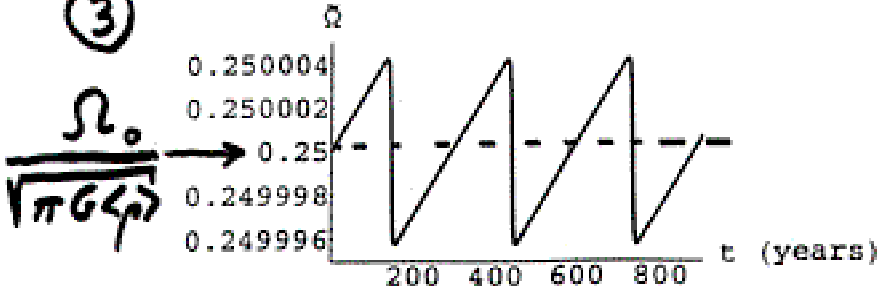
①


V[ln α] (s^-2)
8×10^-17
6×10^-17
4×10^-17
2×10^-17
ln α
-30 -25 -20 -15 -10 -5 0
ln α₀

②


α
0.0001
1.×10^-6
1.×10^-8
1.×10^-10
1.×10^-12
α₀
t (years)
200 400 600 800

③


Ω
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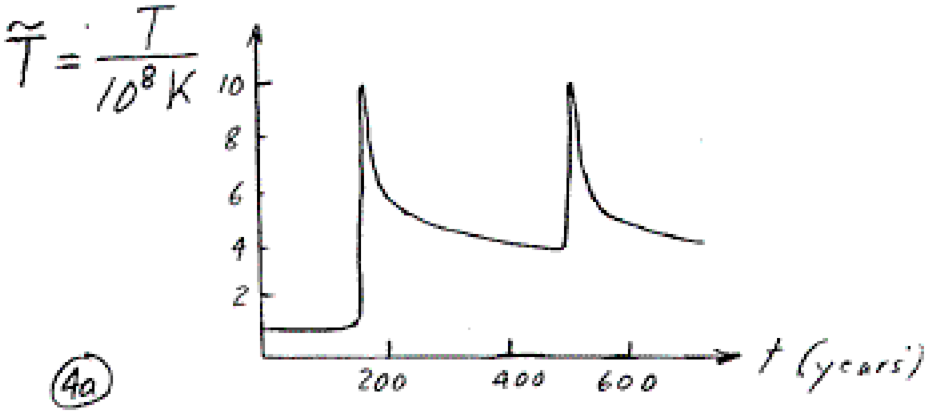
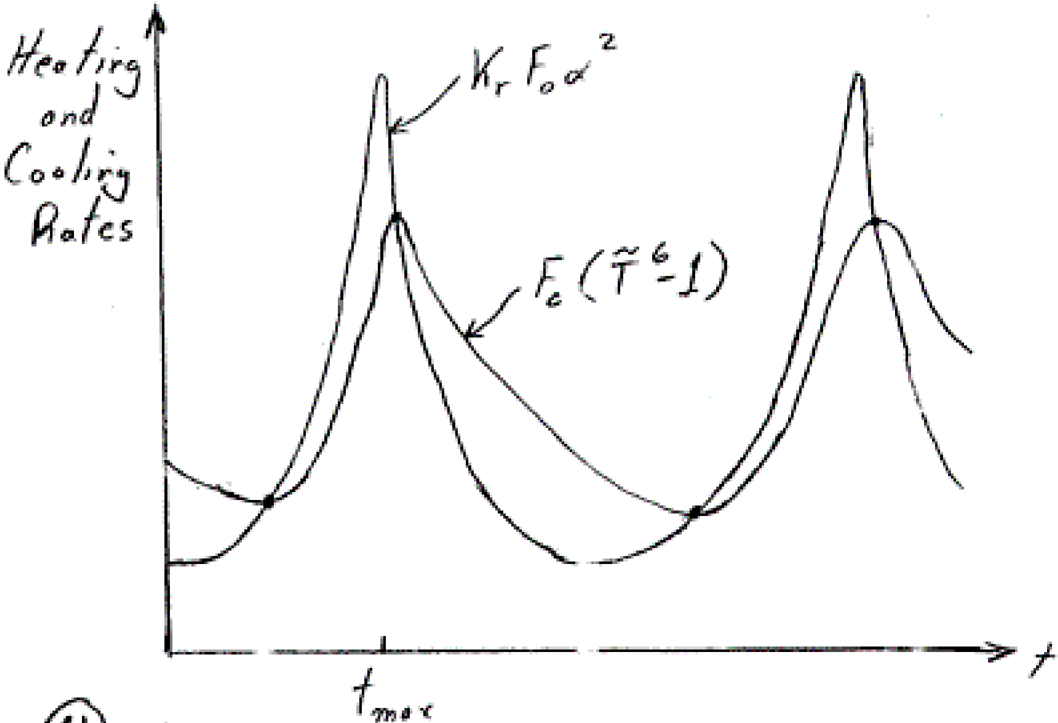
0.250002
0.25
0.249998
0.249996
Ω₀ / √(πG⟨ρ⟩)
t (years)
200 400 600 800



$\tilde{T} = \frac{T}{10^8 K}$
10
8
6
4
2
200
400
600
t (years)

(4a)



Heating and Cooling Rates
$K_r F_o \alpha^2$
$F_c(\tilde{T}^6 - 1)$
t_{max}
t

(4b)