# Optical-mechanical resonances in signal-recycled interferometric gravitational-wave detectors

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based on:

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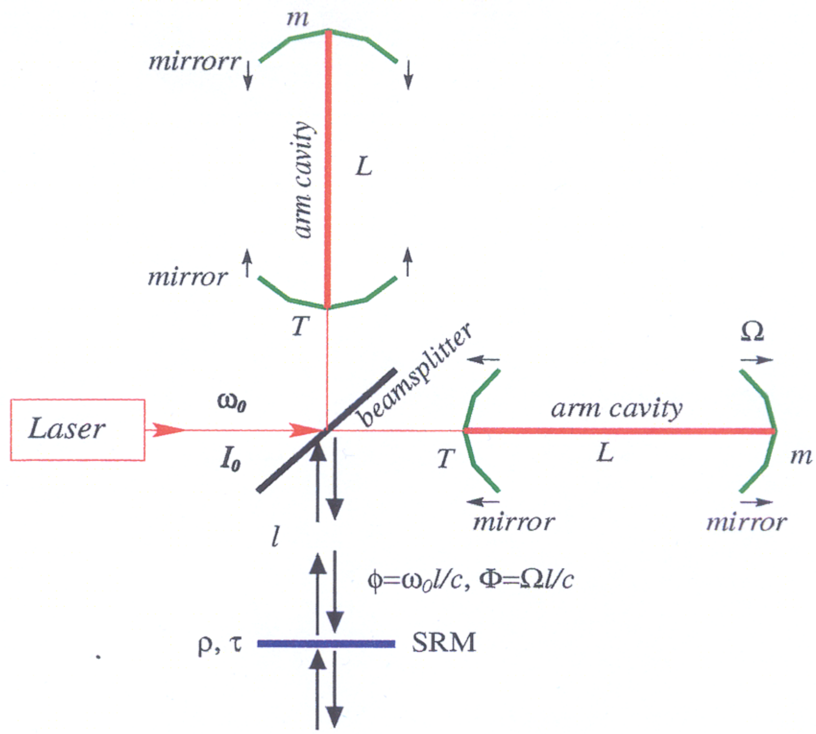
# Optical Configuration




SR Interferometer

| | | |
|---|---|---|
| Carrier ang. freq. | $\omega_0$ | $1.8 \cdot 10^{15}\mathrm{s}^{-1}$ |
| Side-band ang. freq. | $\Omega$ | $\sim 2\pi \times (10 - 10^4)\mathrm{Hz}$ |
| Arm-cavity length | $L$ | 4km |
| ITM power reflectivity | $T$ | 0.033 |
| Arm-cavity linewidth | $\gamma$ | $\frac{Tc}{4L} \approx 2\pi \times 100 \cdot \mathrm{Hz}$ |
| SR-cavity length | $l$ | $\sim 10\mathrm{m}$ |
| SRM reflectivity & transmissivity | $\rho\,,\tau$ | $\rho^2 + \tau^2 = 1$ |
| SR detuning for carrier | $\phi$ | $\frac{\omega_0 l}{c}$ |
| Additional phase for sideband | $\Phi$ | $\frac{\Omega l}{c}$ |
| mirror mass | m | 30kg |
| SQL | $h^2_{SQL}$ | $\frac{8\hbar}{m\,\Omega^2 L^2} \sim 4 \cdot 10^{-48}\mathrm{Hz}^{-1}$@100Hz |
| Laser power at BS | $I_0$ | $\sim 10\mathrm{kW}$ |
| Laser power for Conv. IFO to reach SQL at $\Omega=\gamma$ | $I_{\mathrm{SQL}}$ | $\frac{m\,L^2\,\gamma^4}{4\omega_o} \sim 10\mathrm{kW}$ |

# Approximations made

[Kimble, Levin, Matsko, Thorne & Vyatchanin 00; BC 00-01]

## Arm cavities [KLMTV, 00]:

• Linear order in antisymmetric mode displacement $x(t)$

• Changes in $x(t)$ during one-way propagation in arm cavity neglected:

$\Leftrightarrow$ Final result up to leading order in $\Omega L/c$

$\Omega \sim \gamma = \frac{Tc}{4L} \Rightarrow \frac{\Omega L}{c} \sim \left(\frac{T}{4}\right)\left(\frac{\Omega}{\gamma}\right)$

$\Leftrightarrow$ Up to leading order in $T$

## SR cavity [BC, 00-01]:

• $l \sim 10\text{m}$, i.e., much smaller than arm-cavity length $\Rightarrow$ we ignore the phase gained by GW side-band frequency $\Phi = \frac{\Omega l}{c}$ traveling in SR cavity. [Also notice the convention in defining $l$.]

# Quadrature fields and Two-photon Formalism

**[Caves & Schumaker, Schumaker & Caves 85; Kimble, Levin, Matsko, Thorne & Vyatchanin 00; BC 00-01]**

## Classical Quadrature Fields:

• Electric field within $-2\omega_0 \to 2\omega_0$ [ most cases it's concentrated in a narrow band around $\omega_0$ ]:

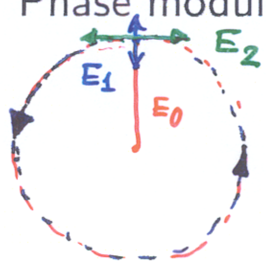
$$E(t) = E_1(t)\cos(\omega_0 t) + E_2(t)\sin(\omega_0 t)$$

$E_1(t)$ and $E_2(t)$: quadrature fields.

• If a monochromatic "carrier field" $E_0 \cos(\omega_0 t)$ superimposed:

$$\begin{aligned} E_{\rm total}(t) &= [E_0 + E_1(t)]\cos(\omega_0 t) + E_2(t)\sin(\omega_0 t) \\ &\approx E_0 \left(1 + \frac{E_1(t)}{E_0}\right) \cos\left[\omega_0 \left(1 + \frac{E_2(t)}{E_0}\right)\right] \end{aligned}$$

$E_1(t) \to$ Amplitude modulation, $E_2(t) \to$ Phase modulation




## Quantization of Quadrature Fields:

• Operator Expansion

$$\hat{E}_j(t) \Rightarrow \int_0^\infty (\hat{a}_j\, e^{-i\Omega t} + \hat{a}_j^\dagger\, e^{i\Omega t})\,\frac{d\Omega}{2\pi} \qquad j = 1, 2$$

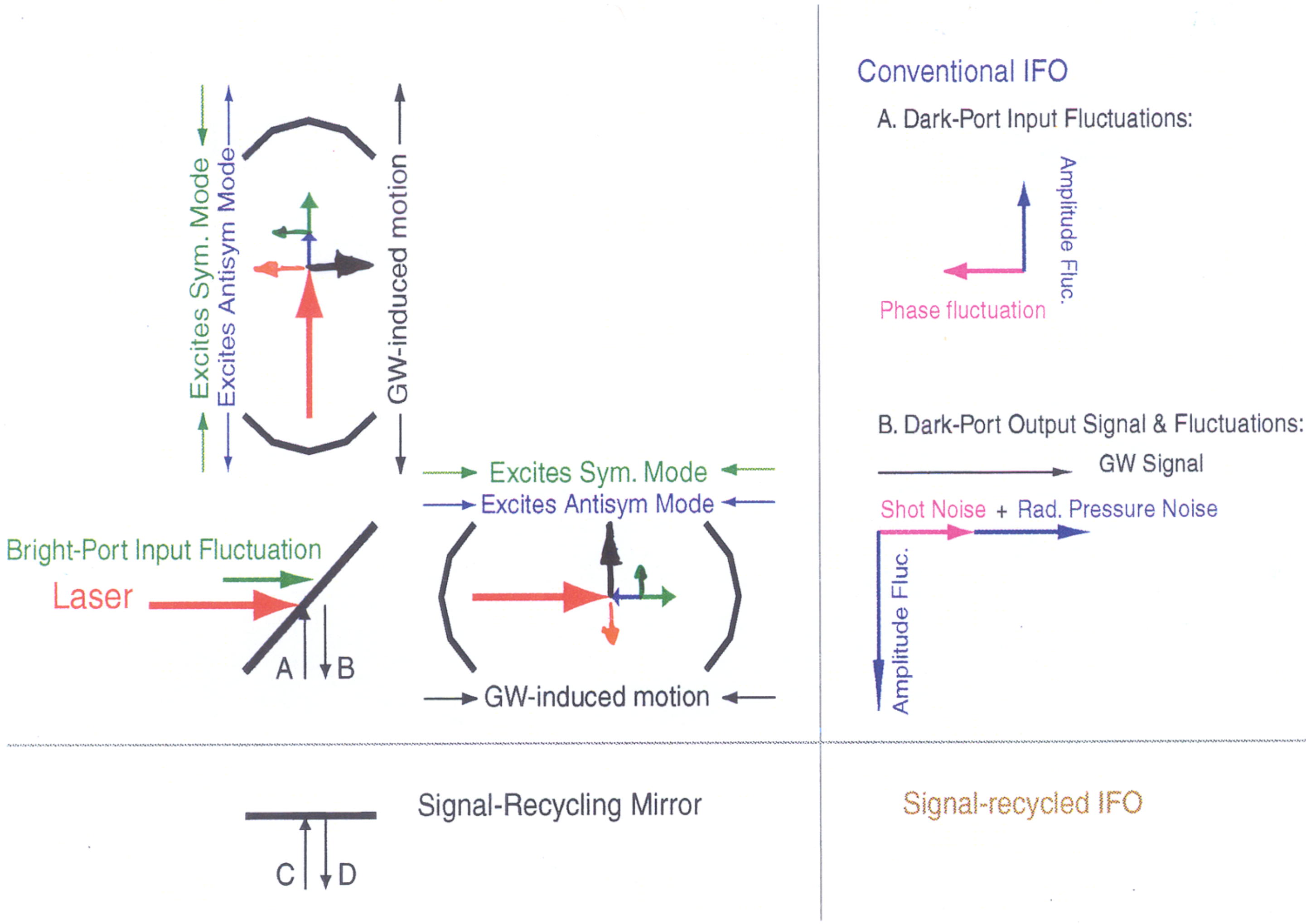
$$\begin{aligned} [\hat{E}_1(t), \hat{E}_1(t')] &= 0 \qquad [\hat{E}_2(t), \hat{E}_2(t')] = 0 \\ [\hat{E}_1(t), \hat{E}_2(t')] &\sim i\delta(t - t') \end{aligned}$$

Similarity between amplitude/phase and position/momentum

• Equations of motion of quantum operators identical to classical ones

# Quadrature Description of LIGO Interferometers

**[Caves, 80s; Kimble, Levin, Matsko, Thorne & Vyatchanin 00; BC 00-01]**




KLMTV: $E_{\rm out}^{\rm conv.} \Leftrightarrow E_{\rm in}^{\rm conv.} + {\rm Signal}$ (B⇔A & Signal)

BC [Based on KLMTV]:

$$E_{\rm out}^{\rm SR} \Leftrightarrow E_{\rm in}^{\rm SR} + {\rm Signal} \text{ (D⇔C \& Signal)}$$

# Noise Spectral Density

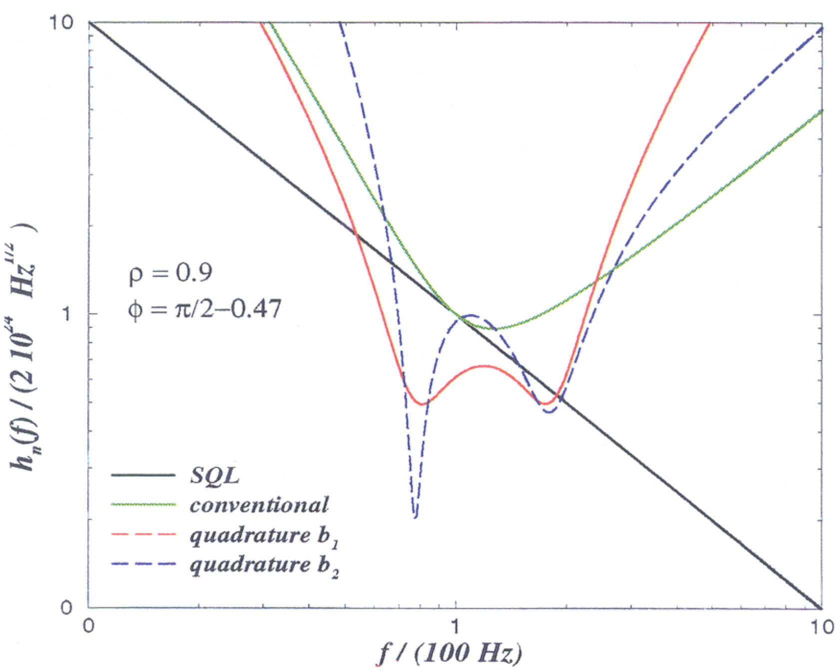
**[Kimble, Levin, Matsko, Thorne & Vyatchanin 00; BC 00-01]**

$\text{Output}(\Omega) = h_{\text{GW}}(\Omega) + \hat{h}_n(\Omega)$ [$\hat{h}_n(\Omega)$ written in terms of input quadrature operators.]

⇒Spectral density [as Fourier transform of autocorrelation function (assuming stationarity) ]:

$$\begin{aligned} & \frac{1}{2} 2\pi\, \delta(\Omega - \Omega')\, S_h(f) \\ = \quad & \langle \text{in} | \hat{h}_n(\Omega)\, \hat{h}_n^\dagger(\Omega') | \text{in} \rangle_{\text{sym}} \\ \equiv \quad & \frac{1}{2} \langle \text{in} | \hat{h}_n(\Omega)\, \hat{h}_n^\dagger(\Omega') + h_n^\dagger(\Omega')\, h_n(\Omega) | \text{in} \rangle \end{aligned}$$

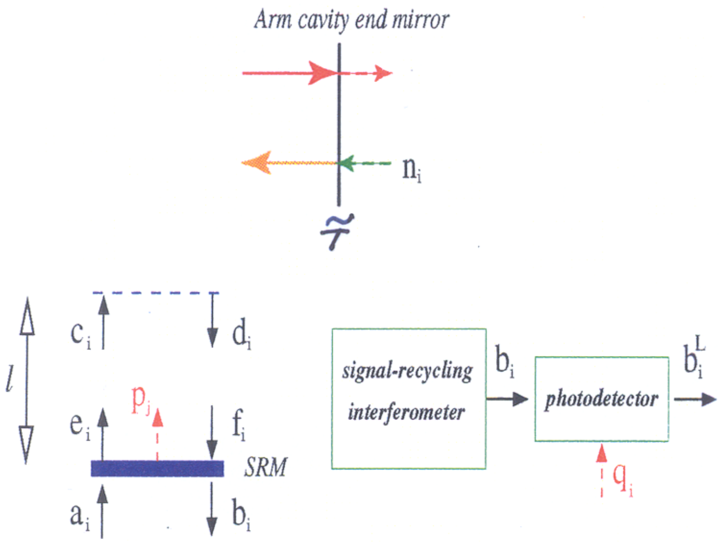
Vacuum state : $\langle 0_a | a_i\, a_{j'}^\dagger | 0_a \rangle_{\text{sym}} = \frac{1}{2} 2\pi\, \delta(\Omega - \Omega')\, \delta_{ij}$

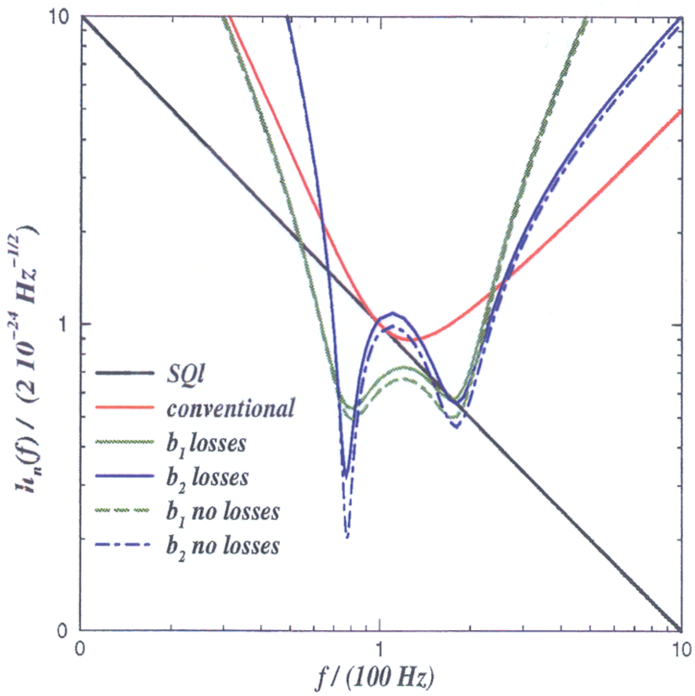



✱ When input is vacuum, classical amplitude/phase fluctuation at SR dark-port also OK. Quantum optical technique *is not* a must.

# Inclusion of optical losses [KLMTV 00; BC 00-01]

## Loss ⇔ Dissipation [smaller signal] + Fluctuation [additional noise]




- Arm cavity: $\mathcal{L} \sim 200 \times 10^{-6}$ $\rightarrow$ loss coefficient in arm-cavity round trip. [Finite end-mirror (power) transmissity $\widetilde{T} = \mathcal{L}$.] Noise operator $n_j$ in vacuum state.

- SR cavity: $\lambda_{\rm SR} \sim 0.02$ $\rightarrow$ fraction of photons lost at each bounce off SR mirror. Additional operator in vacuum state $p_i$ leaking in:

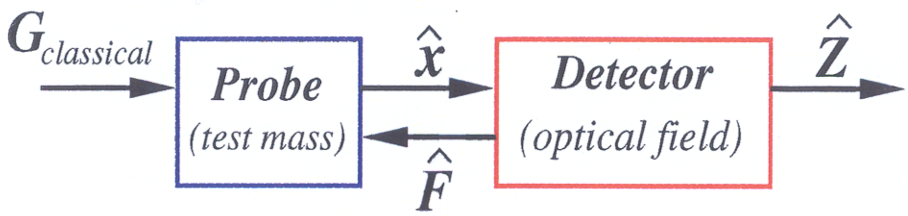
$$e_i = \sqrt{1-\lambda_{\rm SR}} \underbrace{(\tau\, a_i + \rho\, f_i)}_{\text{reflected field when no loss}} + \sqrt{\lambda_{\rm SR}}\, p_i$$

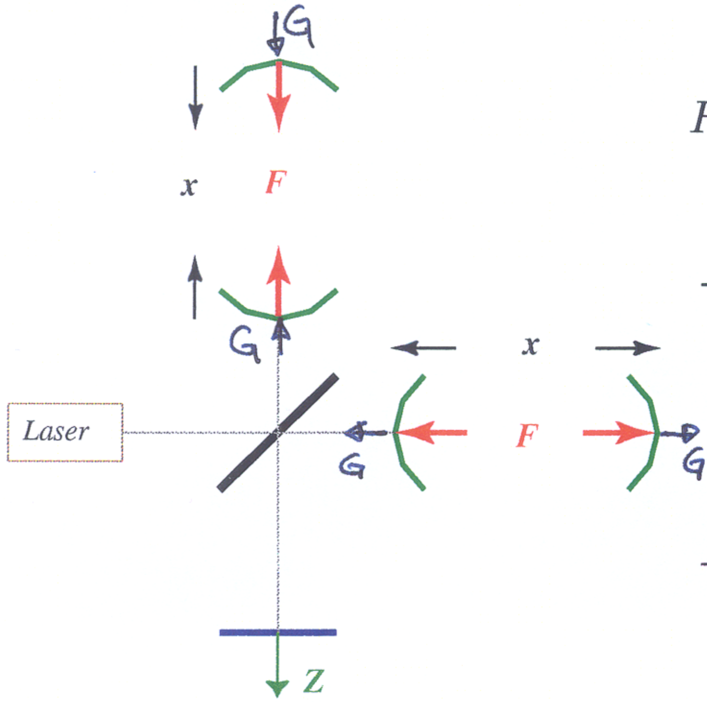
- Photo detection: $\lambda_{\rm PD} \sim 0.1$ $\rightarrow$ photodetection efficiency 90%. Additional operator in vacuum state $q_i$ leaking in:

$$b_i^{\rm L} = \sqrt{1-\lambda_{\rm PD}}\, b_i + \sqrt{\lambda_{\rm PD}}\, q_i$$

# Interferometers as (quantum) linear system

**[KLMTV 00; BC 00-01]**




$$H = H_{\text{opt.}} + H_{\text{mirror}} - \widehat{x}\widehat{F} - \widehat{x}G$$

| | |
|---|---|
| $\widehat{x}$ | Antisym. mode coordinate |
| $\widehat{F}$ | Rad. Pressure force |
| $\widehat{Z}$ | Output field |
| $G$ | Classical GW force |

| | | |
|---|---|---|
| Free Hamiltonians: | $H_{\text{opt}}$ | Optical fields with mirrors fixed |
| | $H_{\text{mirror}}$ | Free mirrors |
| Coupling terms: | $-\widehat{x}\widehat{F}$ | Rad. Pressure/Generation of GW sideband |
| | $-\widehat{x}G$ | GW force |

# (Linear) equations of Motion [Braginsky & Khalili 92; BC 00-01]

Equations of motion of mirror/optical fields under free Hamiltonians (free mass/fixed mirrors) [well-known]:

• Mirror: $\widehat{x}^{(0)}(t) = \widehat{x}_0 + \frac{\widehat{p}_0}{\mu}\, t$

• Optical fields: written in terms of dark-port input quadratures from the dark port.

Coupled evolution: written as **Free evolution + Response**

$$\begin{aligned}
\widehat{Z}(t) &= \widehat{Z}^{(0)}(t) + \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, C_{ZF}(t,t')\,\widehat{x}(t') \\
\widehat{F}(t) &= \widehat{F}^{(0)}(t) + \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, C_{FF}(t,t')\,\widehat{x}(t') \\
\widehat{x}(t) &= \widehat{x}^{(0)}(t) + Lh(t) + \frac{i}{\hbar}\int_{-\infty}^{t} dt'\, C_{xx}(t,t')\,[\widehat{F}(t')]
\end{aligned}$$

Response function (susceptibility): $C_{AB}(t,t') \equiv [\widehat{A}^{(0)}(t), \widehat{B}^{(0)}(t')]$ .

Fourier domain equations of motion [ for time-invariant free Hamiltonians ]:

$$\begin{aligned}
\widehat{Z}(\Omega) &= \widehat{Z}^{(0)}(\Omega) + R_{ZF}(\Omega)\,\widehat{x}(\Omega) \\
\widehat{F}(\Omega) &= \widehat{F}^{(0)}(\Omega) + R_{FF}(\Omega)\,\widehat{x}(\Omega) \\
\widehat{x}(\Omega) &= \widehat{x}^{(0)}(\Omega) + L\,h(\Omega) + R_{xx}(\Omega)\,\widehat{F}(\Omega)
\end{aligned}$$

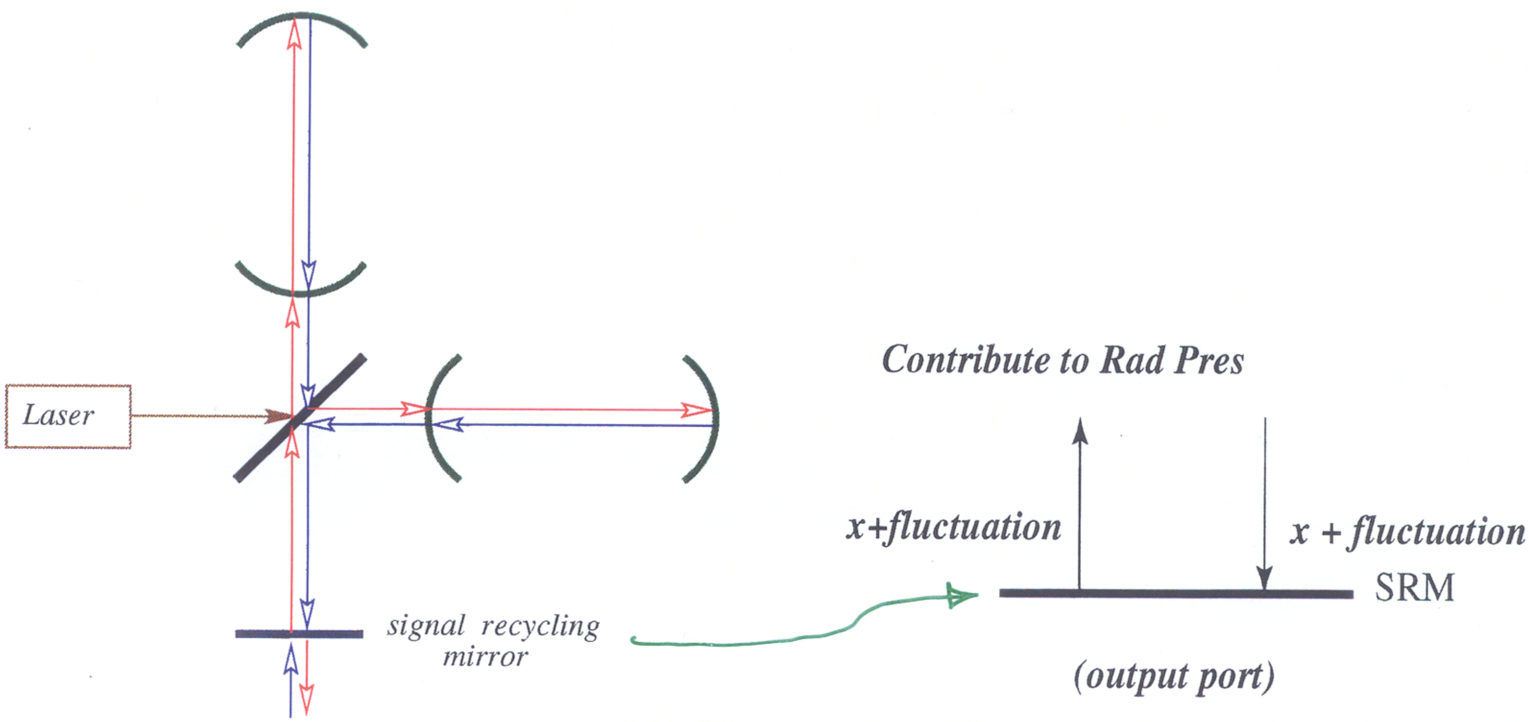
Fourier-domain susceptibility: $R_{AB}(\Omega) = \frac{i}{\hbar}\int_0^{+\infty} d\tau\, e^{i\Omega\tau}\, C_{AB}(0,-\tau)$

# Dynamics of the coupled system

•**LIGO-I:** $\widehat{F}^{(0)} \sim \widehat{E}_1$ at dark-port input, and $R_{FF} = 0 \Rightarrow$

$$\begin{aligned}\widehat{x}(\Omega) &= \widehat{x}^{(0)}(\Omega) + Lh(\Omega) + R_{xx}(\Omega)\widehat{F}^{(0)} \\ \widehat{Z}(\Omega) &= \widehat{Z}^{(0)}(\Omega) + R_{ZF}(\Omega)\widehat{x}(\Omega)\end{aligned}$$

no resonance!




•SR interferometer: [Already known: optical field has one resonance] Optical field fed back into arm cavities also contains displacement information $\Rightarrow$ rad. pressure force depends on test-mass motion$\Rightarrow$ $R_{FF} \neq 0$

$$[\widehat{F}(\Omega) = \widehat{F}^{(0)}(\Omega) + R_{FF}(\Omega)\,\widehat{x}(\Omega)]$$

This will give another resonance.

# Two resonances [Braginsky et al. 97,99; BC, 01]

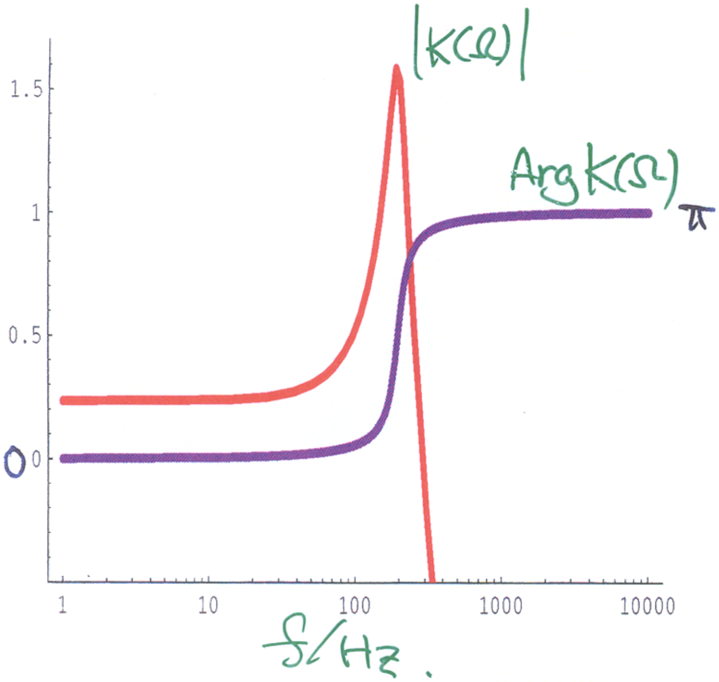
**Double Role of $R_{FF}$**

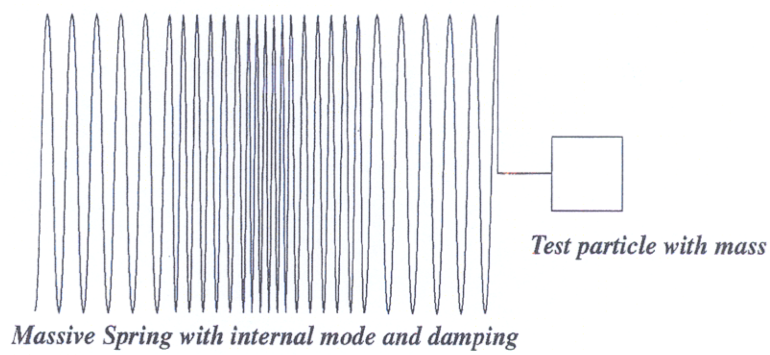
- $\widehat{F}_{\text{Rad.Pressure}} = \widehat{F}^{(0)}(\Omega) + R_{FF}(\Omega)\,\widehat{x}(\Omega)$

[Nonzero response function of optical field → resonance feature due to SR]

- $-\mu\Omega^2\widehat{x}(\Omega) = \widehat{F}^{(0)}(\Omega) + \overbrace{R_{FF}(\Omega)\,\widehat{x}(\Omega)}^{-K(\Omega)\,\widehat{x}(\Omega)} + \text{GW force}$

["spring constant" in Fourier-domain equation of motion of mirrors]




- **Two resonances of the "Optical Spring" detector:**

Low-frequency: $K(\Omega)$ ~ constant real quantity $\Rightarrow \Omega_1 \sim \sqrt{K(0)/\mu} \propto \sqrt{I_{Laser}}$. "Mechanical Resonance"

Higher frequency: resonance in $K(\Omega)$ itself excited $\Rightarrow \Omega_2$. "Optical Resonance"

# Exact Resonance Conditions and Instability

- Resonant frequency ⇔ (complex) frequency at which the mode oscillates: $e^{-i\Omega t}$

$$\mathrm{Im}(\Omega) < 0 \quad \Rightarrow \text{decaying, stable}$$
$$\mathrm{Im}(\Omega) > 0 \quad \Rightarrow \text{growing, unstable}$$

- Equations of motion ⇒ exact resonance condition:

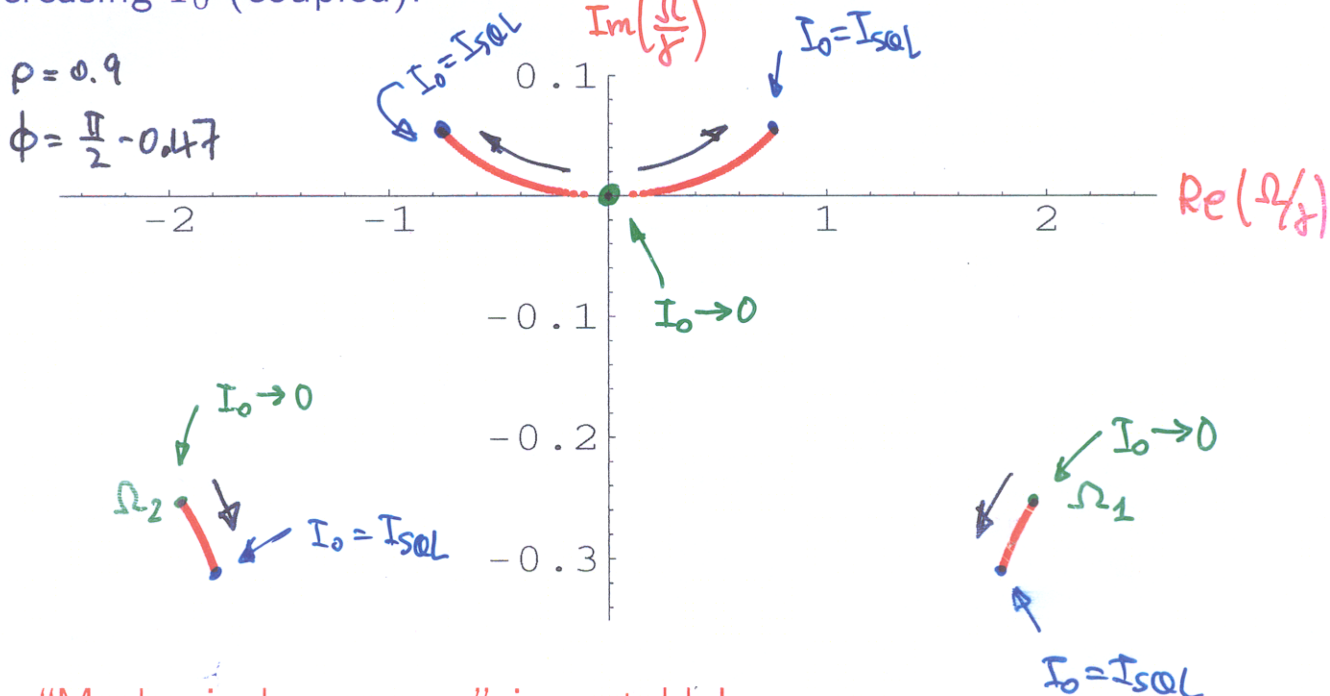
$$1 - R_{xx}(\Omega) R_{\mathrm{FF}}(\Omega) = 0$$
$$\Leftrightarrow \Omega^2(\Omega - \Omega_1)(\Omega - \Omega_2) + \frac{I_0 \gamma^3}{I_{\mathrm{SQL}}}(\Omega_1 - \Omega_2) = 0$$

$\Omega_{1,2} = \pm\Omega_{\mathrm{SR}} - i\epsilon_{\mathrm{SR}}$, the original SR resonant frequencies, stable!

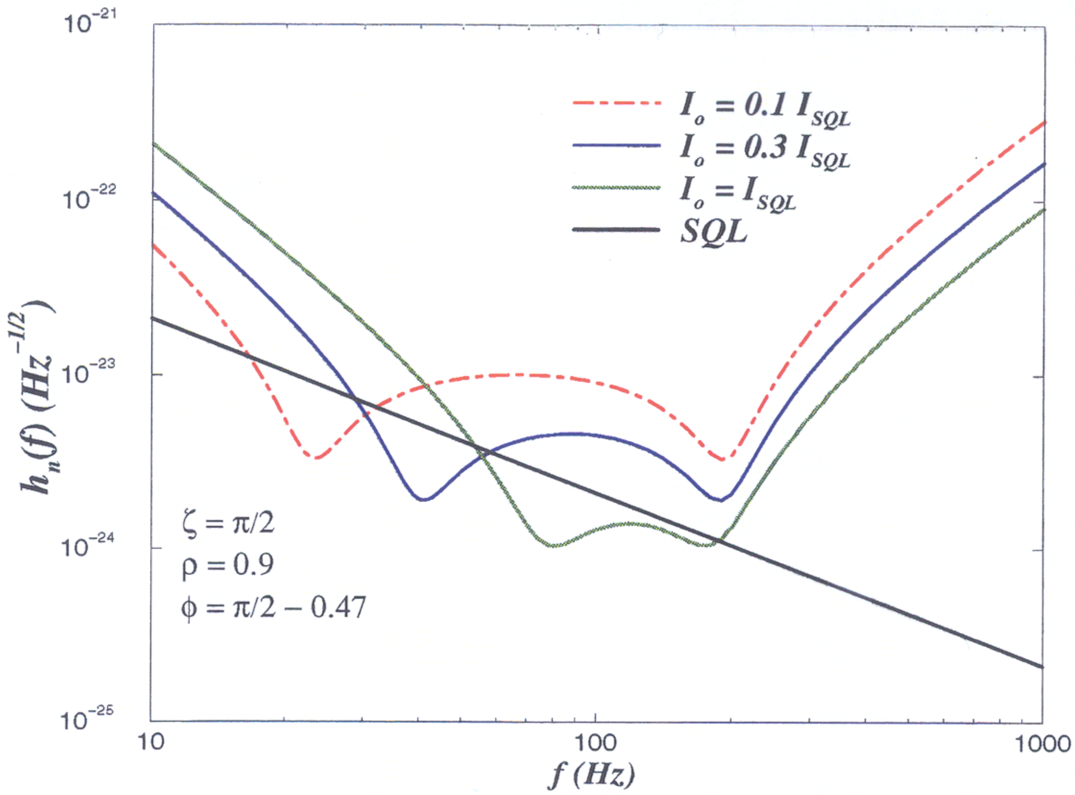
For $I_0 \ll I_{\mathrm{SQL}}$ :(decoupled)

$\Omega = 0$, marginally stable; $\Omega = \Omega_{1,2}$, stable.

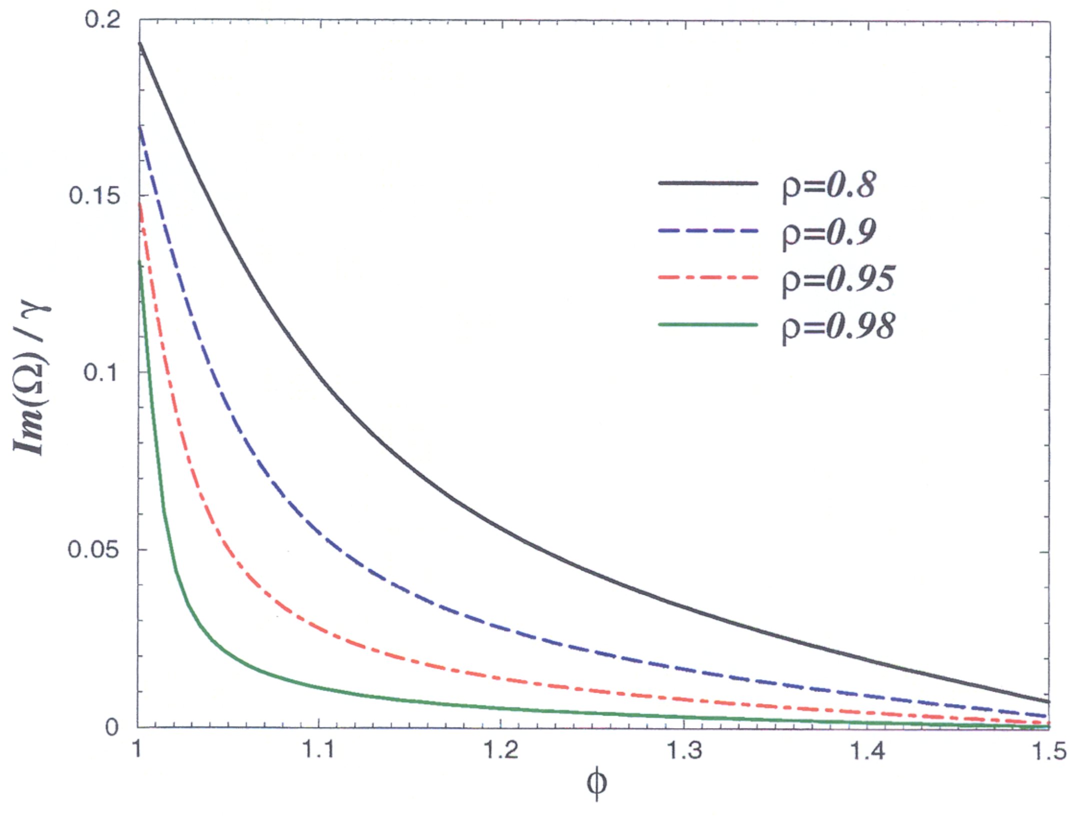
Increasing $I_0$ (coupled):




⇒ "Mechanical resonance" is unstable!

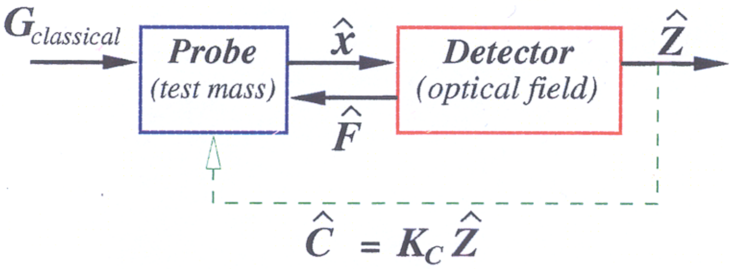


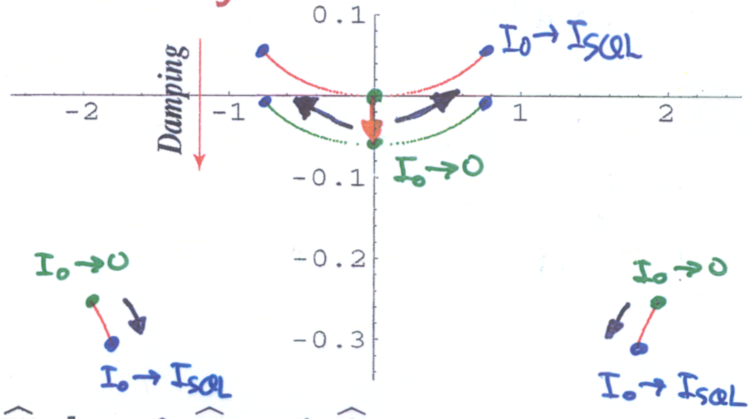
$10^{-21}$
$10^{-22}$
$10^{-23}$
$10^{-24}$
$10^{-25}$
$h_n(f)$ $(Hz^{-1/2})$
$I_o = 0.1\ I_{SQL}$
$I_o = 0.3\ I_{SQL}$
$I_o = I_{SQL}$
SQL
$\zeta = \pi/2$
$\rho = 0.9$
$\phi = \pi/2 - 0.47$
10
100
1000
$f$ (Hz)

0.2
0.15
0.1
0.05
0
$Im(\Omega)/\gamma$
$\rho$=0.8
$\rho$=0.9
$\rho$=0.95
$\rho$=0.98
1
1.1
1.2
1.3
1.4
1.5
$\phi$

# Example of a control system




$$\widehat{H} = [(\widehat{H}_{\mathcal{P}} - \widehat{x}\, G) + \widehat{H}_{\mathcal{D}}] - \widehat{x}\, \widehat{F} - \widehat{x}\, \widehat{C}$$

$$\widehat{C}(t) = \int_{-\infty}^{t} dt'\, K_C(t-t')\, \widehat{Z}(t')$$

Solving in Fourier Domain:

$$\begin{aligned}
\widehat{x} &= \widehat{x}^{(0)} + L\,h + R_{xx}\,[\widehat{F} + \widehat{C}] \\
\widehat{F} &= \widehat{F}^{(0)} + R_{FF}\,\widehat{x} \\
\widehat{Z} &= \widehat{Z}^{(0)} + R_{ZF}\,\widehat{x} \\
\widehat{C} &= K_C \widehat{Z}
\end{aligned}$$

- Naive example: "damping" the effective response of $\widehat{x}$ to $\widehat{F}$. $\widehat{C} \propto \widehat{Z}$ contains $\widehat{x} \Rightarrow$ dynamics changed. Effectively:

$$R^{C}_{xx} = \frac{R_{xx}}{1 - K_C R_{xx} R_{ZF}} \overset{\text{require}}{\Longleftrightarrow} -\frac{4}{m(\Omega + i\lambda)^2}$$

This works for only for $\pi/2 \le \zeta \le \pi$, resulting control force:

$$C(t) = c_0\, Z(t) + c_1 \dot{Z}(t) + c_2 \ddot{Z}(t) + c_3 \int_{-\infty}^{t} dt' e^{-\Lambda(t-t')} Z(t')$$

$\Lambda \sim$ wide range, depending on homodyne phase.

- The right way to find control schemes??

# Noise of the controlled system

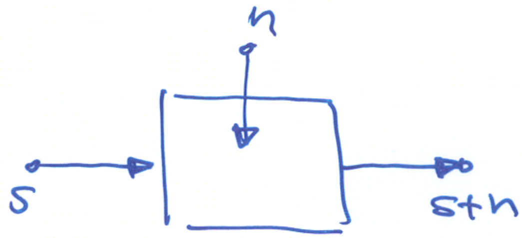
Solving the equations of motion in Fourier domain

$\Rightarrow$ output = (factor) $\cdot$ (uncontrolled output)

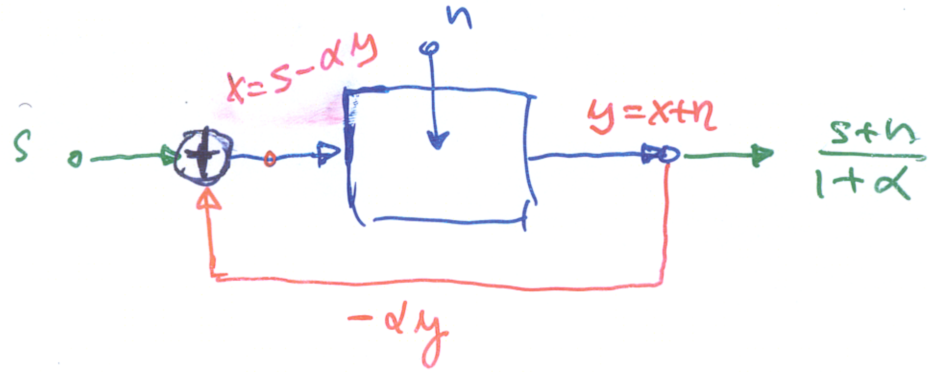
$\Rightarrow$ Noise spectral density same as evaluated from uncontrolled output!

Explanation:

- uncontrolled :




- controlled :




$$s - \alpha y + n = y \Rightarrow \boxed{y = \frac{s+n}{1+\alpha}}$$

signal & noise always come w/ same ratio !

Electronics noise will be added. Possible existence of quantum limit: fluctuation-dissipation theorem. [Braginsky, private communication]

# Conclusions

- Two-photon formalism applied to SR interferometers.

- Noise spectral density evaluated, beating the SQL by factor 2 for $\Delta f \sim f$.

- Effect of optical loss investigated.

- Interesting dynamical effect, "Optical Spring" with two resonances.

- Instabilities present, control system preliminarily analyzed.

- RF modulation/demodulation schemes