Data Analysis III: Stochastic Background

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Scope of This Lecture

- Ground-based detectors (IFOs & bars)

- Cross-correlation measurements

- Assume uncorrelated noise $\gg$ stochastic GW signal

Outline

I Optimally Filtered Cross-Correlation Statistic

II Overlap Reduction Function

III Specific Detector Pairs

Conventions

- One-sided PSD (so $\langle \widetilde{h}(f)^* \widetilde{h}(f') \rangle = \frac{1}{2}\delta(f - f')P(f)$)
- Detector response is $h(t) = d^{ab} h_{ab}(t, \vec{x})$
  - $d^{\text{ifo}}_{ab} = \frac{1}{2}(\widehat{u}_a\widehat{u}_b - \widehat{v}_a\widehat{v}_b)$
  - $d^{\text{bar}}_{ab} = \widehat{u}_a\widehat{u}_b$
- Overlap reduction function is

$$\gamma_{12}(f) = \frac{5}{2}\Gamma_{12}(f) = \frac{5}{8\pi} \sum_{A=+,\times} \int_{S^2} d\widehat{\Omega}\, e^{i2\pi f \widehat{\Omega}\cdot\Delta\vec{x}/c}\, F_{1A}(\widehat{\Omega})\, F_{2A}(\widehat{\Omega})$$

- Detector response $F_{1A}(\widehat{\Omega}) = d_{ab} e^{ab}_A(\widehat{\Omega})$
- Work with $h_0^2 \Omega_{\text{GW}}(f) = \frac{h_0^2}{\rho_{\text{crit}}} \frac{d\rho_{\text{GW}}}{d \ln f} = \frac{4\pi^2 h_0^2}{3H_0^2} f^3 S_h(f)$

I. Cross-correlation statistic

- Basic idea: look for correlations between detectors
  - Detector 1: $h_1 = s_1 + n_1$, Detector 2: $h_2 = s_2 + n_2$

- Assume noise uncorrelated with signal & between detectors

- Cross-correlation:

$$\langle h_1 h_2 \rangle \;=\; \langle n_1 n_2 \rangle + \langle n_1 s_2 \rangle + \langle s_1 n_2 \rangle + \langle s_1 s_2 \rangle$$

  only surviving term is from stochastic GW signal

Cross-Correlation Statistic (cont'd)

$$\begin{aligned} Y_Q &= \int dt_1\, dt_2\, h_1(t_1)\, Q(t_1 - t_2) h_2(t_2) \\ &= \int df\, \tilde{h}_1^*(f)\, \tilde{Q}(f)\, \tilde{h}_2(f) \end{aligned}$$

- Mean from $\langle \tilde{h}_1^*(f) \tilde{h}_2(f') \rangle = \delta(f - f')\, \frac{3H_0^2}{20\pi^2} |f|^{-3}\, \Omega_{GW}(f) \gamma_{12}(|f|)$:

$$\langle Y \rangle = \frac{3H_0^2}{20\pi^2} T \int df |f|^{-3}\, \Omega_{GW}(f) \gamma_{12}(|f|) \tilde{Q}(f)$$

- Variance from $\langle \tilde{h}_i(f)^*\, \tilde{h}_i(f') \rangle = \frac{1}{2} \delta(f - f') P_i(f)$:

$$\langle Y^2 \rangle = \frac{T}{4} \int df\, P_1(f) \left| \tilde{Q}(f) \right|^2 P_2(f)$$

(Assume environmental noise dominates variance)

Optimal Filter

Choose $\widetilde{Q}(f)$ to maximize signal-to-noise ratio:

$$\mathsf{SNR}^2 = \frac{\langle Y \rangle^2}{\langle Y^2 \rangle} = \left(\frac{3H_0^2}{20\pi^2}\right)^2 T \frac{\left(\int df\, A(f)^* P_1(f) P_2(f) \widetilde{Q}(f)\right)^2}{\int df\, \widetilde{Q}(f)^* P_1(f) P_2(f) \widetilde{Q}(f)}$$

this is accomplished by choosing

$$\widetilde{Q}(f) \propto A(f) = \frac{f^{-3}\Omega_{\mathsf{GW}}(f)\gamma_{12}(f)}{P_1(f)P_2(f)}$$

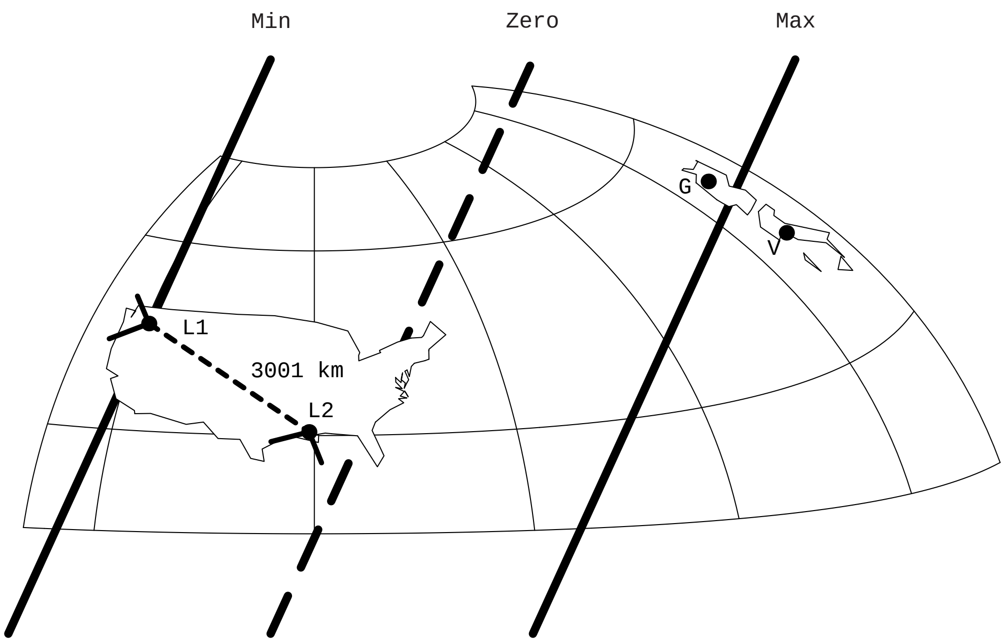
& gives

$$\mathsf{SNR}^2 = \left(\frac{3H_0^2}{20\pi^2}\right)^2 T \int df \frac{(f^{-3}\Omega_{\mathsf{GW}}(f)\gamma_{12}(f))^2}{P_1(f)P_2(f)}$$

Note role of signal, geometry & noise

II. Overlap Reduction Function

Depends on alignment of detectors (polarization sensitivity)
Frequency dependence from cancellations when $\lambda \lesssim |\Delta\vec{x}|$



(figure from Allen & Romano, gr-qc/9710117)

Overlap Reduction Function (cont'd)

$$\gamma_{12}(f) = d_1^{ab} d_2^{cd} \frac{5}{8\pi} \sum_{A=+,\times} \int_{S^2} d\hat{\Omega}\; e^{i2\pi f \hat{\Omega}\cdot\Delta\vec{x}/c}\; e_{Aab}(\hat{\Omega}) e_{Acd}(\hat{\Omega})$$

$$= d_{1ab} d_2^{cd} \frac{5}{4\pi} \int_{S^2} d\hat{\Omega}\; P^{\mathsf{TT}}{}^{ab}_{cd}(\hat{\Omega}) e^{i2\pi f \hat{\Omega}\cdot\Delta\vec{x}/c}$$

where $P^{\mathsf{TT}}{}^{ab}_{cd}(\hat{\Omega}) = \frac{1}{2}\sum_{A=+,\times} e_A^{ab}(\hat{\Omega}) e_{Acd}(\hat{\Omega})$ is a projection operator onto traceless symmetric tensors transverse to $\hat{\Omega}$.

Note we can replace each d_{ab} with its traceless part

$$D_{ab} = d_{ab} - \frac{1}{3}\delta_{ab}\, d_c^c$$

$\exists$ a closed-form solution for $\gamma(f)$ i.t.o. Bessel functions; look at the simpler form of $\gamma(0)$ . . .

Coïncident Overlap Reduction Function

$$\gamma(0) = d_{1ab} d_2^{cd} \frac{5}{4\pi} \int_{S^2} d\hat{\Omega}\ P^{\mathsf{TT}}{}^{ab}_{cd}(\hat{\Omega})$$

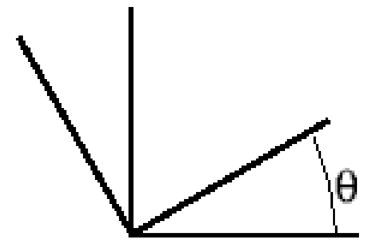
But $\int_{S^2} d\hat{\Omega}\ P^{\mathsf{TT}}{}^{ab}_{cd}(\hat{\Omega}) \propto P^{\mathsf{T}}{}^{ab}_{cd}$ (projection onto traceless symmetric tensors) by symmetry, & since $P^{\mathsf{T}}{}^{ab}_{ab} = 5$ & $P^{\mathsf{TT}}{}^{ab}_{ab}(\hat{\Omega}) = 2$, the proportionality constant must be $(2/5)4\pi$, so

$$\gamma(0) = 2 D_{1ab} D_2^{ab}$$

This holds in the $f \to 0$ limit, or for coïncident detectors.

Coïncident Interferometers

Geometry described by angle θ between IFOs

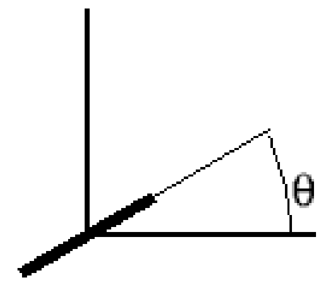


$$D_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \frac{1}{2}\begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos^2\theta\sin^2\theta & 0 \\ 2\cos^2\theta\sin^2\theta & \sin^2\theta - \cos^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma = 2D_{1ab}D_2^{ab} = \cos 2\theta$$

Coïncident Interferometer & Bar

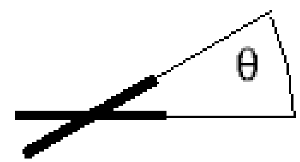
Geometry described by angle θ between IFO & bar 

$$D_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} \cos^2\theta - \frac{1}{3} & \cos^2\theta\sin^2\theta & 0 \\ \cos^2\theta\sin^2\theta & \sin^2\theta - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$\gamma = 2D_{1ab}D_2^{ab} = \cos 2\theta$$

Coïncident Bars

Geometry described by angle θ between bars 

$$D_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$D_2 = \begin{pmatrix} \cos^2\theta - \frac{1}{3} & \cos^2\theta \sin^2\theta & 0 \\ \cos^2\theta \sin^2\theta & \sin^2\theta - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

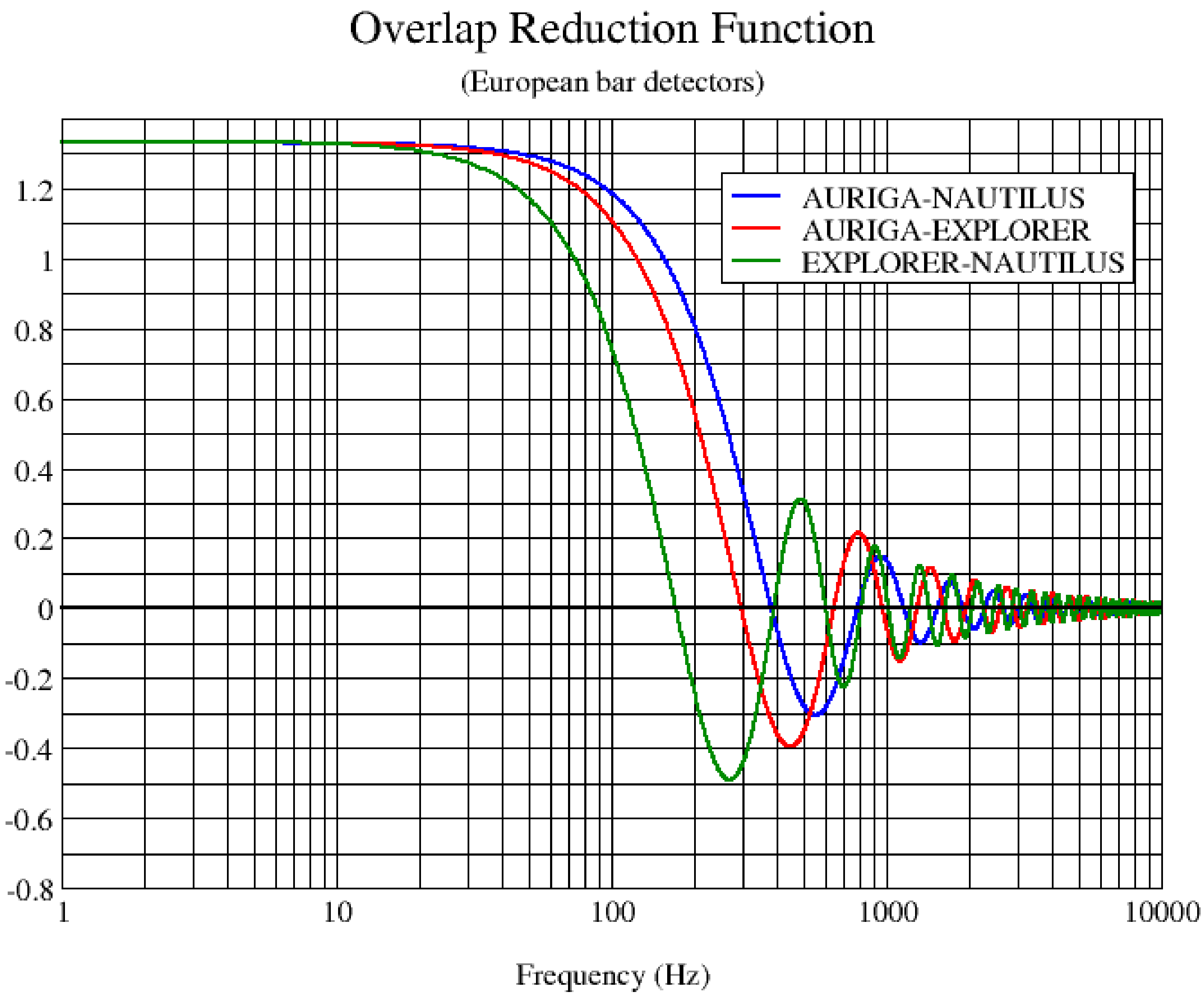
$$\gamma = 2 D_{1ab} D_2^{ab} = \frac{1}{3}(1+3\cos 2\theta)$$

Note this means for parallel bars, $\gamma(0) = \frac{4}{3}$

III. Specific Correlation Experiments

- European Bars: EXPLORER/NAUTILUS/AURIGA
- Correlations with LIGO Livingston (aka LLO or LIGO-LA)
- Correlations with VIRGO

(Overlap reduction function plots made with Erick Vallerino)



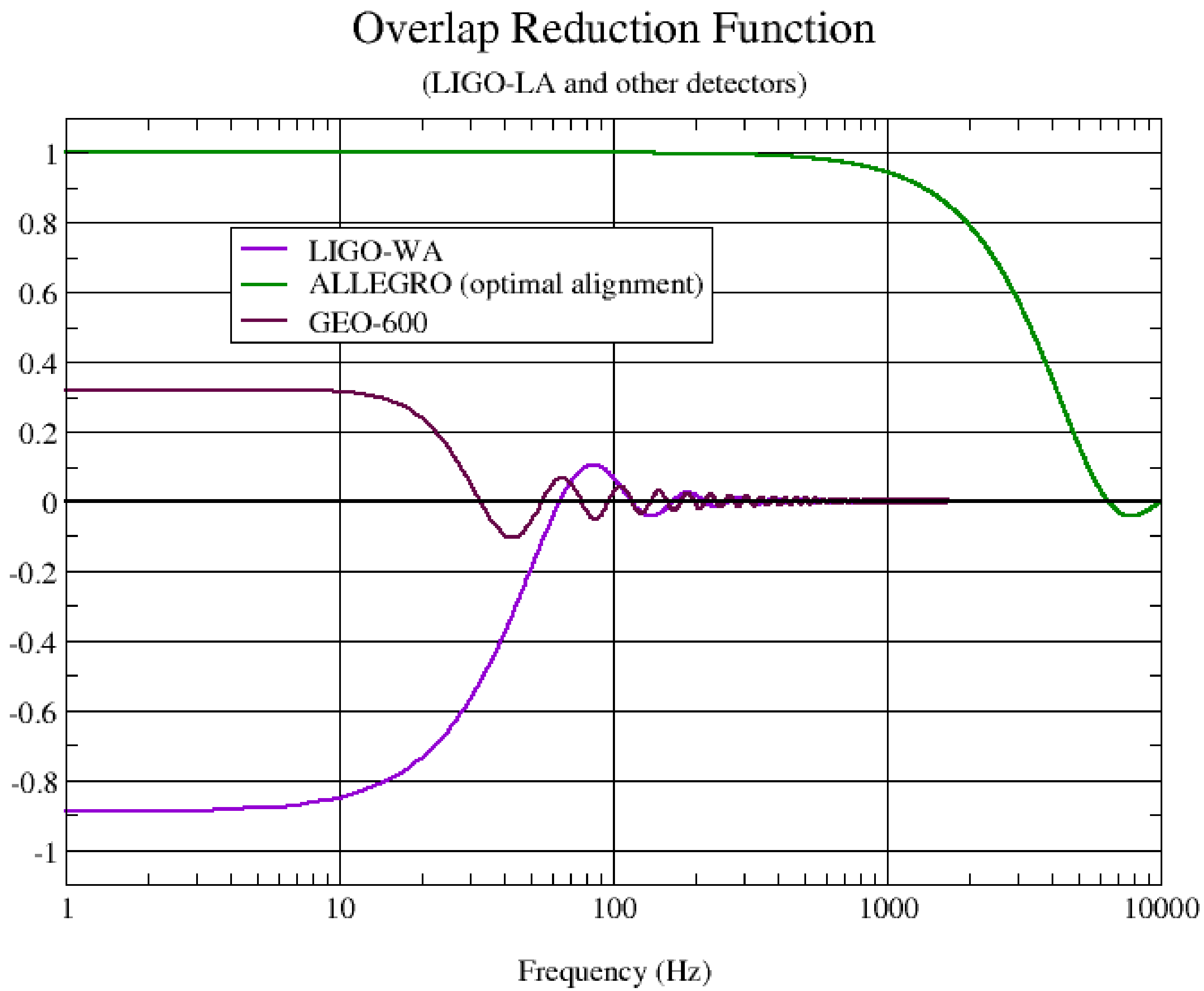
Overlap Reduction Function
(European bar detectors)
AURIGA-NAUTILUS
AURIGA-EXPLORER
EXPLORER-NAUTILUS
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
1
10
100
1000
10000
Frequency (Hz)

Correlations Between European Bar Detectors

- Important value is $\gamma(907\,\text{Hz})$

- $10\% < \gamma(907\,\text{Hz}) < 20\%$ for all bars

- Current best upper limit: correlation between EXPLORER & NAUTILUS bars (Astone et al, 1999): $\Omega_{\text{GW}}(907\,\text{Hz}) \leq 60$


Overlap Reduction Function
(LIGO-LA and other detectors)
LIGO-WA
ALLEGRO (optimal alignment)
GEO-600
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
1
10
100
1000
10000
Frequency (Hz)

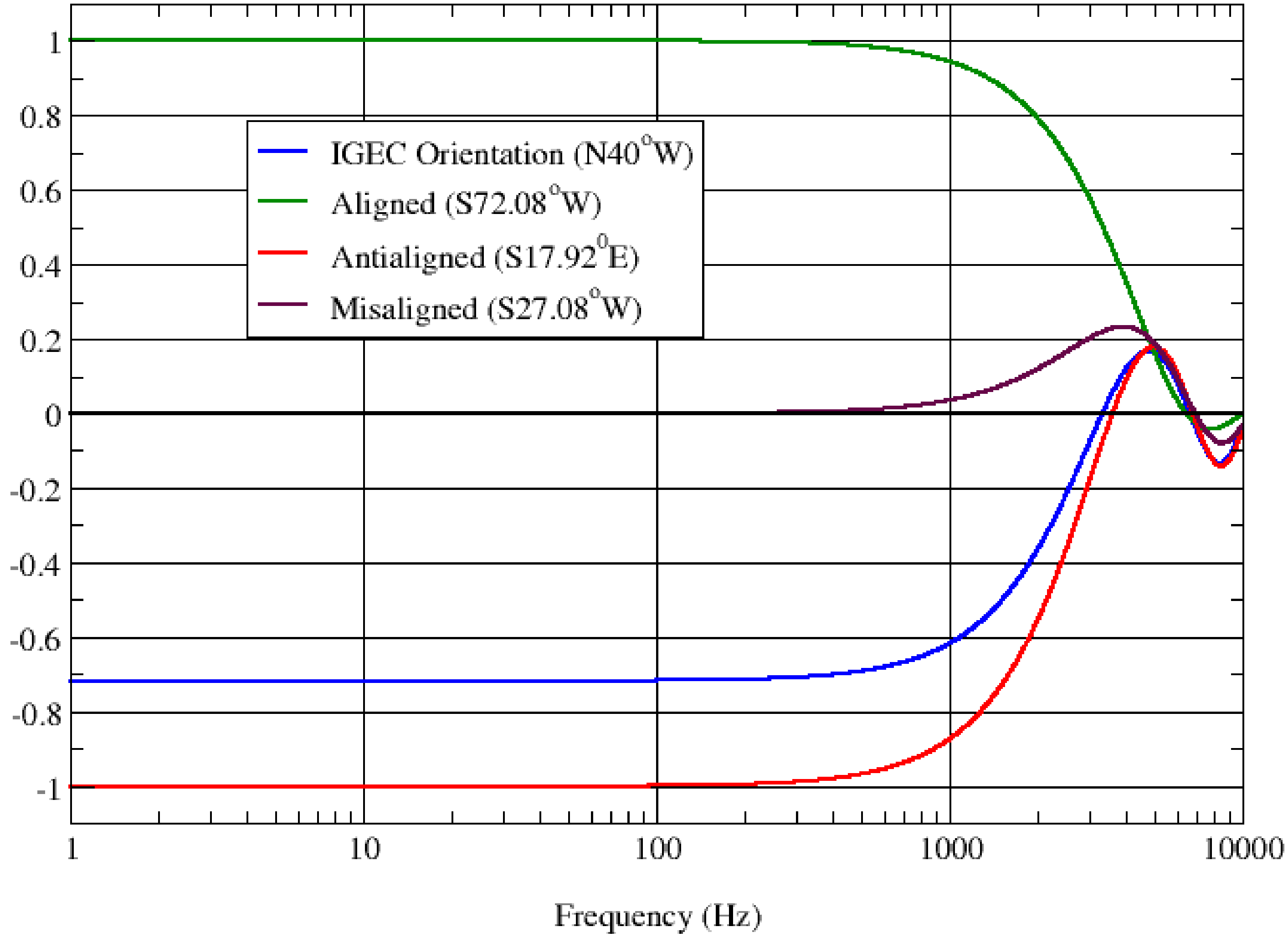
LIGO-LIGO Correlations

- Tradeoff between noise spectrum & overlap reduction fcn $\longrightarrow$ most of the correlations should come from lower frequencies (50–250Hz)
- If amplitude spectral density has same shape as design sensitivity, just scaled up, can set a 90% confidence level upper limit on constant $\Omega_{\text{GW}}(f)$ around

$$\Omega_{\text{GW}}(f) \lesssim 6 \times 10^{-6} \times \left(\frac{17\,\text{days}}{T}\right)^{1/2} \times \left(\frac{(\text{LHO ASD})(\text{LLO ASD})}{(\text{LIGO-1 ASD})^2}\right)$$

Overlap Reduction Function

(LIGO-LA and ALLEGRO)

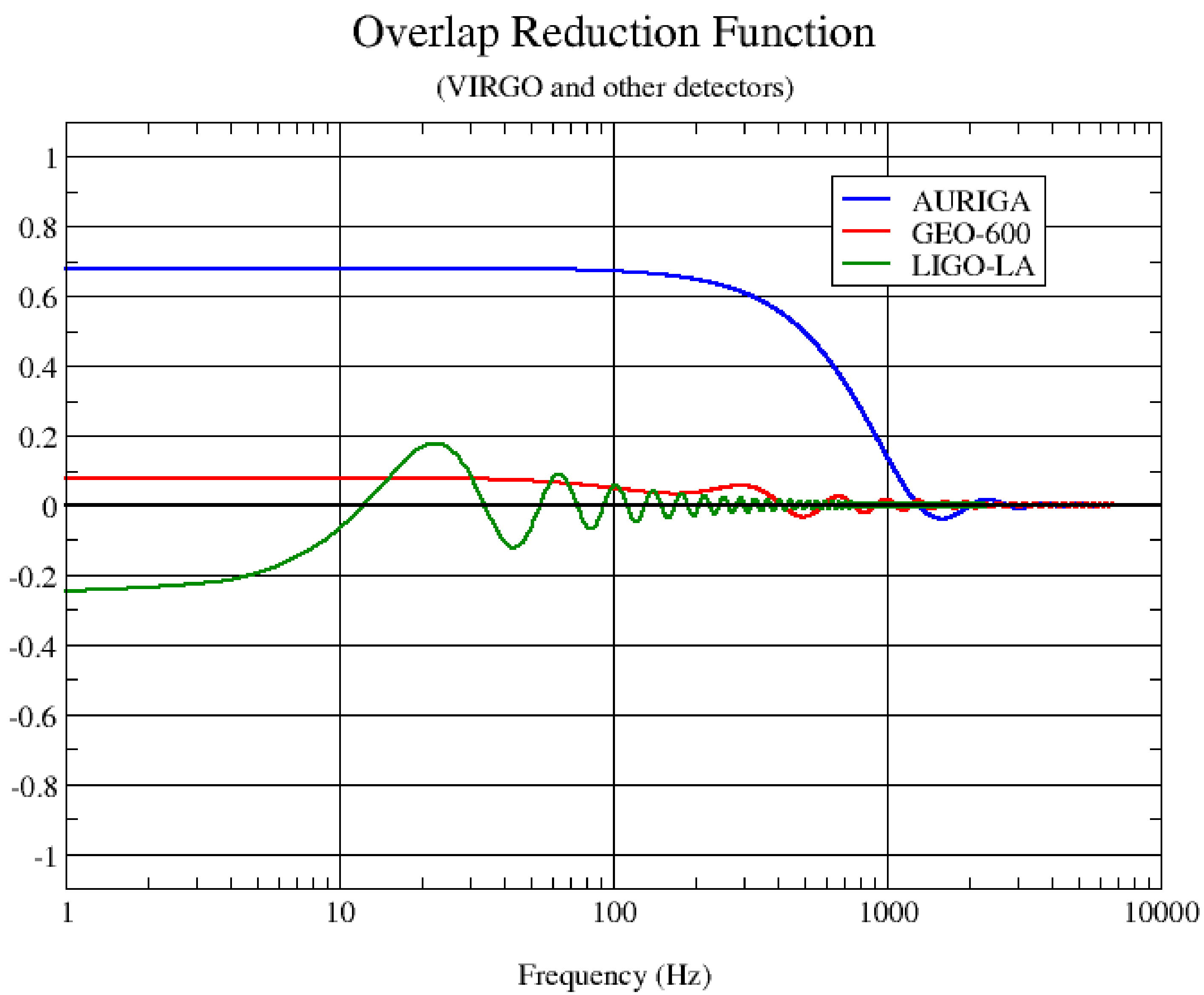


LIGO-ALLEGRO Correlations

- ALLEGRO and LIGO-Livingston only 40km apart
- For optimal alignment, $\gamma(900\,\text{Hz}) \approx 95\%$
- W/No correlated noise & ALLEGRO bandwidth & noise as in PRD **54**, 1264 (1996) could set a 90% confidence level upper limit around

$$\Omega_{\text{GW}}(900\,\text{Hz}) \lesssim 0.2 \times \left(\frac{17\,\text{days}}{T}\right)^{1/2} \times \left(\frac{\text{LLO ASD}(900\,\text{Hz})}{10^{-22}\,\text{Hz}^{-1/2}}\right)$$

- Comparing correlations for different ALLEGRO orientations can distinguish stoch BG from correlated noise. (Finn & Lazzarini 2001)

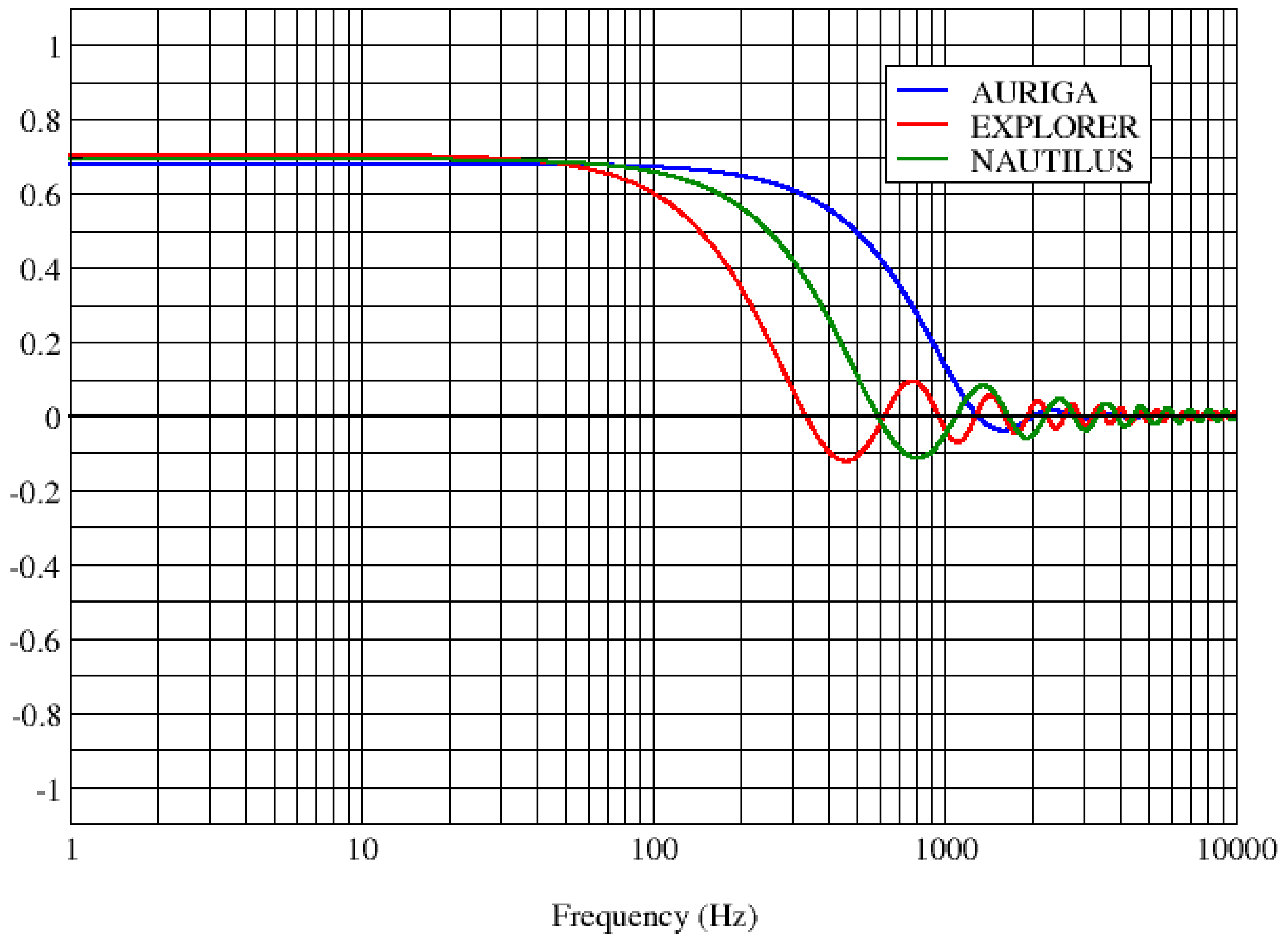

Overlap Reduction Function
(VIRGO and other detectors)
AURIGA
GEO-600
LIGO-LA
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
1
10
100
1000
10000
Frequency (Hz)

Correlations Involving VIRGO

- GEO-600 probes a different polarization
- Other detectors far away
- Low-frequency sensitivity potentially helpful for corr w/IFOs
- Some potential for correlations with bars (but note bar resonance near onset of high-frequency cancellations)

Overlap Reduction Function

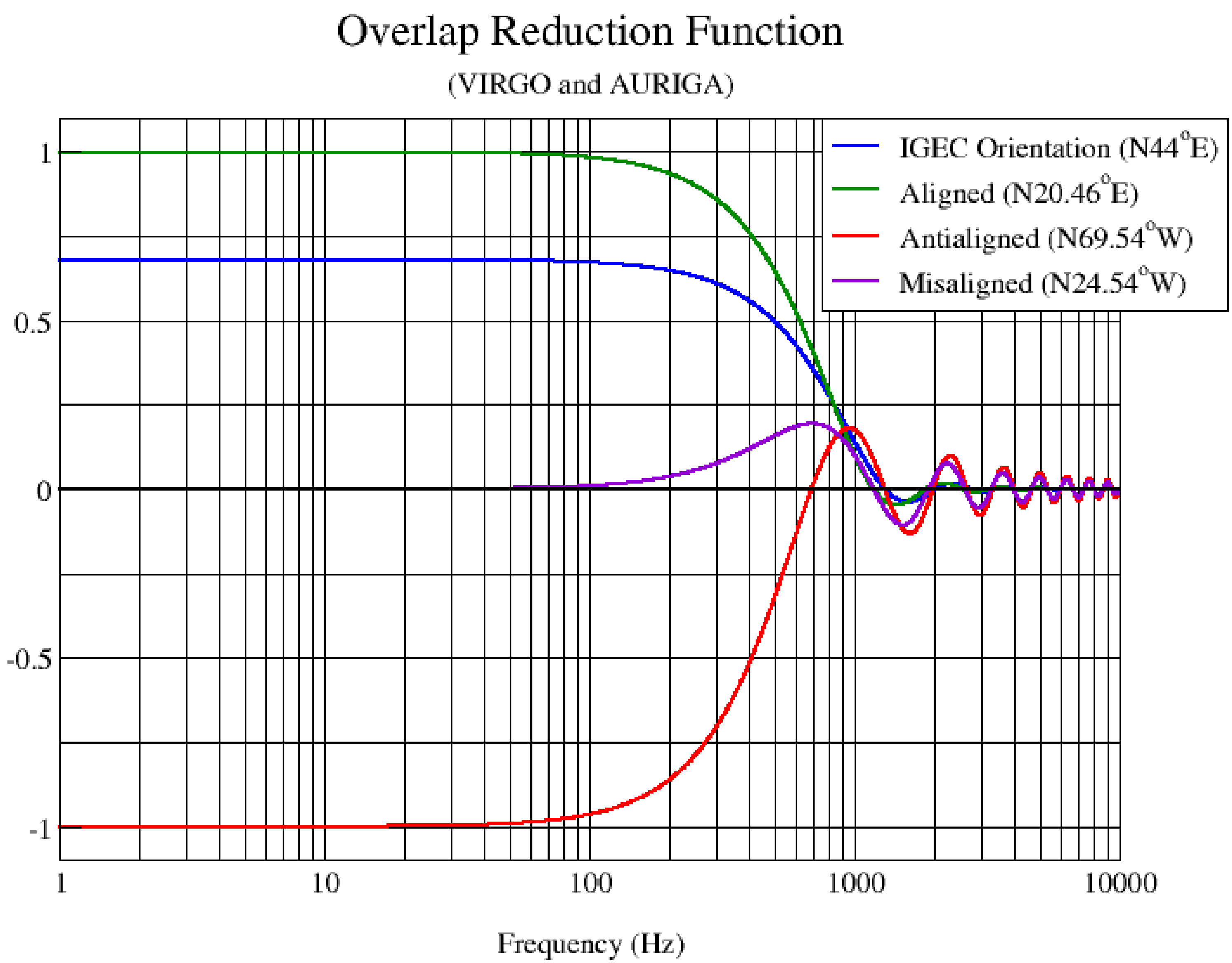
(VIRGO and European bar detectors)



Correlations Between VIRGO & Existing Bars

- AURIGA closest (223 km)

- Existing bars slightly too far away for best overlap at current resonant frequency


Overlap Reduction Function
(VIRGO and AURIGA)
IGEC Orientation (N44°E)
Aligned (N20.46°E)
Antialigned (N69.54°W)
Misaligned (N24.54°W)
1
0.5
0
-0.5
-1
1
10
100
1000
10000
Frequency (Hz)

Orientation Dep of VIRGO-AURIGA Overlap Reduction Function

- At current bar resonant frequency, VIRGO-AURIGA GW correlations are $\sim$ independent of AURIGA orientation $\longrightarrow$ Can't modulate VIRGO-AURIGA GW signal á la LLO-ALLEGRO

Coda: Current Observational Upper Limits

- Current best upper limit: correlation between EXPLORER & NAUTILUS bars (Astone et al, 1999): $\Omega_{\mathrm{GW}}(907\,\mathrm{Hz}) \le 60$
- Upper limit from single bar (Astone et al, 1996): $\Omega_{\mathrm{GW}}(907\,\mathrm{Hz}) \le 100$
- Correlation between Garching & Glasgow prototype IFOs (Compton et al, 1994): $\Omega_{\mathrm{GW}}(f) \lesssim 3 \times 10^5$

LLO-LHO & LLO-ALLEGRO upper limit calculations underway with data from 2002 Jan LIGO E7 Engineering Run (LIGO-GEO also to follow)

Summary

- To detect a stochastic GW background, look for a **cross-correlation** among detectors

- Maximize signal-to-noise using an **optimal filter** $\widetilde{Q}(f) \propto \frac{f^{-3}\Omega_{\mathrm{GW}}(f)\gamma_{12}(f)}{P_1(f)P_2(f)}$

- **Overlap Reduction Function** $\gamma_{12}(f)$ determines role of **observing geometry** (distance & orientation)

References

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   ICTP lecture: gr-qc/0008027
2. JTW et al, 2001 Amaldi proc: gr-qc/0110019
3. B. Allen, Les Houches lecture: gr-qc/9604033
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7. K. Compton et al, 1994 MG7 Proceedings
8. LSC Stochastic Upper Limits Page:
   http://feynman.utb.edu/~joe/research/stochastic/upperlimits/
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10. Finn & Lazzarini, PRD: gr-qc/0104040;
    Poster: LIGO graphical presentation G010246-00-E