Data Analysis III:
Stochastic Background

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Lecture presented at IInd VIRGO/SIGRAV Summer School

Cascina, Italy 2003 May 30

G030323-00-Z

Scope of This Lecture

- Isotropic spin-2 (not scalar) background
  (cosmological or astrophysical)

- Ground-based detectors
  (IFOs & bars − extension to spheres straightforward)

- Cross-correlation measurements

- Assume uncorrelated noise $\gg$ stochastic GW signal

<u>Outline</u>

I Optimally Filtered Cross-Correlation Statistic

II Overlap Reduction Function

III Specific Detector Pairs

Conventions

- One-sided PSD (so $\langle \tilde{h}(f)^* \tilde{h}(f') \rangle = \frac{1}{2}\delta(f-f')P(f)$)
- Detector response is $h(t) = d^{ab}\,h_{ab}(t, \vec{x})$
  - $d_{ab}^{\text{ifo}} = \frac{1}{2}(\hat{u}_a\hat{u}_b - \hat{v}_a\hat{v}_b)$ (spin-2 modes of spheres similar)
  - $d_{ab}^{\text{bar}} = \hat{u}_a\hat{u}_b$

- Overlap reduction function is

$$\gamma_{12}(f) = \frac{5}{2}\Gamma_{12}(f) = \frac{5}{8\pi}\sum_{A=+,\times}\int_{S^2}d\hat{\Omega}\,e^{i2\pi f\hat{\Omega}\cdot\Delta\vec{x}/c}\,F_{1A}(\hat{\Omega})\,F_{2A}(\hat{\Omega})$$

- Detector response $F_A(\hat{\Omega}) = d_{ab}e_A^{ab}(\hat{\Omega})$
- Work with $h_{100}^2\Omega_{\text{GW}}(f) = \frac{h_{100}^2}{\rho_{\text{crit}}}\frac{d\rho_{\text{GW}}}{d\ln f} = \frac{4\pi^2 h_{100}^2}{H_0^2}f^3 S_h(f)$

I. Cross-correlation statistic

- Basic idea: look for correlations between detectors

  - Detector 1: $h_1 = s_1 + n_1$, Detector 2: $h_2 = s_2 + n_2$

- Assume noise uncorrelated with signal & between detectors

- Cross-correlation:

$$\langle h_1 h_2 \rangle \; = \; \langle n_1 n_2 \rangle + \langle n_1 s_2 \rangle + \langle s_1 n_2 \rangle + \langle s_1 s_2 \rangle$$

only surviving term is from stochastic GW signal

Cross-Correlation Statistic (cont'd)

$$Y_Q = \int dt_1\, dt_2\, h_1(t_1)\, Q(t_1 - t_2) h_2(t_2)$$

$$= \int df\, \tilde{h}_1^*(f)\, \tilde{Q}(f)\, \tilde{h}_2(f)$$

- Mean from
$$\langle \tilde{h}_1^*(f) \tilde{h}_2(f') \rangle = \delta(f - f') \frac{3H_0^2}{20\pi^2} |f|^{-3}\, \Omega_{\mathsf{GW}}(f) \gamma_{12}(|f|):$$

$$\langle Y \rangle = \frac{3H_0^2}{20\pi^2} T \int df |f|^{-3}\, \Omega_{\mathsf{GW}}(f) \gamma_{12}(|f|) \tilde{Q}(f)$$

- Variance from $\langle \tilde{h}_i(f)^* \tilde{h}_i(f') \rangle = \frac{1}{2}\delta(f - f') P_i(f):$

$$\langle Y^2 \rangle = \frac{T}{4} \int df\, P_1(f) \left| \tilde{Q}(f) \right|^2 P_2(f)$$

(Assume environmental noise dominates variance)

Optimal Filter

Choose $\widetilde{Q}(f)$ to maximize signal-to-noise ratio:

$$\text{SNR}^2 = \frac{\langle Y \rangle^2}{\langle Y^2 \rangle} = \left(\frac{3H_0^2}{20\pi^2} \right)^2 T \frac{\left(\int df\, A(f)^* P_1(f) P_2(f) \widetilde{Q}(f) \right)^2}{\int df\, \widetilde{Q}(f)^* P_1(f) P_2(f) \widetilde{Q}(f)}$$

this is accomplished by choosing

$$\widetilde{Q}(f) \propto A(f) = \frac{f^{-3} \Omega_{\text{GW}}(f) \gamma_{12}(f)}{P_1(f) P_2(f)}$$
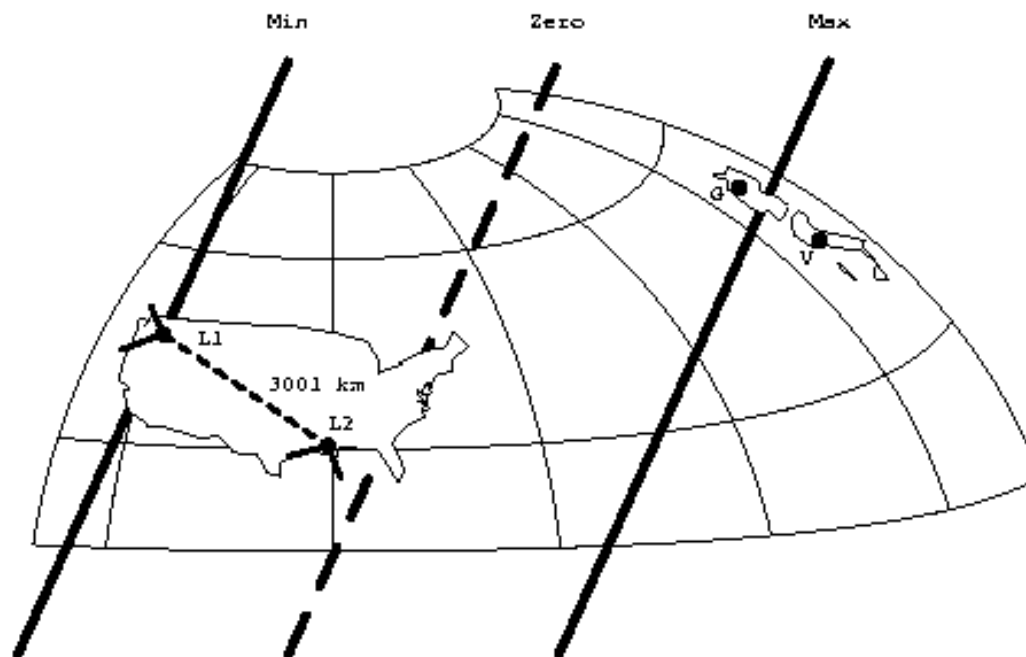
& gives

$$\text{SNR}^2 = \left(\frac{3H_0^2}{20\pi^2} \right)^2 T \int df \frac{(f^{-3} \Omega_{\text{GW}}(f) \gamma_{12}(f))^2}{P_1(f) P_2(f)}$$

Note role of signal, geometry & noise

II. Overlap Reduction Function

Depends on alignment of detectors (polarization sensitivity)
Frequency dependence from cancellations when $\lambda \lesssim |\Delta \vec{x}|$

(figure from Allen & Romano, gr-qc/9710117)

Overlap Reduction Function (cont'd)

$$\gamma_{12}(f) = d_1^{ab} d_2^{cd} \frac{5}{8\pi} \sum_{A=+,\times} \int_{S^2} d\hat{\Omega} \; e^{i2\pi f \hat{\Omega} \cdot \Delta \vec{x}/c} \; e_{Aab}(\hat{\Omega}) e_{Acd}(\hat{\Omega})$$

$$= d_{1ab} d_2^{cd} \frac{5}{4\pi} \int_{S^2} d\hat{\Omega} \; P^{\mathsf{TT}\,ab}_{\quad cd}(\hat{\Omega}) e^{i2\pi f \hat{\Omega} \cdot \Delta \vec{x}/c}$$

where $P^{\mathsf{TT}\,ab}_{\quad cd}(\hat{\Omega}) = \frac{1}{2}\sum_{A=+,\times} e_A^{ab}(\hat{\Omega}) e_{Acd}(\hat{\Omega})$ is a projection operator onto traceless symmetric tensors transverse to $\hat{\Omega}$.

Note we can replace each d_{ab} with its traceless part

$$D_{ab} = d_{ab} - \frac{1}{3}\delta_{ab}\, d_c^c$$

$\exists$ a closed-form solution for $\gamma(f)$ i.t.o. Bessel functions; look at the simpler form of $\gamma(0)$ . . .

Coïncident Overlap Reduction Function

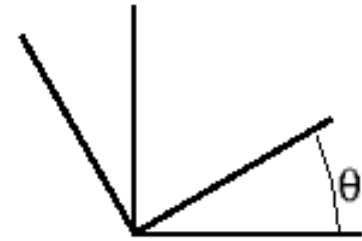$$\gamma(0) = d_{1ab}d_2^{cd}\frac{5}{4\pi}\int_{S^2}d\widehat{\Omega}\ P^{\mathsf{TT}ab}_{cd}(\widehat{\Omega})$$

But $\int_{S^2}d\widehat{\Omega}\ P^{\mathsf{TT}ab}_{cd}(\widehat{\Omega}) \propto P^{\mathsf{T}ab}_{cd}$ (projection onto traceless symmetric tensors) by symmetry, & since $P^{\mathsf{T}ab}_{ab}= 5$ & $P^{\mathsf{TT}ab}_{ab}(\widehat{\Omega})= 2$, the proportionality constant must be $(2/5)4\pi$, so

$$\gamma(0) = 2D_{1ab}D_2^{ab}$$

This holds in the $f \to 0$ limit, or for coïncident detectors.

Coïncident Interferometers

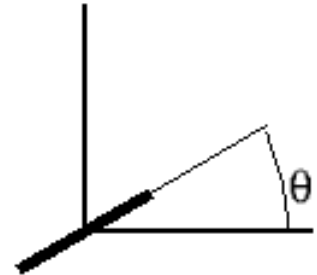Geometry described by angle θ between IFOs

$$D_1 = \tfrac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \tfrac{1}{2} \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos^2\theta\sin^2\theta & 0 \\ 2\cos^2\theta\sin^2\theta & \sin^2\theta - \cos^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma = 2D_{1ab}D_2^{ab} = \cos 2\theta$$

Coïncident Interferometer & Bar

Geometry described by angle θ between IFO & bar

$$D_1 = \tfrac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} \cos^2\theta - \tfrac{1}{3} & \cos^2\theta \sin^2\theta & 0 \\ \cos^2\theta \sin^2\theta & \sin^2\theta - \tfrac{1}{3} & 0 \\ 0 & 0 & -\tfrac{1}{3} \end{pmatrix}$$

$$\gamma = 2 D_{1ab} D_2^{ab} = \cos 2\theta$$

Coïncident Bars

Geometry described by angle θ between bars

$$D_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$D_2 = \begin{pmatrix} \cos^2\theta - \frac{1}{3} & \cos^2\theta\sin^2\theta & 0 \\ \cos^2\theta\sin^2\theta & \sin^2\theta - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$\gamma = 2D_{1ab}D_2^{ab} = \frac{1}{3}(1 + 3\cos 2\theta)$$

Note this means for parallel bars, $\gamma(0) = \frac{4}{3}$
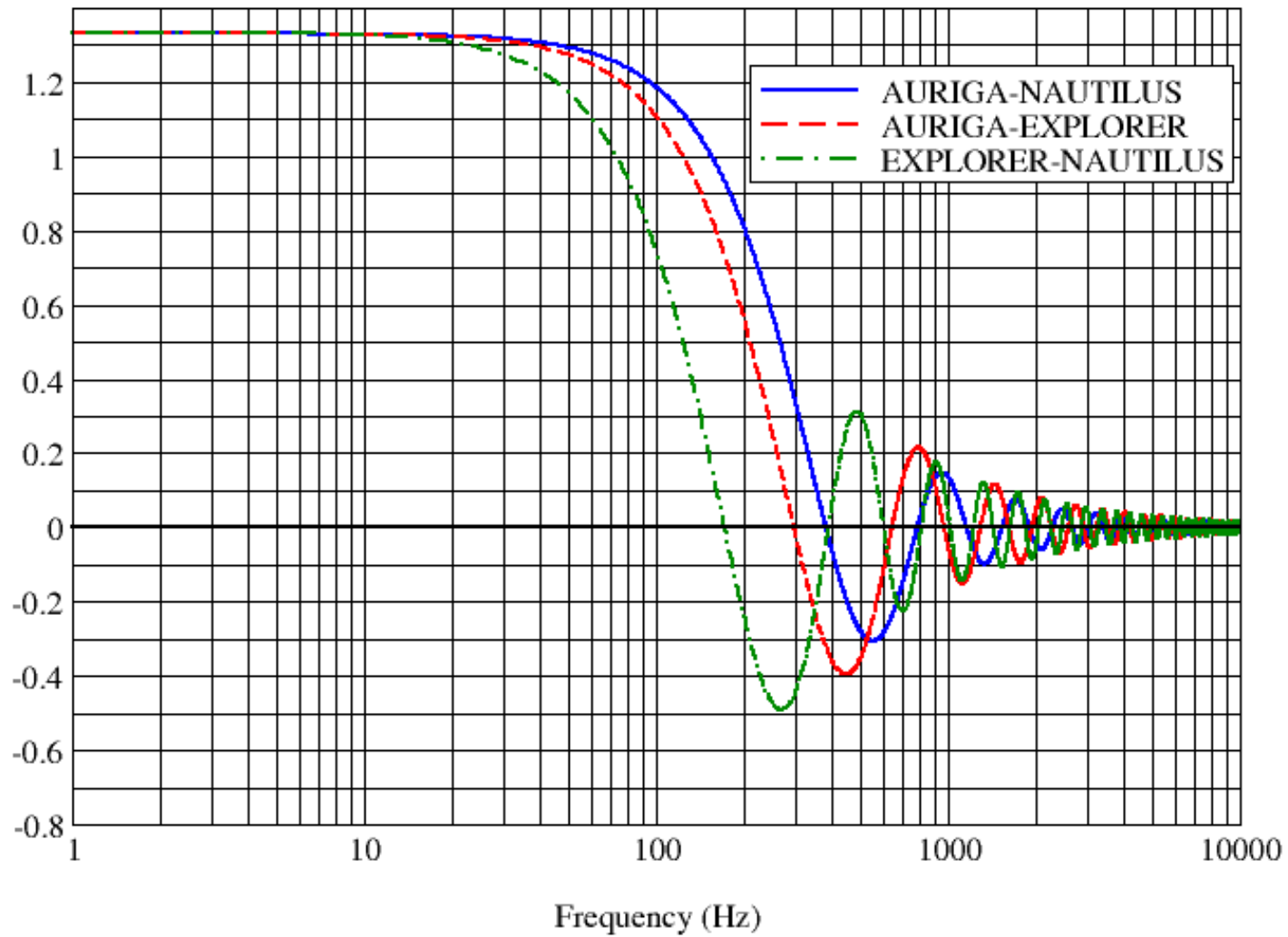
Cartoon courtesy of E. Coccia, NAUTILUS Group (Rome)

III. Specific Correlation Experiments

- European Bars: EXPLORER/NAUTILUS/AURIGA

- Correlations with LIGO Livingston (aka LLO or LIGO-LA)

- Correlations with VIRGO

(Overlap reduction function plots made with Erick Vallerino)
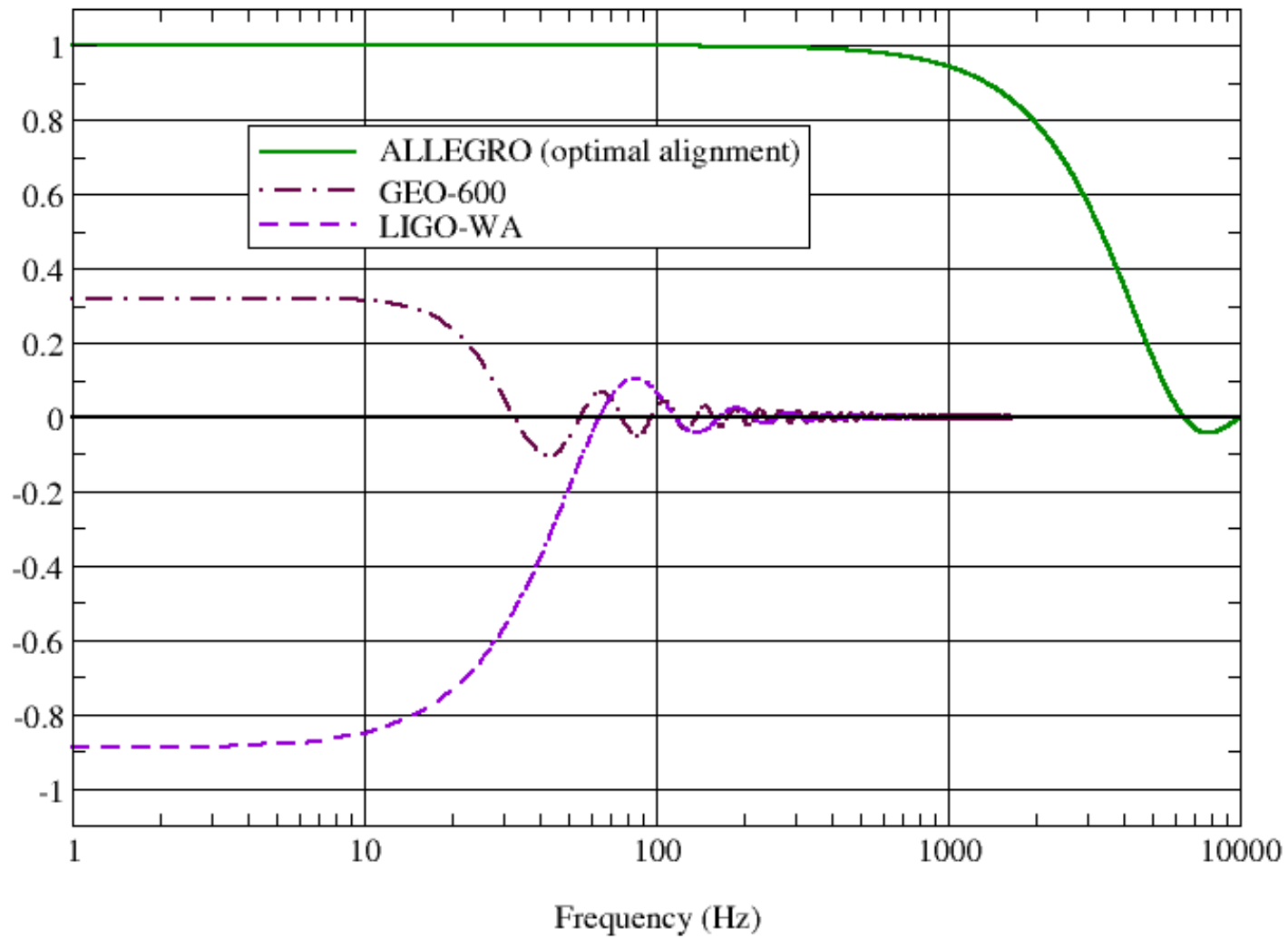
Overlap Reduction Function

(European bar detectors)

Frequency (Hz)

Correlations Between
European Bar Detectors

- Important value is $\gamma(907\,\mathrm{Hz})$

- $10\% < \gamma(907\,\mathrm{Hz}) < 20\%$ for all bars

- Current best upper limit: correlation between
  EXPLORER & NAUTILUS bars (Astone et al, 1999):
  $h_{100}^2 \Omega_{\mathrm{GW}}(907\,\mathrm{Hz}) \leq 60$

Overlap Reduction Function

(LIGO-LA and other detectors)

Legend:
- ALLEGRO (optimal alignment) — green solid line
- GEO-600 — dark red dash-dot line
- LIGO-WA — purple dashed line

Y-axis: 1, 0.8, 0.6, 0.4, 0.2, 0, -0.2, -0.4, -0.6, -0.8, -1
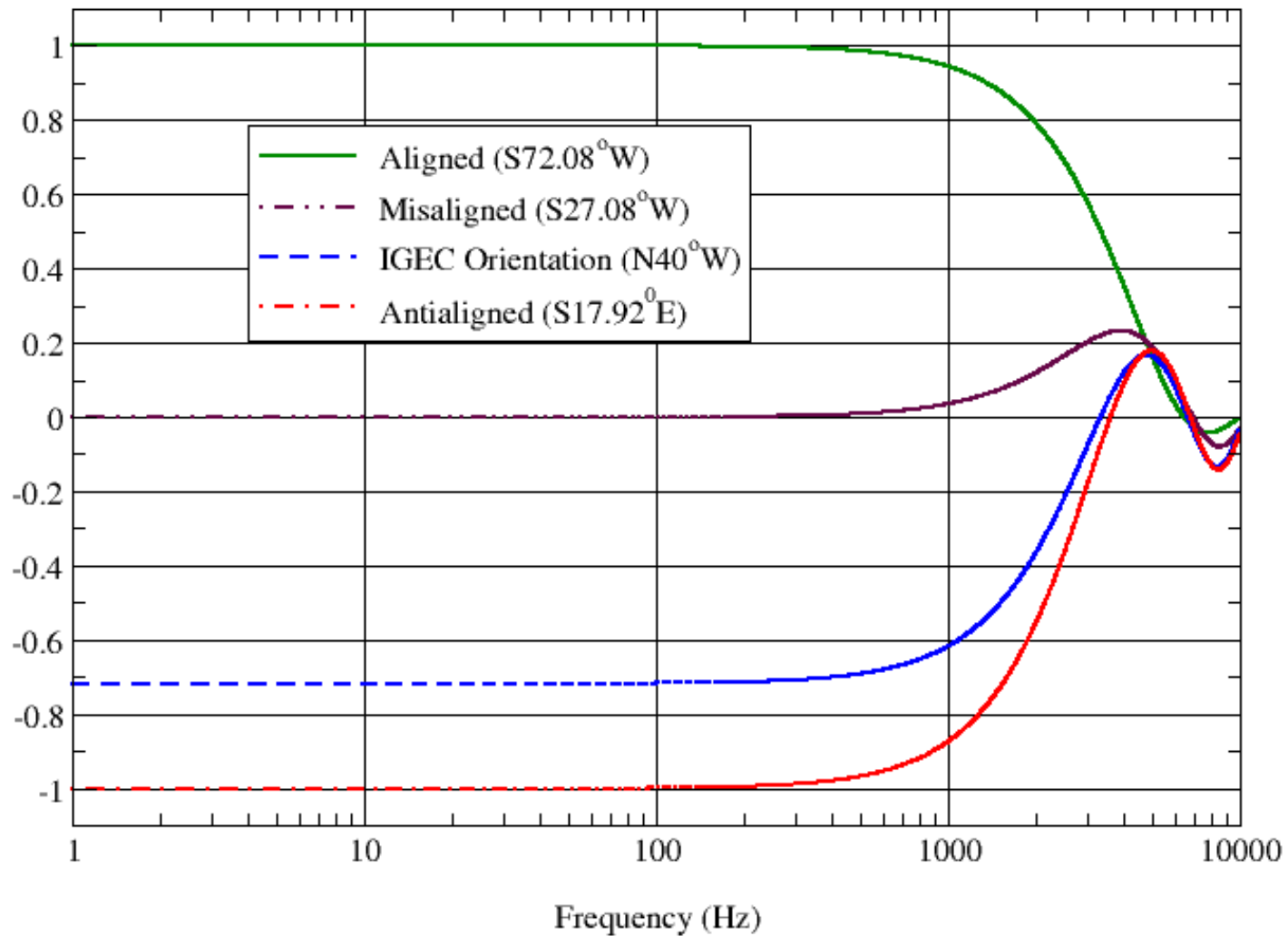
X-axis: Frequency (Hz) — 1, 10, 100, 1000, 10000

LIGO-LIGO Correlations

- Tradeoff between noise spectrum & overlap reduction fcn
  → most of the correlations should come from
  lower frequencies (50–250Hz)
- If amplitude spectral density has same shape as
  design sensitivity, just scaled up, can set a 90% confidence
  level upper limit on constant $\Omega_{\mathsf{GW}}(f)$ around

$$h_{100}^2 \Omega_{\mathsf{GW}}(f) \lesssim 10^{-6} \times \left(\frac{1\,\mathsf{yr}}{T}\right)^{1/2} \times \left(\frac{(\mathsf{LHO\ ASD})(\mathsf{LLO\ ASD})}{(\mathsf{LIGO\text{-}1\ ASD})^2}\right)$$

Overlap Reduction Function

(LIGO-LA and ALLEGRO)

Legend:
- Aligned (S72.08°W) — green solid line
- Misaligned (S27.08°W) — dark red dash-dot-dot line
- IGEC Orientation (N40°W) — blue dashed line
- Antialigned (S17.92°E) — red dash-dot line

Frequency (Hz)

LIGO-ALLEGRO Correlations

- ALLEGRO and LIGO-Livingston only 40 km apart
- For optimal alignment, $\gamma(900\,\text{Hz}) \approx 95\%$
- W/No correlated noise & ALLEGRO bandwidth around 30 Hz could set a 90% confidence level upper limit around

$$h_{100}^2 \Omega_{\text{GW}}(900\,\text{Hz}) \lesssim 10^{-2} \times \left(\frac{1\,\text{yr}}{T}\right)^{1/2} \times \left(\frac{\text{LLO ASD}(900\,\text{Hz})}{10^{-22}\,\text{Hz}^{-1/2}}\right)$$

- Comparing correlations for different ALLEGRO orientations can distinguish stoch BG from correlated noise.
  (Finn & Lazzarini 2001)

Overlap Reduction Function
(VIRGO and other detectors)

Frequency (Hz)

Legend:
- AURIGA (blue dashed)
- GEO-600 (red solid)
- LIGO-LA (green dash-dot)
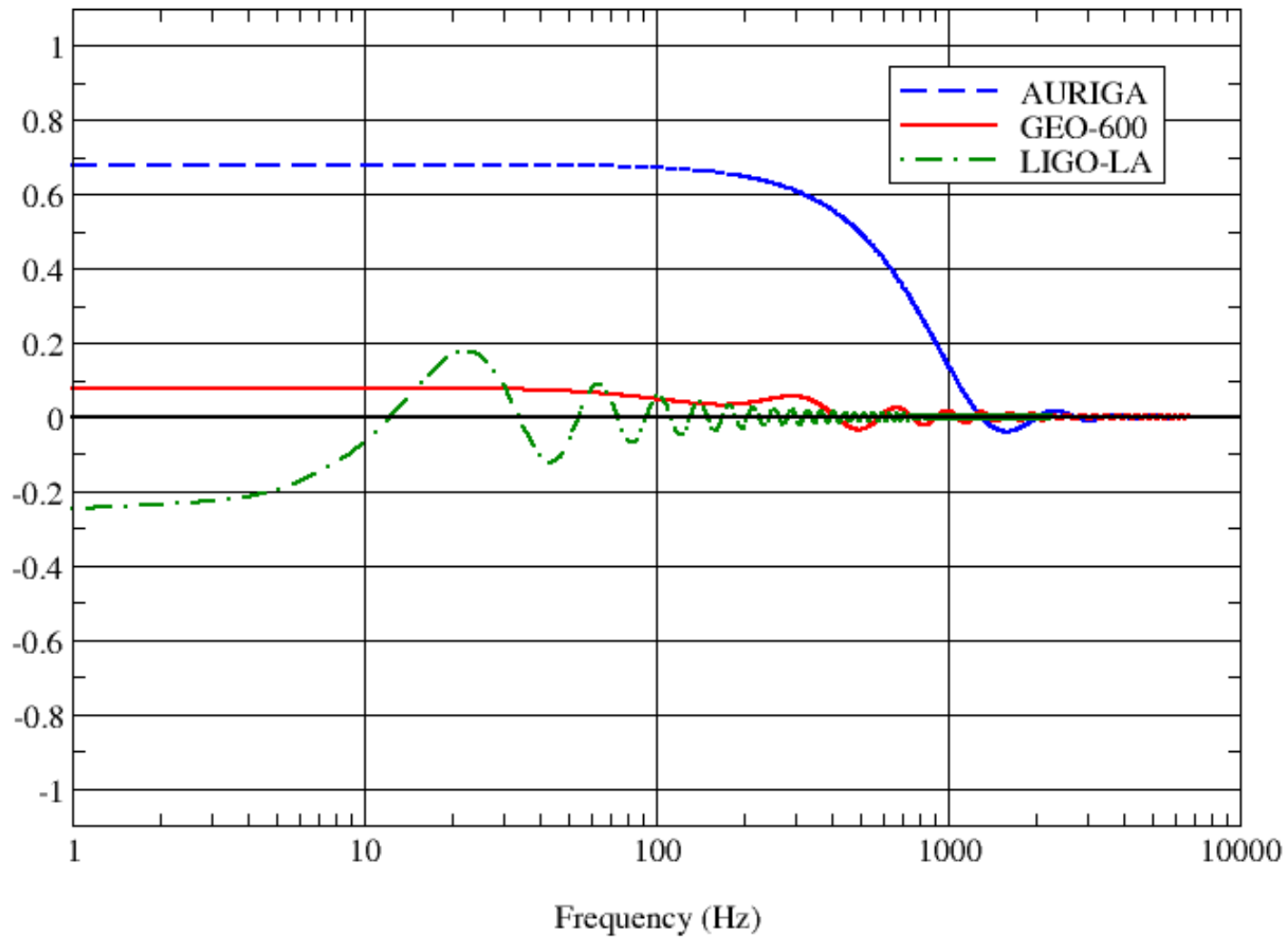
Correlations Involving VIRGO

- GEO-600 probes a different polarization

- Other interferometers far away

- Low-frequency sensitivity potentially helpful for corr w/IFOs

- Some potential for correlations with bars (but note
  bar resonance near onset of high-frequency cancellations)

Overlap Reduction Function
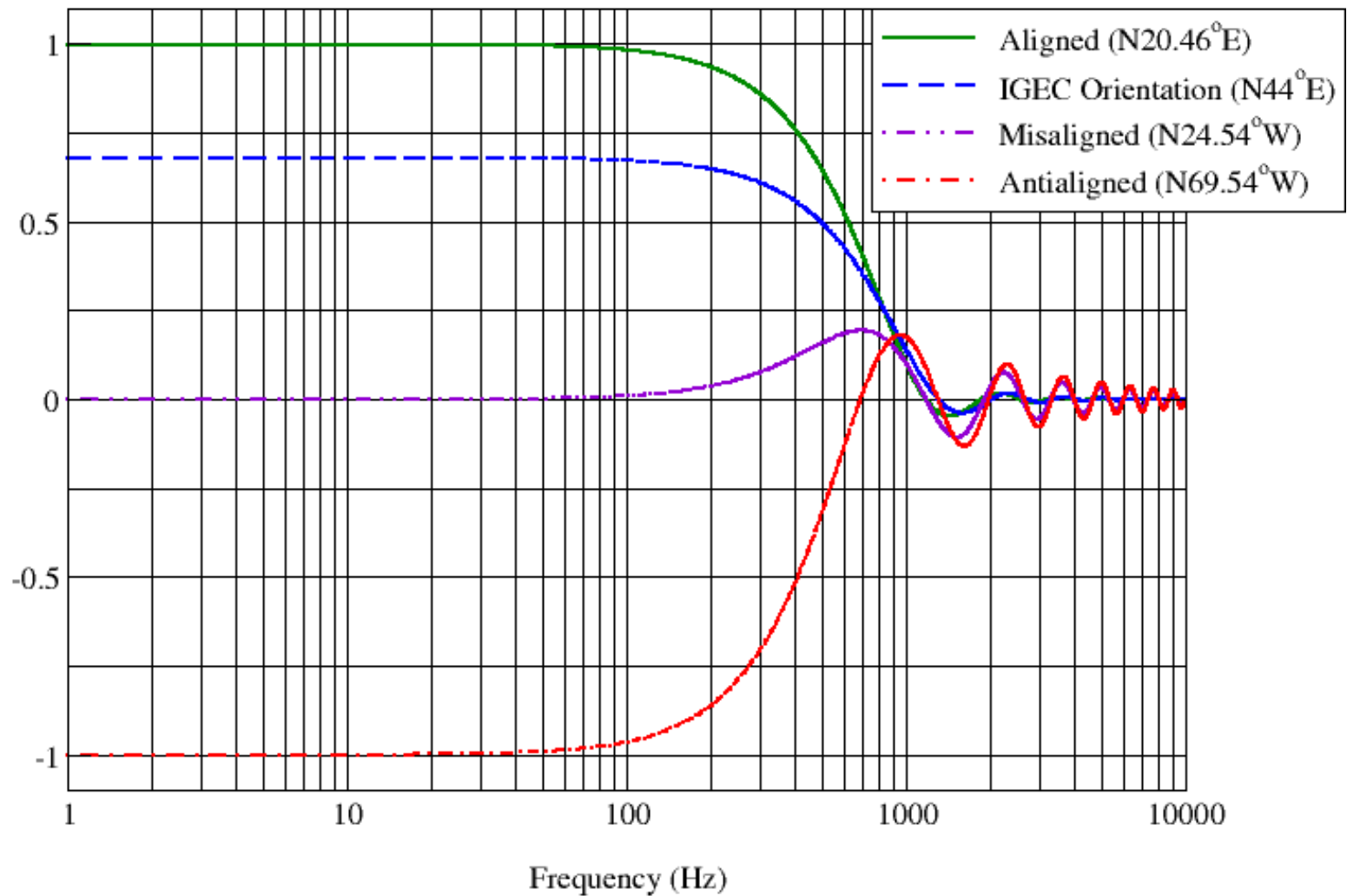
(VIRGO and European bar detectors)
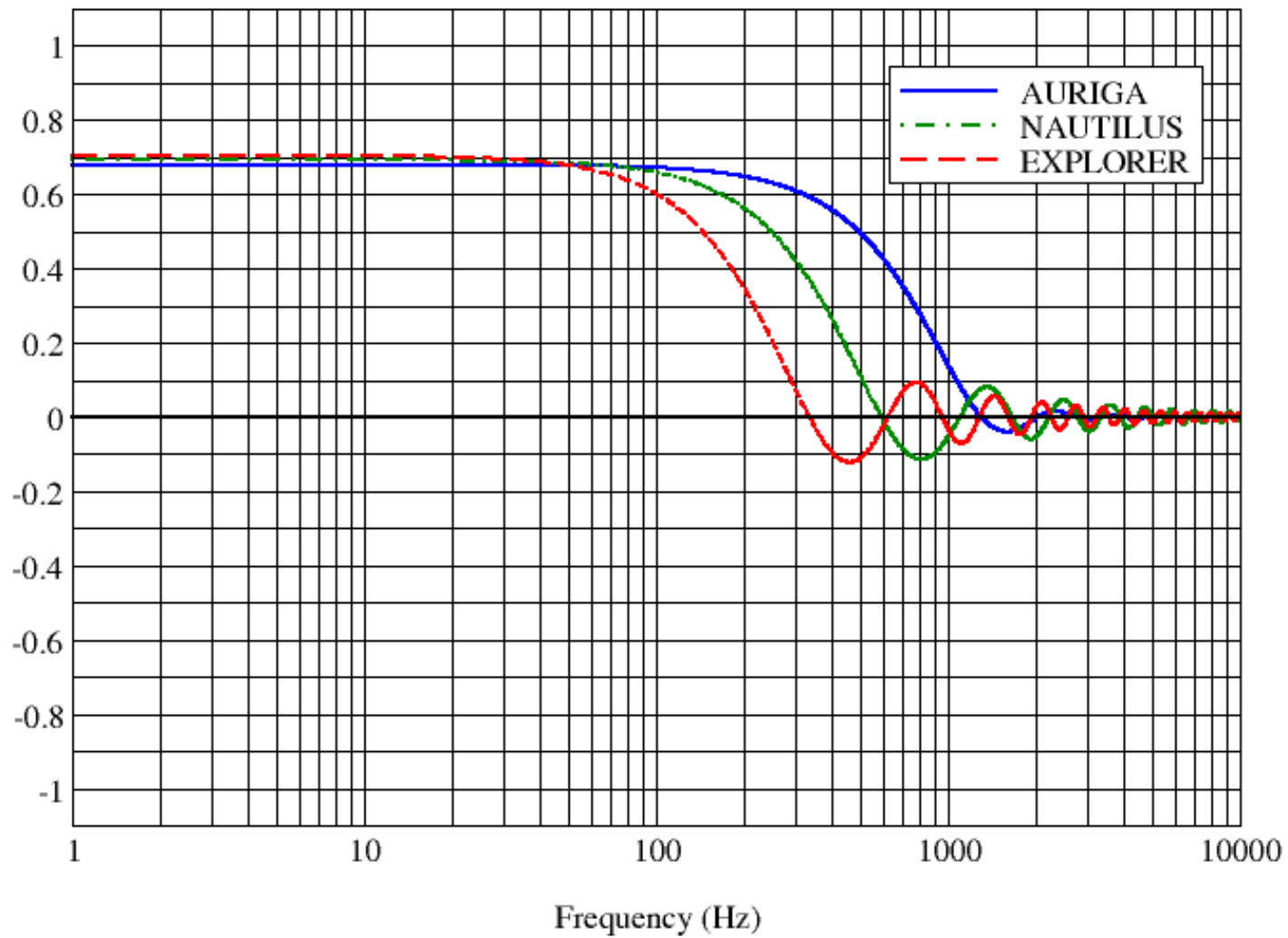
Frequency (Hz)

Correlations Between VIRGO & Existing Bars

- AURIGA closest (223 km)

- Existing bars slightly too far away for best overlap at current resonant frequency

Overlap Reduction Function

(VIRGO and AURIGA)

Legend:
- Aligned (N20.46°E)
- IGEC Orientation (N44°E)
- Misaligned (N24.54°W)
- Antialigned (N69.54°W)

Frequency (Hz)

Orientation Dep of VIRGO-AURIGA Overlap Reduction Function

- At current bar resonant frequency, VIRGO-AURIGA GW
  correlations are $\sim$ independent of AURIGA orientation
  $\longrightarrow$ Can't modulate VIRGO-AURIGA GW signal
  à la LLO-ALLEGRO

Coda: Current Observational Upper Limits

- Current best upper limit: correlation between EXPLORER & NAUTILUS bars (Astone et al, 1999): $h_{100}^2 \Omega_{\text{GW}}(907\,\text{Hz}) \leq 60$

- Upper limit from single bar (Astone et al, 1996): $h_{100}^2 \Omega_{\text{GW}}(907\,\text{Hz}) \leq 100$

- Correlation between Garching & Glasgow prototype IFOs (Compton et al, 1994): $h_{100}^2 \Omega_{\text{GW}}(f) \lesssim 3 \times 10^5$

- Limit from LLO & LHO Engineering (E7) Data: $h_{100}^2 \Omega_{\text{GW}}(f) \lesssim 8 \times 10^4$

- Science (S1) results from LLO-LHO coming soon!

- LLO-ALLEGRO & LIGO-GEO analyses underway/planned

Summary

- To detect a stochastic GW background, look for a **cross-correlation** among detectors

- Maximize signal-to-noise using an **optimal filter**
$$\tilde{Q}(f) \propto \frac{f^{-3}\Omega_{\text{GW}}(f)\gamma_{12}(f)}{P_1(f)P_2(f)}$$

- **Overlap Reduction Function** $\gamma_{12}(f)$ determines role of **observing geometry** (distance & orientation)

References

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