Perturbation method in the assessment of radiation reaction in the capture of stars by black holes

Alessandro D.A.M. Spallicci†§, Sofiane Aoudia†

† ILGA UMR 6162 CNRS, Observatoire de la Côte d'Azur, Nice

Abstract. This work deals with the motion of a radially falling star in Schwarzschild geometry and correctly identifies radiation reaction terms by the perturbative method. The results are: i) identification of all terms up to first order in perturbations and second in trajectory deviation, including radiation reaction terms; ii) renormalisation of divergent terms by the zeta Riemann and Hurwitz functions. The work implements a method previously identified by one of the authors and corrects some current misconceptions and results.

MSC: 83C10 Equations of motion 83C57 Black holes 70F05 Two-body problem

1. Introduction

Radiation reaction, still partially outstanding problem in general relativity, is of most concern for gravitational waves detectors. Its influence is manifest on the waveforms where a phase mismatch of the templates with the signals may cause loss of detection. We analyse a radially falling star, m, captured by a massive black hole, M, by perturbative methods:

- The motion is studied in strong gravity, the perturbation being based on the m/M ratio. There is no use of energy balance and adiabatic hypothesis. The former is the imposition, not the rightful outcome, of the equality of the energy radiated with the energy loosed by the system. The latter can't be evoked since the particle immediately has to react to the radiation emitted, contrarily to inspiral motion where radiation reaction time scale is larger than the orbital period.
- Radial fall is an idealisation of the capture scenario, but applicable to final plunging. Furthermore, most of the radiation, and thus reaction, occurs close to the horizon where inspiral has ceased.

2. The metric, the perturbation scheme and the geodesic equation

Perturbation method for analysis of radiation reaction has been previously proposed [1] - [3]. The metric is the sum of the Schwarzschild metric and the perturbations:

$$\eta_{\mu\nu} = \begin{pmatrix} f & 0 \\ 0 & -\dfrac{1}{f} \end{pmatrix} \quad h_{\mu\nu} = \begin{pmatrix} -fH_0 & -H_1 \\ -H_1 & -\dfrac{1}{f}H_2 \end{pmatrix} \quad \eta^{\mu\nu} = \begin{pmatrix} \dfrac{1}{f} & 0 \\ 0 & -f \end{pmatrix} \quad h^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{f}H_0 & H_1 \\ H_1 & -fH_2 \end{pmatrix}$$

§ Bd. de l'Observatoire, BP 4229, 06304 Nice France. Email spallicci@obs-nice.fr. This research work was supported by ESA with a G. Colombo Senior Research Fellowship to A. Spallicci.

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$$g_{tt} = f(1-H_0) \quad g_{tr} = g_{rt} = -H_1 \quad g_{rr} = -\frac{1}{f}(1+H_2)$$

$$g^{tt} = \frac{1}{f}(1+H_0) \quad g^{tr} = g^{rt} = -H_1 \quad g^{rr} = -f(1-H_2)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad f = \frac{r-2M}{r}$$

The position of the particle $r_e = r_p + \Delta r_p$ is given by the unperturbed trajectory in the unperturbed field r_p and by several contributions, among which radiation reaction, given by the unperturbed and perturbed field, that generate a trajectory deviation Δr_p. The field is developed in Taylor series around the *real* position of the particle: $g_{\mu\nu}(r_e) = g_{\mu\nu}(r_p) + \Delta r_p \left(\partial g_{\mu\nu}/\partial r\right)_{r_p}$. The geodesic is only dependent upon radial and time coordinates:

$$\Gamma^t_{rr}\left(\frac{dr}{dt}\right)^3 + \left(2\Gamma^t_{tr} - \Gamma^r_{rr}\right)\left(\frac{dr}{dt}\right)^2 + \left(\Gamma^t_{tt} - 2\Gamma^r_{tr}\right)\left(\frac{dr}{dt}\right) - \Gamma^r_{tt} \tag{1}$$

$$\Gamma^t_{rr} = \frac{1}{2}g^{tt}(2g_{tr,r} - g_{rr,t}) + \frac{1}{2}g^{tr}g_{rr,r} \qquad \Gamma^t_{tr} = \frac{1}{2}g^{tt}g_{tt,r} + \frac{1}{2}g^{tr}g_{rr,t}$$

$$\Gamma^r_{rr} = \frac{1}{2}g^{rr}g_{rr,r} + \frac{1}{2}g^{rt}(2g_{tr,r} - g_{rr,t}) \qquad \Gamma^t_{tt} = \frac{1}{2}g^{tt}g_{tt,t} + \frac{1}{2}g^{tr}\left(2g_{rt,t} - g_{tt,r}\right)$$

$$\Gamma^r_{tr} = \frac{1}{2}g^{rr}g_{rr,t} + \frac{1}{2}g^{rt}g_{tt,r} \qquad \Gamma^r_{tt} = \frac{1}{2}g^{rr}\left(2g_{rt,t} - g_{tt,r}\right) + \frac{1}{2}g^{rt}g_{tt,t}$$

In absence of the weak field hyphothesis, h is not limited in amplitude, but the following justifies that solely the terms in *Tab. 1* are to be retained. We suppose:

$$\frac{[h^{(1)}]^2}{\eta} \simeq \frac{h^{(2)}}{\eta} \ll \frac{h^{(1)}}{\eta} < \frac{\Delta \ddot{r}_p}{\ddot{r}_p}$$

since $\Delta\ddot{r}_p$, acceleration trajectory deviation, is the sum of two types of contributions. The former is given by the Schwarzschild metric, the latter by the perturbations h:

$$\Delta\ddot{r}_p = \Delta\ddot{r}_p(\eta) + \Delta\ddot{r}_p(h) \tag{2}$$

The acceleration is dependent upon $h^{(1)}$ derivatives which are not necessarily small, especially in the last phase of the trajectory. In conclusion, the terms proportional to $h^{(1)}, \Delta r_p, \Delta\dot{r}_p$, $h^{(1)}$ derivatives and $\Delta r_p^2, \Delta\dot{r}_p^2, \Delta r_p\Delta\dot{r}_p$ are retained, while those to $[h^{(1)}]^2$, and $h^{(2)}$ are neglected, as the second order terms in trajectory deviation when multiplied by first order perturbations. It is thus a development at first order in perturbations and second order in trajectory deviation or mixed terms. Previous work [4]-[5] was limited to first order. We write the geodesic equation in the following form:

$$\Delta\ddot{r}_p = \alpha_1\Delta r_p + \alpha_2\Delta\dot{r}_p + \alpha_3\Delta r_p^2 + \alpha_4\Delta\dot{r}_p^2 + \alpha_5\Delta r_p\Delta\dot{r}_p + \alpha_6 + \alpha_7\Delta r_p + \alpha_8\Delta\dot{r}_p \tag{3}$$

The physical significance of the terms is essential‖. The terms $\alpha_{1,2,3,4,5}$ arise from the pure Schwarzschild metric and they may be alternatively interpreted as

‖ The α_1 term correspond to the A term of [4]-[5], apart of an error in the quoted publications. The A term, when corrected, amounts to

$$A = \frac{2M}{r^2}\left[\frac{1}{r} - \frac{3M}{r^2} - \frac{3(r-M)}{r^2(r-2M)^2}\dot{r}_p^2\right] \tag{4}$$

The α_2 term correspond to the B term, α_6 correspond to C, while the terms $\alpha_{3,4,5,7,8}$ are neglected. The terms $\alpha_{1,2,6}$ do not represent radiation reaction.

representing the geodesic deviation of two particles separated on the radial axis by a Δr_p distance. In the scenario of a single falling particle, they represent the *unperturbed* Schwarzschild metric influence calculated on the *perturbed* particle trajectory, i.e. the *real* position ($\alpha_{1,3}$), velocity ($\alpha_{2,4}$) or both (α_5). The α_6 term is the the lowest order containing the perturbations. It represents the perturbation influence calculated at the position and velocity of the particle in the *unperturbed* trajectory and thus not radiation reaction. The α_7 term represents the perturbation influence calculated on the *real* position of the particle in the *perturbed* trajectory. Finally, the α_8 term represents the perturbation influence calculated on the *real* velocity of the particle in the *perturbed* trajectory. The latter two terms contain the lowest order radiation reaction terms.

3. Black hole polar perturbations equation

Zerilli [6] - [9] found the equation for polar perturbations and studied the emitted radiation adding a source term, a freely falling test mass m into the the black M. The equation is written in terms of the wavefunction ψ_l for each l-pole component, the tortoise coordinate r^*, the polar potential $V_l(r)$, the 2^l-pole source component $S_l(r,t)$:

$$\frac{d^2\psi_l(r,t)}{dr^{*2}} - \frac{d^2\psi_l(r,t)}{dt^2} - V_l(r)\psi_l(r,t) = S_l(r,t) \qquad r^* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2} \qquad \lambda = \frac{1}{2}(l-1)(l+2)$$

$$S_l = \frac{\left(1 - \frac{2M}{r}\right)4M\sqrt{(2l+1)\pi}}{(\lambda+1)(\lambda r + 3M)} \times$$

$$\left\{ r\left(1 - \frac{2M}{r}\right)^2 \delta'[r - r_p(t)] - \left(\lambda + 1 - \frac{M}{r} - \frac{6Mr}{\lambda r + 3M}\right)\delta[r - r_p(t)]\right\}$$

where $r_p(t)$, geodesic in *unperturbed Schwarzschild metric*, is the inverse of:

$$t = -4M\left(\frac{r}{2M}\right)^{1/2} - \frac{4M}{3}\left(\frac{r}{2M}\right)^{3/2} - 2M\ln\left[\left(\sqrt{\frac{r}{2M}} - 1\right)\left(\sqrt{\frac{r}{2M}} + 1\right)^{-1}\right] \qquad (5)$$

The perturbations around the particle are (Regge-Wheeler gauge $H_0^l = H_2^l$):

$$H_0^l = -\frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2(\lambda+1)r^3}{r^2(\lambda r + 3M)^2}\psi + \frac{3M^2 - \lambda M r + \lambda r^2}{r(\lambda r + 3M)}\psi_{,r} + (r - 2M)\psi_{,rr} \quad (6)$$

$$H_1^l = r\psi_{,rt} - \frac{3M^2 + 3\lambda M r - \lambda r^2}{(r - 2M)(\lambda r + 3M)^2}\psi_{,t} \qquad (7)$$

The *unperturbed* velocity is given by (r_0 is the test mass position at start):

$$\dot{r}_p = -\left(1 - \frac{2M}{r_p}\right)\left(\frac{2M}{r_p} - \frac{2M}{r_0}\right)^{1/2}\left(1 - \frac{2M}{r_0}\right)^{-1/2} \qquad (8)$$

4. Renormalisation

The infinite sum over the finite multipole components contributions leads to the problem of dealing infinities in the results. For ever larger l the metric perturbations tend to an asymptotic behaviour. In other words, the curves representing each metric perturbation component for each l, accumulate over the $l \to \infty$ curve. Thus the subtraction from each mode of the $l \to \infty$ leads to a convergent series. We extend the application of the Riemann zeta function for renormalisation [4]- [5] to all pertinent terms of the geodesic of *Tab. 1.* Instead, mode-sum renormalisation is planned in the near future. For $L = l + 0.5$, the wavefunction and its derivatives assume the following forms at large L or l [5], [10]¶ when averaged around the particle at r_p:

$$\bar{\psi} \simeq 4\sqrt{2\pi} m L^{-2.5} \qquad \bar{\psi}_{,r} \simeq -\frac{6\sqrt{2\pi} m (r_0 - 2M)}{r_0 (r_p - 2M)} L^{-2.5} \qquad \bar{\psi}_{,rr} \simeq \frac{4\sqrt{2\pi} m (r_0 - 2M)}{r_0 (r_p - 2M)^2} L^{-0.5}$$

$$\bar{\psi}_{,rrr} \simeq \frac{4\sqrt{2\pi} m (r_0 - 2M)}{r_0 (r_p - 2M)^3} \left[\frac{5(r_0 - 2M)}{2r_0} + \frac{9M}{r_p} - 6 \right] L^{-0.5}$$

$$\bar{\psi}_{,t} \simeq \frac{6\sqrt{2\pi} m \sqrt{r_0 - 2M} \dot{r}_p}{\sqrt{r_0} r_p} L^{-2.5} \qquad \bar{\psi}_{,tr} \simeq -\frac{4\sqrt{2\pi} m \sqrt{r_0 - 2M} \dot{r}_p}{\sqrt{r_0} r_p (r_p - 2M)} L^{-0.5}$$

$$\bar{\psi}_{,trr} = -\frac{4\sqrt{2\pi} m \sqrt{r_0 - 2M} \dot{r}_p}{\sqrt{r_0} r_p (r_p - 2M)^2} \left[\frac{5(r_0 - 2M)}{2r_0} + \frac{9M}{r_p} - 4 \right] L^{-0.5}$$

Using the above equations, eqs.(6,7) and recasting $H_{1,t}$ as function of $\bar{\psi}, \ \bar{\psi}_{,r}, \ \bar{\psi}_{,rr}, \ \bar{\psi}_{,rrr}$, the α_6 term for large L or l around the particle is:

$$\alpha_6 = \sum_{l=0}^{\infty} \alpha_6^l \qquad \alpha_6^l = \alpha_6^a L^0 + \alpha_6^b L^{-2} + \alpha_6^c L^{-4} + O(L^{-6}) \tag{9}$$

The term $\alpha_6^a L^0$ needs + renormalisation [12]. The Riemann zeta function [13] and its generalisation, the Hurwitz zeta function [14], are defined by:

$$\zeta(s) = \sum_{l=1}^{\infty} (l)^{-s} \qquad \zeta(s, a) = \sum_{l=0}^{\infty} (l + a)^{-s} \tag{10}$$

where in our case $a = 0.5$. Thus:

$$\zeta(s, 0.5) = \sum_{l=0}^{\infty} (l + 0.5)^{-s} = 2^s \left[\sum_{l=0}^{\infty} (2l + 1)^{-s} \right] \tag{11}$$

Due to the imparity of the term in braces, eq.(11)is rewritten as:

$$\zeta(s, 0.5) = 2^s \left\{ \zeta(s) - \left[\sum_{l=0}^{\infty} (2l)^{-s} \right] \right\} = 2^s \left(1 - 2^{-s}\right) \zeta(s) = (2^s - 1)\, \zeta(s) \tag{12}$$

Some special values of the Hurwitz functions are:

$$\zeta(-2, 0.5) = 0 \qquad \zeta(0, 0.5) = 0 \qquad \zeta(2, 0.5) = \frac{1}{2}\pi^2 \qquad \zeta(4, 0.5) = \frac{1}{6}\pi^4 \tag{13}$$

¶ The derivation of such expressions, quoted from a paper in preparation by Barack and Lousto, as referred by [5] and [11], is yet unpublished.
+ The term $\alpha_6^a L^0$ differs from b of eq.(13) in [5] which is again different from the value given in [10].

The latter values when applied to eq.(9), give:

$$\alpha_6 = \alpha_6^a \sum_{l=0}^{\infty} (l+0.5)^0 + \alpha_6^b \sum_{l=0}^{\infty} (l+0.5)^{-2} + \alpha_6^c \sum_{l=0}^{\infty} (l+0.5)^{-4} + [0(l+0.5)^{-6}] =$$

$$\alpha_6^a \zeta(0,0.5) + \alpha_6^b \zeta(2,0.5) + \alpha_6^c \zeta(4,0.5) + [0(l+0.5)^{-6}] = \frac{1}{2}\pi^2 \alpha_6^b + \frac{1}{6}\pi^4 \alpha_6^c + [0(l+0.5)^{-6}] \quad (14)$$

For the renormalisation of $\alpha_{7,8}$ terms, the expressions: $\bar{\psi}_{,tt} \bar{\psi}_{,ttt} \bar{\psi}_{,ttr} \bar{\psi}_{,trrr} \bar{\psi}_{,rrrr}$ are deducted [12] operating on the averaged wavefunctions and derivatives, and the homogeneous wave equation. The latter is recast as [1] - [2]:

$$\frac{1}{\rho^2} \frac{d^2 \psi_l(r,t)}{dr^2} - \frac{d^2 \psi_l(r,t)}{dt^2} + \frac{\rho - 1}{r \rho^2} \frac{d \psi_l(r,t)}{dr} - V_l(r) \psi_l(r,t) = 0 \quad (15)$$

where $\rho = dr^*/dr$. Deriving sequentially eq.(15), we get the ψ needed derivatives. The latter are evaluated for $L \rightarrow \infty$ and when inserted in the $\alpha_{7,8}$ terms, result into:

$$\alpha_7 = \sum_{l=0}^{\infty} \alpha_7^l \qquad \alpha_7^l = \alpha_7^a L^2 + \alpha_7^b L^0 + \alpha_7^c L^{-2} + \alpha_7^d L^{-4} + O(L^{-6}) \quad (16)$$

$$\alpha_8 = \sum_{l=0}^{\infty} \alpha_8^l \qquad \alpha_8^l = \alpha_8^a L^0 + \alpha_8^b L^{-2} + \alpha_8^c L^{-4} + O(L^{-6}) \quad (17)$$

Normalisation of eqs. (16,17) leads to:

$$\alpha_7 = \frac{1}{2}\pi^2 \alpha_7^c + \frac{1}{6}\pi^4 \alpha_7^d + [0(l+0.5)^{-6}] \qquad \alpha_8 = \frac{1}{2}\pi^2 \alpha_8^b + \frac{1}{6}\pi^4 \alpha_8^c + [0(l+0.5)^{-6}] \quad (18)$$

5. Conclusions

We have obtained the following results: i) calculation and determination of all terms up to first order in perturbations and second in trajectory deviation, contributing to the trajectory of a radially falling test mass in Schwarzschild geometry; ii) renormalisation of all divergent terms stemmed from the infinite sum of finite angular momentum dependent components by the zeta Riemann and Hurwitz functions; iii) correction and improvements of previously published results.

6. References

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Table 1. Classification of geodesic terms.

$\Gamma^t_{rr}\left(\frac{dr}{dt}\right)^3$	$2\Gamma^t_{tr}\left(\frac{dr}{dt}\right)^2$	$-\Gamma^r_{rr}\left(\frac{dr}{dt}\right)^2$	$\Gamma^t_{tt}\left(\frac{dr}{dt}\right)$	$-2\Gamma^r_{tr}\left(\frac{dr}{dt}\right)$	$-\Gamma^r_{tt}$
	$\eta^{tt}\eta_{tt,r}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}\eta_{rr,r}\dot{r}_p^2$			$\frac{1}{2}\eta^{rr}\eta_{tt,r}$
		Unperturbed Schwarzschild terms			
	$\eta^{tt}_{,r}\eta_{tt,r}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}_{,r}\eta_{rr,r}\dot{r}_p^2$			$\frac{1}{2}\eta^{rr}_{,r}\eta_{tt,r}$
	$\eta^{rr}\eta_{tt,rr}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}\eta_{rr,rr}\dot{r}_p^2$			$\frac{1}{2}\eta^{rr}\eta_{tt,rr}$
		α_1 *terms*			
	$2\eta^{tt}\eta_{tt,r}\dot{r}_p$	$-\eta^{rr}\eta_{rr,r}\dot{r}_p$			
		α_2 *terms*			
	$\eta^{tt}_{,r}\eta_{tt,rr}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}_{,r}\eta_{rr,rr}\dot{r}_p^2$			$\frac{1}{2}\eta^{rr}_{,r}\eta_{tt,rr}\Delta r_p^2$
		α_3 *terms*			
	$\eta^{tt}\eta_{tt,r}$	$-\frac{1}{2}\eta^{rr}_{,r}\eta_{rr,rr}$			
		α_4 *terms*			
	$2\eta^{tt}_{,r}\eta_{tt,r}\dot{r}_p$	$-\eta^{rr}_{,r}\eta_{rr,r}\dot{r}_p$			
	$2\eta^{tt}\eta_{tt,rr}\dot{r}_p$	$-\eta^{rr}\eta_{rr,rr}\dot{r}_p$			
		α_5 *terms*			
$\eta^{tt}h_{tr,r}\dot{r}_p^3$	$h^{tt}\eta_{tt,r}\dot{r}_p^2$	$-\frac{1}{2}h^{rr}\eta_{rr,r}\dot{r}_p^2$	$\frac{1}{2}\eta^{tt}h_{tt,t}\dot{r}_p$	$-\eta^{rr}h_{rr,t}\dot{r}_p$	$-\eta^{rr}h_{rt,t}$
$-\frac{1}{2}\eta^{tt}h_{rr,t}\dot{r}_p^3$	$\eta^{tt}h_{tt,r}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}h_{rr,r}\dot{r}_p^2$	$-\frac{1}{2}h^{tr}\eta_{tt,r}\dot{r}_p$	$-h^{rt}\eta_{tt,r}\dot{r}_p$	$\frac{1}{2}h^{rr}\eta_{tt,r}$
$\frac{1}{2}h^{tr}\eta_{rr,r}\dot{r}_p^3$					$\frac{1}{2}\eta^{rr}h_{tt,r}$
		α_6 *terms*			
$\eta^{tt}_{,r}h_{tr,r}\dot{r}_p^3$	$h^{tt}_{,r}\eta_{tt,r}\dot{r}_p^2$	$-\frac{1}{2}h^{rr}_{,r}\eta_{rr,r}\dot{r}_p^2$	$\frac{1}{2}\eta^{tt}_{,r}h_{tt,t}\dot{r}_p$	$-\eta^{rr}_{,r}h_{rr,t}\dot{r}_p$	$-\eta^{rr}_{,r}h_{rt,t}$
$\eta^{tt}h_{tr,rr}\dot{r}_p^3$	$\eta^{tt}_{,r}h_{tt,r}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}_{,r}h_{rr,r}\dot{r}_p^2$	$\frac{1}{2}\eta^{tt}h_{tt,tr}\dot{r}_p$	$-\eta^{rr}h_{rr,tr}\dot{r}_p$	$-\eta^{rr}h_{rt,tr}$
$-\frac{1}{2}\eta^{tt}_{,r}h_{rr,t}\dot{r}_p^3$	$h^{tt}\eta_{tt,rr}\dot{r}_p^2$	$-\frac{1}{2}h^{rr}\eta_{rr,rr}\dot{r}_p^2$	$-\frac{1}{2}h^{tr}_{,r}\eta_{tt,r}\dot{r}_p$	$-h^{rt}_{,r}\eta_{tt,r}\dot{r}_p$	$\frac{1}{2}h^{rr}_{,r}\eta_{tt,r}$
$-\frac{1}{2}\eta^{tt}h_{rr,tr}\dot{r}_p^3$	$\eta^{tt}h_{tt,rr}\dot{r}_p^2$	$-\frac{1}{2}\eta^{rr}h_{rr,rr}\dot{r}_p^2$	$-\frac{1}{2}h^{tr}\eta_{tt,rr}\dot{r}_p$	$-h^{rt}\eta_{tt,rr}\dot{r}_p$	$\frac{1}{2}\eta^{rr}_{,r}h_{tt,r}$
$\frac{1}{2}h^{tr}_{,r}\eta_{rr,r}\dot{r}_p^3$					$\frac{1}{2}h^{rr}\eta_{tt,rr}$
$\frac{1}{2}h^{tr}\eta_{rr,rr}\dot{r}_p^3$					$\frac{1}{2}\eta^{rr}h_{tt,rr}$
		α_7 *terms*			
$3\eta^{tt}h_{tr,r}\dot{r}_p^2$	$2h^{tt}\eta_{tt,r}\dot{r}_p$	$-h^{rr}\eta_{rr,r}\dot{r}_p$	$\frac{1}{2}\eta^{tt}h_{tt,t}$	$-\eta^{rr}h_{rr,t}$	
$-\frac{3}{2}\eta^{tt}h_{rr,t}\dot{r}_p^2$	$2\eta^{tt}h_{tt,r}\dot{r}_p$	$-\eta^{rr}h_{rr,r}\dot{r}_p$	$-\frac{1}{2}h^{tr}\eta_{tt,r}$	$-h^{rt}\eta_{tt,r}$	
$\frac{3}{2}h^{tr}\eta_{rr,r}\dot{r}_p^2$					
		α_8 *terms*			