Treatment of Grating Pairs Using Plane-Wave Approximation

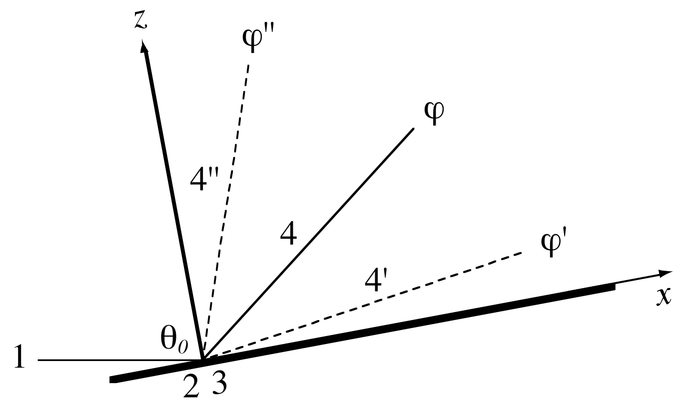
LIGO-G040194-00-Z

Yanbei Chen

based on

- Talks of Stacy Wise and Volker Quetschke
- E.B. Treacy, IEEE J. of Quant Elec, 9, QE-5 (1969)]
- Discussions made at the configuration sessions

Single grating, single plane wave



1. Incident wave
   $\exp[ik\sin\theta_0 x - ik\cos\theta_0 z]$

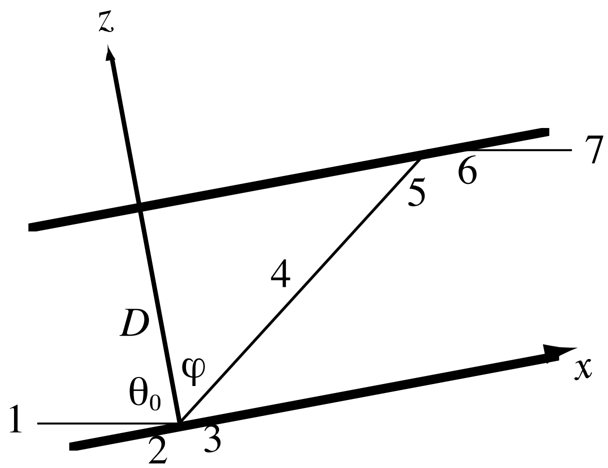
2. On the surface (ingoing wave)
   $\exp[ik\sin\theta_0 x]$

3. Immediately after (outgoing wave)
   $$\exp[ik\sin\theta_0 x]\exp[2ih(x)]$$
   $$\Rightarrow \quad \exp[ik\sin\theta_0 x]\sum_n C_n\exp[in\alpha x]$$

4. Outgoing waves
   $$\sum_n C_n\exp[i(k\sin\theta_0+n\alpha)x+ik\cos\varphi_n z]\,,\quad (k\sin\theta_0+n\alpha)^2+k^2\cos^2\varphi_n = k^2$$

We only consider one of the non-zero orders!

A Pair of Gratings: I



5. At the surface of the second grating

$$\exp[ik\sin\theta_0 x + in\alpha x + ik\cos\varphi D]$$

6. Immediately after second grating

$$\begin{aligned} & \exp[ik\sin\theta_0 x + in\alpha x + ik\cos\varphi D]\exp[-in\alpha x] \\ = \; & \exp[ik\sin\theta_0 x + ik\cos\varphi D] \end{aligned}$$

7. Final outgoing wave

$$\exp[ik\sin\theta_0 x - ik\cos\theta_0(z-D) + ik\cos\varphi D]$$

8. Phase of the outgoing plane wave:

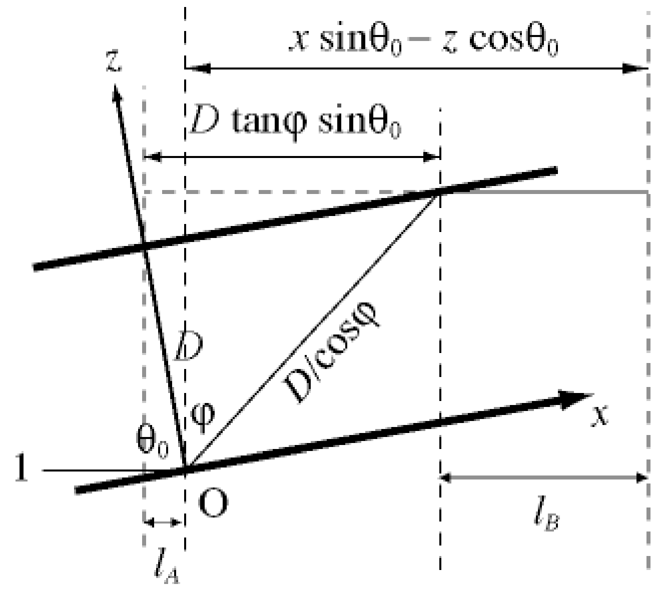
$$\Phi(\omega; x, z) = \frac{\omega}{c}[x\sin\theta_0 - (z-D)\cos\theta_0 + D\cos\varphi]$$

$$\Rightarrow \quad \frac{\partial\Phi}{\partial\omega} = \frac{x\sin\theta_0 - (z-D)\cos\theta_0}{c} + \frac{D}{c}\left[\cos\varphi - \omega\frac{d\varphi}{d\omega}\sin\varphi\right]$$

A Pair of Gratings: II

$$k \sin\theta_0 + n\alpha = k \sin\varphi \Rightarrow \omega \frac{d\varphi}{d\omega} = \frac{\sin\theta_0 - \sin\varphi}{\cos\varphi}$$

$$\Rightarrow \quad \frac{\partial\Phi}{\partial\omega} = \frac{x\sin\theta_0 - (z-D)\cos\theta_0}{c} + \frac{D}{c}\left[\cos\varphi - \omega\frac{d\varphi}{d\omega}\sin\varphi\right] = \frac{x\sin\theta_0 - z\cos\theta_0}{c} + \frac{D\cos\theta_0}{c} + \frac{D}{c}\left[\frac{1}{\cos\varphi} - \tan\varphi\sin\theta_0\right]$$

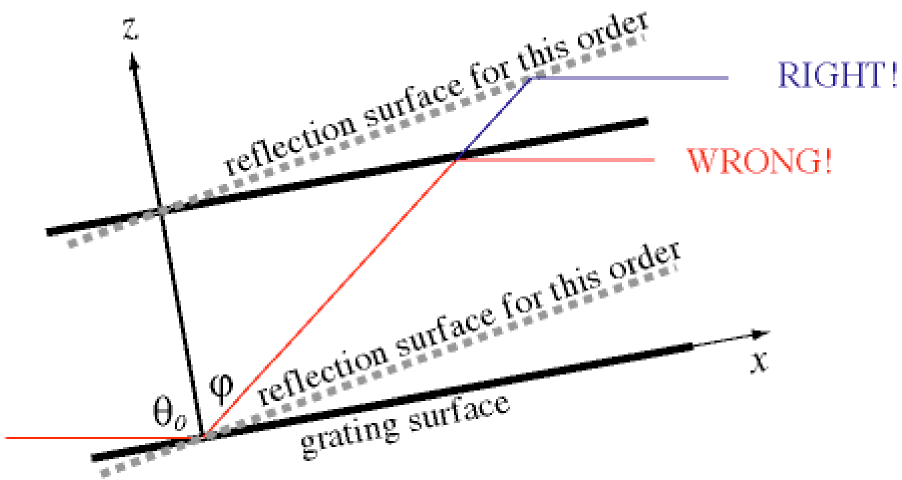


From geometry, this is indeed

$$\frac{\partial\Phi}{\partial\omega} = \frac{D/\cos\varphi + l_B}{c}$$

$$= \frac{L(\omega)}{c} \qquad (> 0)$$

What‘s wrong with our previous understanding?



- The grating is not simply reflecting the plane wave, but instead imposes an x-dependent phaseshift.
- At first sight, this phaseshift is oscillatory, i.e. $2h(x)$ in page 2. But for any particular diffraction order, this phase shift is linear $(n\alpha x)$ in x!
- As a consequence, we can effectively think of the grating pair as a pair of mirrors with frequency dependent orientation [see Treacy‘s first derivation] --- this is why our original thought was completely wrong.

Treacy's second derivation: I

- Consider a trial impulse with frequency spectrum $A(\omega)$, centered at ω_0 . If, at the output port of the two-grating system,

$$\int d\omega A(\omega) e^{-i\omega t} \Rightarrow \int d\omega A(\omega) e^{-i\omega t + i\Phi(\omega)}$$

- Then the Stationary-Phase Approximation implies a pulse delay of

$$\tau = \left.\frac{\partial \Phi}{\partial \omega}\right|_{\omega_0}$$

- **Interestingly, this shows that, for plane waves and assuming no amplitude modulation, the white-light effect should not exist!**

- Now, back to our grating system, can we actually show that $\tau = L(\omega)$?

Treacy's second derivation: II

- Now suppose we have an impulse at the input port (with various values of $\mathbf{k}$)

$$\int A_{\mathbf{k}} \exp[-i\omega(\mathbf{k})t + i\mathbf{k}\cdot\mathbf{x}]d^3\mathbf{k}$$

- Pulse peak trajectory, before hitting the grating, is given by (SPA)

$$\frac{\partial}{\partial \mathbf{k}}[-\omega(\mathbf{k})t + \mathbf{k}\cdot\mathbf{x}] = 0 \Rightarrow x = \frac{\mathbf{k}}{|\mathbf{k}|}ct \quad (t < 0)$$

- After the grating, $\mathbf{k}$-component will be diffracted to $\mathbf{k}$'

$$\int A_{\mathbf{k}} \exp[-i\omega(\mathbf{k})t + i\mathbf{k}'\cdot\mathbf{x}]d^3\mathbf{k} = \int A_{\mathbf{k}} \exp[-i\omega(\mathbf{k}')t + i\mathbf{k}'\cdot\mathbf{x}]d^3\mathbf{k}\,, \quad |\mathbf{k}'| = |\mathbf{k}|\,.$$

- The peak trajectory is given by (SPA again)

$$\frac{\partial}{\partial \mathbf{k}}[-\omega(\mathbf{k}')t + \mathbf{k}'\cdot\mathbf{x}] = 0 \Leftrightarrow \frac{\partial}{\partial \mathbf{k}'}[-\omega(\mathbf{k}')t + \mathbf{k}'\cdot\mathbf{x}] = 0 \Rightarrow x = \frac{\mathbf{k}'}{|\mathbf{k}'|}ct \quad (t > 0)$$

- The pulse travels along $\mathbf{k}$' with speed c: same direction and speed as the phase front!!

  ⇒ Travel times is $\tau = L(\omega)$!

Summary

- A grating pair *can* be analyzed qualitatively by elementary scalar-wave optics, yielding $\partial\Phi/\partial\omega = L(\omega)/c$, which, being always positive, is fundamentally different from our old understanding of $\Phi = \omega\, L(\omega)\, /c$.
- Two methods can be used to analyze the grating-pair system, as given by Treacy:
  - When using monochromatic plane waves, one has to be careful about defining the planes of reflection: they are not parallel to the grating surfaces! [Origin of our long-time mistake.]
  - One can also inject pulses into the grating system as *gedanken experiments*.
    - One has a pulse delay of $\tau = \partial\Phi/\partial\omega$. This already means that white-light cavity cannot be made.
    - For a grating pair, **the pulse travels in the same way as the phase front**, so $\tau = L(\omega)$