# An Efficient Algorithm for Computing the Time Resolved Full Sky Cross Power in an Interferometer with Omni-directional Elements

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**Abstract**

Both time- and frequency-domain algorithms will be presented for the fast, approximate, numeric evaluation of the cross-correlation integral that appears in a variety of imaging applications. These algorithms possess a number of useful features. Their operation counts are linear in the amount (duration) of data to be processed, which allows them to be run quickly on small amounts of data, thereby allowing one to resolve time-dependence in the sources being imaged. The algorithms also provide means by which to systematically increase or decrease their accuracy for increases and decreases in speed respectively, that is to say one can choose to obtain the result to lower accuracy and thereby obtain the result more quickly or with more modest computer requirements. The frequency-domain implementation is particularly fast, and on a 32-bit $1.8\,\mathrm{GHz}$ Pentium M computer has demonstrated the ability to synthesize full-sky cross-power images for a single baseline up to harmonic order 19, with an SNR of $26\,\mathrm{dB}$, at speeds of over $800\,\mathrm{ksample/s}$.

# Measuring Cross Power Incident Upon a Network

- Two instruments at $\vec{r}_1$ and $\vec{r}_2$.

- Instruments produce outputs $g_1(t)$ and $g_2(t)$.

- Choose a direction $\hat{s}$.

- Delay-correct outputs to the origin.

- Integrate their product for a time $T$ centred on time $t$.

- Repeat for new $\hat{s}$ until sky is mapped.

- Repeat for new $t$.

$\vec{r}_1 \cdot \hat{s}$

$\hat{s}$

$\vec{r}_2$

$\vec{r}_1$

$g_1(t)$
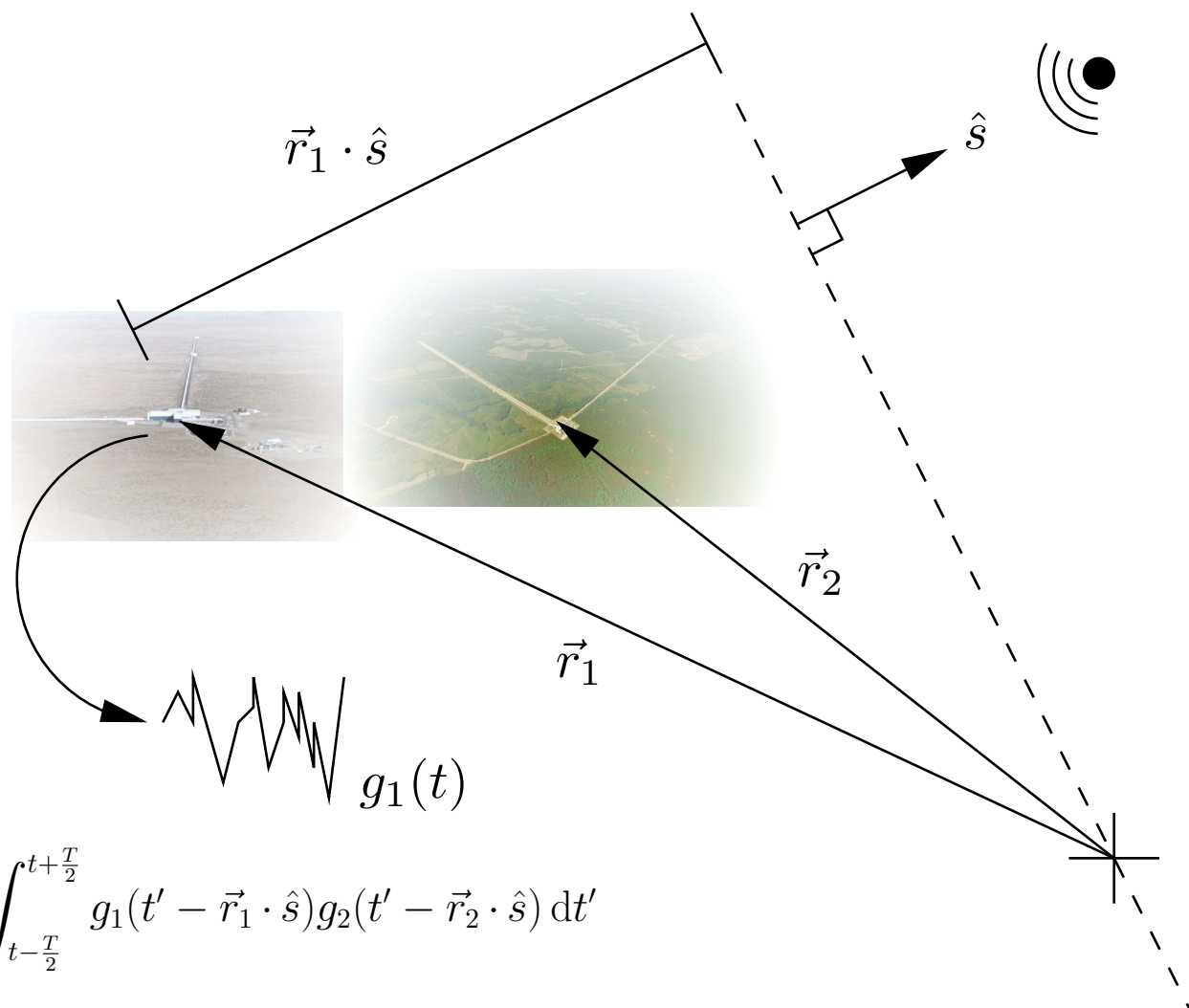
$$\xi_{1,2}(\hat{s}) = \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} g_1(t' - \vec{r}_1 \cdot \hat{s}) g_2(t' - \vec{r}_2 \cdot \hat{s})\, \mathrm{d}t'$$

$$\xi_{1,2}(\hat{s}) = \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} g_1(t' - \vec{r}_1 \cdot \hat{s}) g_2(t' - \vec{r}_2 \cdot \hat{s}) \, \mathrm{d}t'$$

# Comments

• Brute force approach is computationally costly.

• van Cittert-Zernike theorem provides a quick technique for narrow band and narrow field of view case.

• Generalization to full-sky case is available (Perley?).

• But still really only for narrow frequency bands, and may also be computationally costly.

• We need:

    – large bandwidth (comparable to centre frequency),

    – wide field of view (full sky),

    – and also ability to generate many images, each for a short interval of time (to search for transients).

$$\xi_{1,2}(\hat{s}) = \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} g_1(t' - \vec{r}_1 \cdot \hat{s}) g_2(t' - \vec{r}_2 \cdot \hat{s}) \, \mathrm{d}t'$$

# Getting Started

- Each time series is a vector,

$$\vec{g} = \begin{bmatrix} \vdots \\ g(t_j) \\ \vdots \end{bmatrix},$$

  where $t_j = j\Delta t$.

- Geometric delay is a linear transformation,

$$\vec{g}(\hat{s}) = \mathbb{T}(\vec{r} \cdot \hat{s}) \cdot \vec{g}.$$

- Integrated cross power is the inner product

$$\xi_{1,2}(\hat{s}) = \frac{1}{N}\vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s}) \cdot \vec{g}_2.$$

$$\xi_{1,2}(\hat{s}) = \frac{1}{N}\vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s}) \cdot \vec{g}_2.$$

# Observations

- $g(t)$ are band-limited functions of time, so $\xi(\hat{s})$ is an approximately "band-limited" function on the sphere.

- We can choose a basis for $\xi(\hat{s})$:

  – mesh of samples over the sphere,

  – vector of spherical harmonic coefficients.

- Want to avoid recomputing $\mathbb{T}$ for different $t$.

- Want to find a basis for $\vec{g}$ in which $\mathbb{T}$ is diagonal or band diagonal.

$$\xi_{1,2}(\hat{s}) = \frac{1}{N} \vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s}) \cdot \vec{g}_2.$$

# Geometric Delay

- The discrete samples $\vec{g}$ are time delayed by:

    - convolving with the $\mathrm{sinc}$ interpolating kernel
    - to construct a continuous function of time

$$g_i(t) = \sum_{k=-\infty}^{\infty} g_i(k\Delta t) \, \mathrm{sinc}[(t - k\Delta t)/\Delta t].$$

    - which is evaluated at the delayed sample times, $t_j = j\Delta t - \vec{r}_i \cdot \hat{s}$,

$$g_i(t_j; \vec{r}_i \cdot \hat{s}) = \sum_{k=-\infty}^{\infty} g_i(k\Delta t) \, \mathrm{sinc}[(j\Delta t - \vec{r}_i \cdot \hat{s} - k\Delta t)/\Delta t]$$

- So the matrix elements of $\mathbb{T}$ are

$$\mathbb{T}_{jk}(\vec{r}_i \cdot \hat{s}) = \mathrm{sinc}(j - k - \vec{r}_i \cdot \hat{s}/\Delta t).$$

- The elements of $\mathbb{T}$ decay as the inverse of their distance from the diagonal.

$$\xi_{1,2}(\hat{s}) = \frac{1}{N} \vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s}) \cdot \vec{g}_2.$$

# Approximations

- Set to 0 elements of $\mathbb{T}$ far from the diagonal
  - Makes $\mathbb{T}$ an $N_{\mathcal{T}}$-point interpolation
- $\xi(\hat{s})$ and $\mathbb{T}(\vec{r} \cdot \hat{s})$ are both exactly band-limited:
  - assume that above some spherical harmonic order(s), all coefficients in their expansions are 0.
- Sky is not rotating:
  - avoids recomputing $\mathbb{T}$
  - good if duration of integration small
  - do long integration by summing a sequence of snap-shots
- Each approximation provides an adjustable knob:
  - $N_{\mathcal{T}}$: how many elements to retain in $\mathbb{T}$,
  - $l_{\mathcal{T}}$: to what harmonic order to expand $\mathbb{T}(\vec{r} \cdot \hat{s})$,
  - $l_{\xi}$: to what harmonic order to expand $\xi(\hat{s})$,
  - the maximum time to integrate without splitting into snap-shots.

$$\xi_{1,2}(\hat{s}) = \frac{1}{N} \vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s}) \cdot \vec{g}_2.$$

# Time-Domain Technique

- Each row of $\mathbb{T}$ is the same as the row above it, shifted to the right by 1 element — only need to know one row.

- For each instrument, compute the harmonic expansion of each of the $N_{\mathcal{T}}$ elements in the middle row of $\mathbb{T}(\vec{r} \cdot \hat{s})$.

$$\mathbb{T}_{jk}(\vec{r} \cdot \hat{s}) \rightarrow \mathbb{T}_{jk}^{(lm)}(\vec{r})$$

- In an Earth-fixed co-ordinate system, $\vec{r}$ is a constant so $\mathbb{T}_{jk}^{(lm)}(\vec{r})$ is a number (not a function of anything).

- Compute one column of

$$\vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{(lm)\,\mathrm{T}}(\vec{r}_1)$$

- Compute corresponding row of

$$\mathbb{T}^{(lm)}(\vec{r}_2) \cdot \vec{g}_2$$

- Compute the product to obtain one sample of $\xi^{(lm)}$, repeat and sum.

# Product of Harmonic Expansions

- In the harmonic domain:

$$
G^{(lm)}(t_j) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sum_{l_1=\max\{0,l-l_{g_2}\}}^{\min\{l_{g_1},l+l_{g_2}\}} \sqrt{2l_1+1} \sum_{\substack{l_2=|l-l_1|, \\ l_1+l_2+l \text{ even}}}^{\min\{l_{g_2},l+l_1\}} \sqrt{2l_2+1} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \times
$$

$$
\sum_{m_1=\max\{-l_1,m-l_2\}}^{\min\{+l_1,m+l_2\}} g_1^{(l_1 m_1)}(t_j) g_2^{(l_2,(m-m_1))}(t_j) \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m-m_1 & -m \end{pmatrix}.
$$

- $\mathrm{O}(l^{<5})$, or reduced to $\mathrm{O}(l^{<3})$ for azimuthally-symmetric functions.

- Or use the FSHT of Healy et al. to transform to the image domain, compute the product, and transform back. $\mathrm{O}(l^2 \log^2 l)$.

# Summing Images (Baselines or Snap-Shots)

- Rotate coefficients in the harmonic domain using Wigner $D$-matrices to sky-fixed co-ordinate system.
  - Pre-compute matrices for rotation from baseline co-ordinate system to sky-fixed at 0 h GMST. $\mathrm{O}(l^3)$
  - Rotate around Earth's spin axis by multiplying by $\mathrm{e}^{im\mathrm{GMST}}$. $\mathrm{O}(l^2)$
- Sum coefficients from different images.

$$\xi_{1,2}(\hat{s}) = \frac{1}{N}\vec{g}_1^{\mathrm{T}} \cdot \mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s}) \cdot \vec{g}_2.$$

# Frequency Domain

- The matrix multiplication

$$\mathbb{T}(\vec{r} \cdot \hat{s}) \cdot \vec{g}$$

  is the convolution of $\vec{g}$ with an impulse response function (the middle row of $\mathbb{T}$).

- Fourier transform $\vec{g}$ and the middle row of $\mathbb{T}$, and compute convolution in the frequency domain.

- Equivalent to a co-ordinate transformation that diagonalizes $\mathbb{T}$.

- In time domain, difficult to pre-compute $\mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s})$ but now we can and it's trivial.

- Speed increases:

  - Convolution goes from $\mathrm{O}(N_{\mathcal{T}}^2)$ to $\mathrm{O}(N_{\mathcal{T}} \log N_{\mathcal{T}})$,
  - By pre-computing $\mathbb{T}^{\mathrm{T}}(\vec{r}_1 \cdot \hat{s}) \cdot \mathbb{T}(\vec{r}_2 \cdot \hat{s})$, the "product of functions on the sphere" problem goes from $\mathrm{O}(l^3)$ to $\mathrm{O}(0)$.

- Price:

  - Must use a pre-determined number of samples (integration time).
  - Must be careful with noise resulting from the periodicity implied by the Fourier transform.

# Result

- Can be written, frequency bin by frequency bin, as

$$\xi_{1,2}[q]^{(lm)} = \tilde{g}_1[q]\tilde{g}_2^*[q] \left( \frac{1}{N} \tilde{\mathbb{T}}_q(\vec{r_1} \cdot \hat{s}) \tilde{\mathbb{T}}_q^*(\vec{r_2} \cdot \hat{s}) \right)^{(lm)}$$

- Each frequency bin $q$ provides

  - a vector of harmonic co-efficients describing
  - the angular distribution of cross power
  - integrated for time $T$
  - at $t$
  - $\rightarrow$ a time/frequency/direction decomposition of the detector outputs.

- Summing over the frequency bins and undoing the harmonic expansion to get the total cross power shows another interpretation:
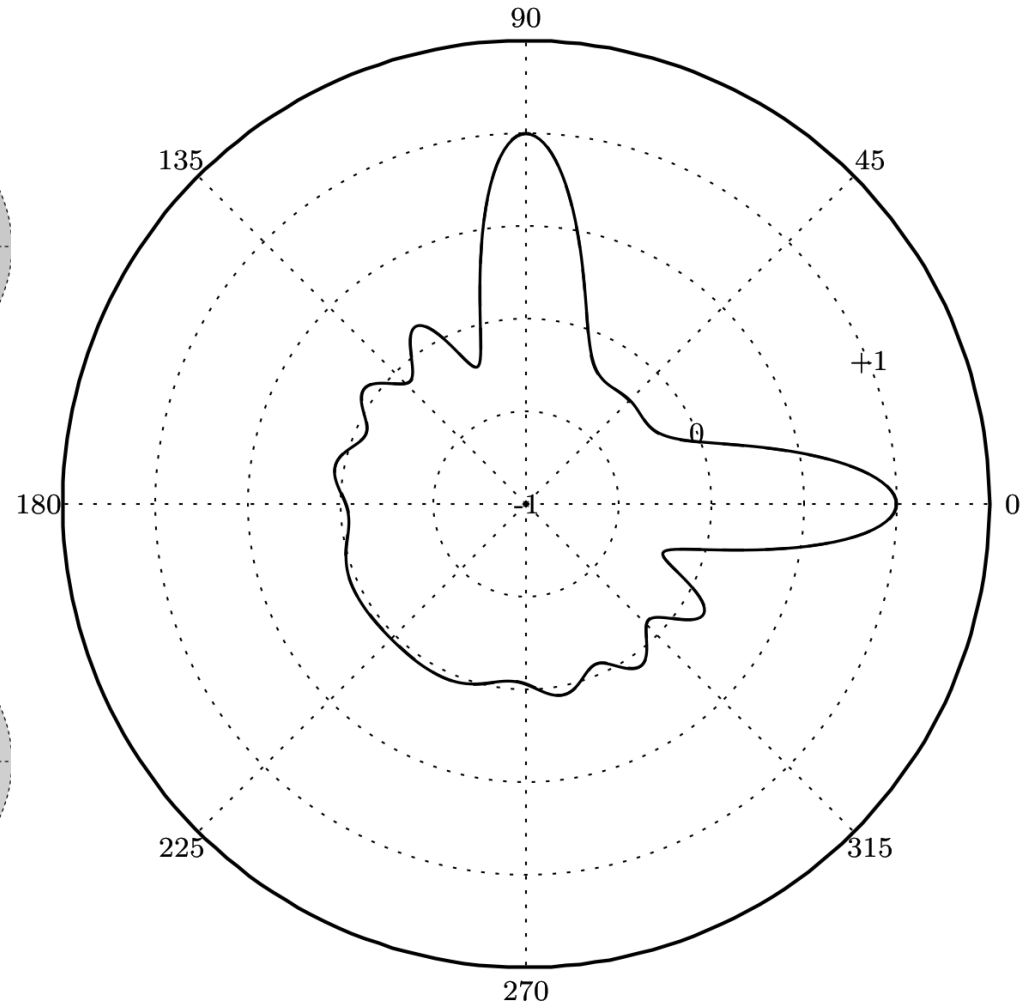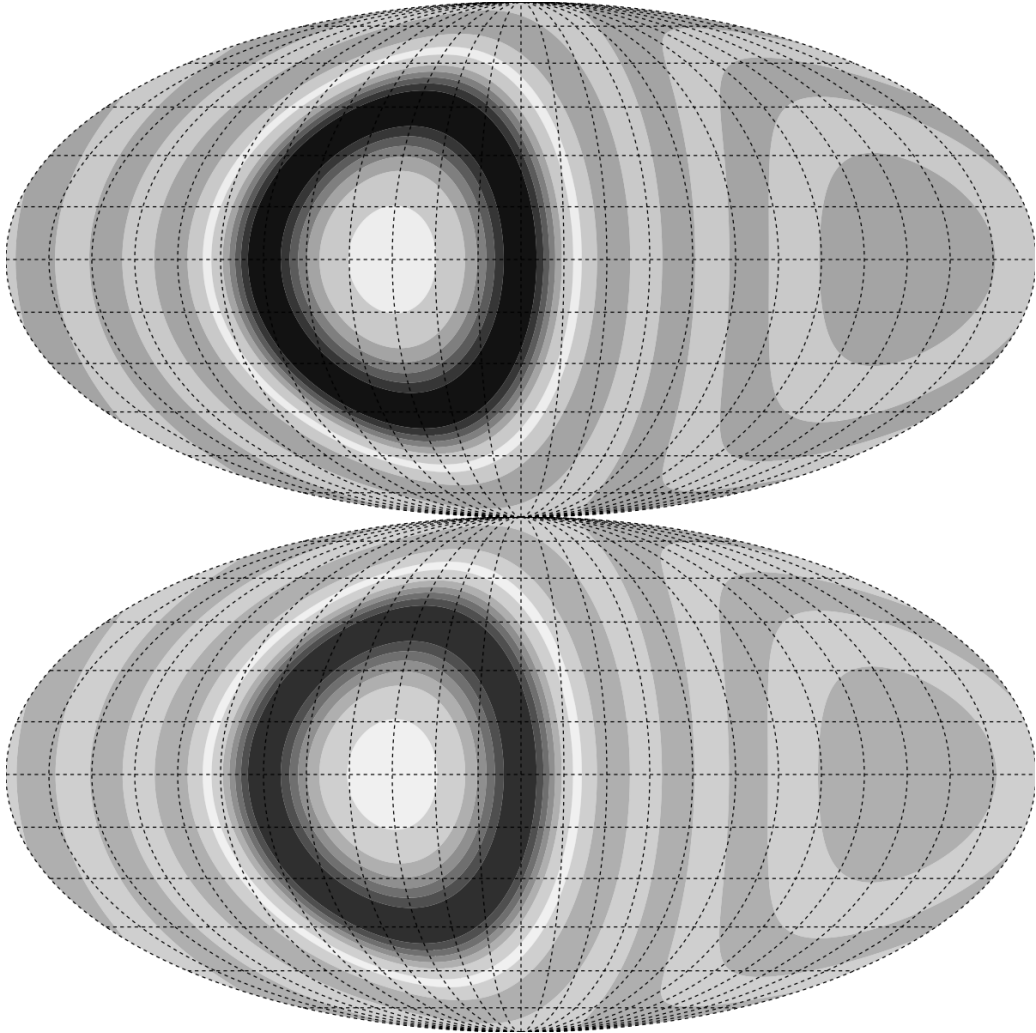
$$\xi_{1,2}(\hat{s}) = \frac{1}{N} \sum_q \tilde{g}_1[q]\tilde{g}_2^*[q] \left( \frac{1}{N} \tilde{\mathbb{T}}_q(\vec{r_1} \cdot \hat{s}) \tilde{\mathbb{T}}_q^*(\vec{r_2} \cdot \hat{s}) \right)$$

- Each frequency bin in the Fourier transform of the convolution of the instrument outputs provides the complex amplitude for exactly one mode of the brightness distribution on the sphere.
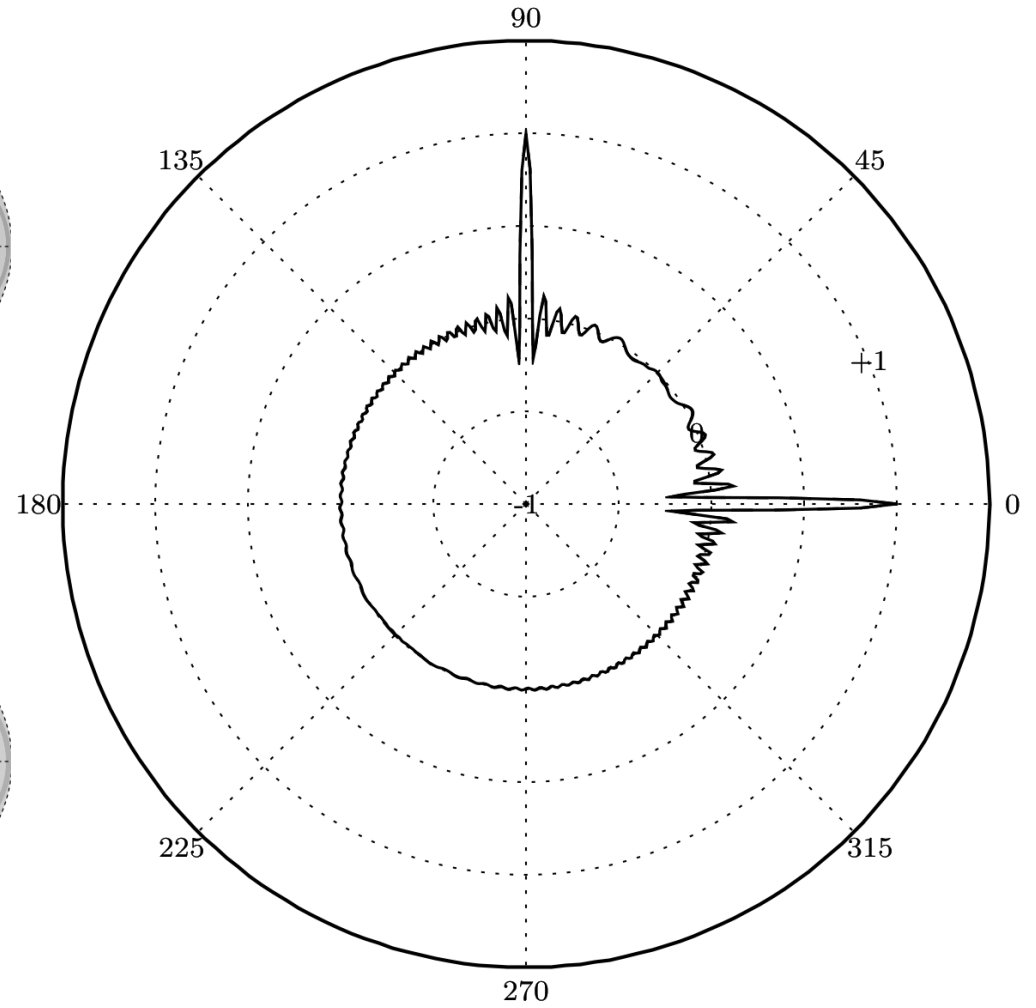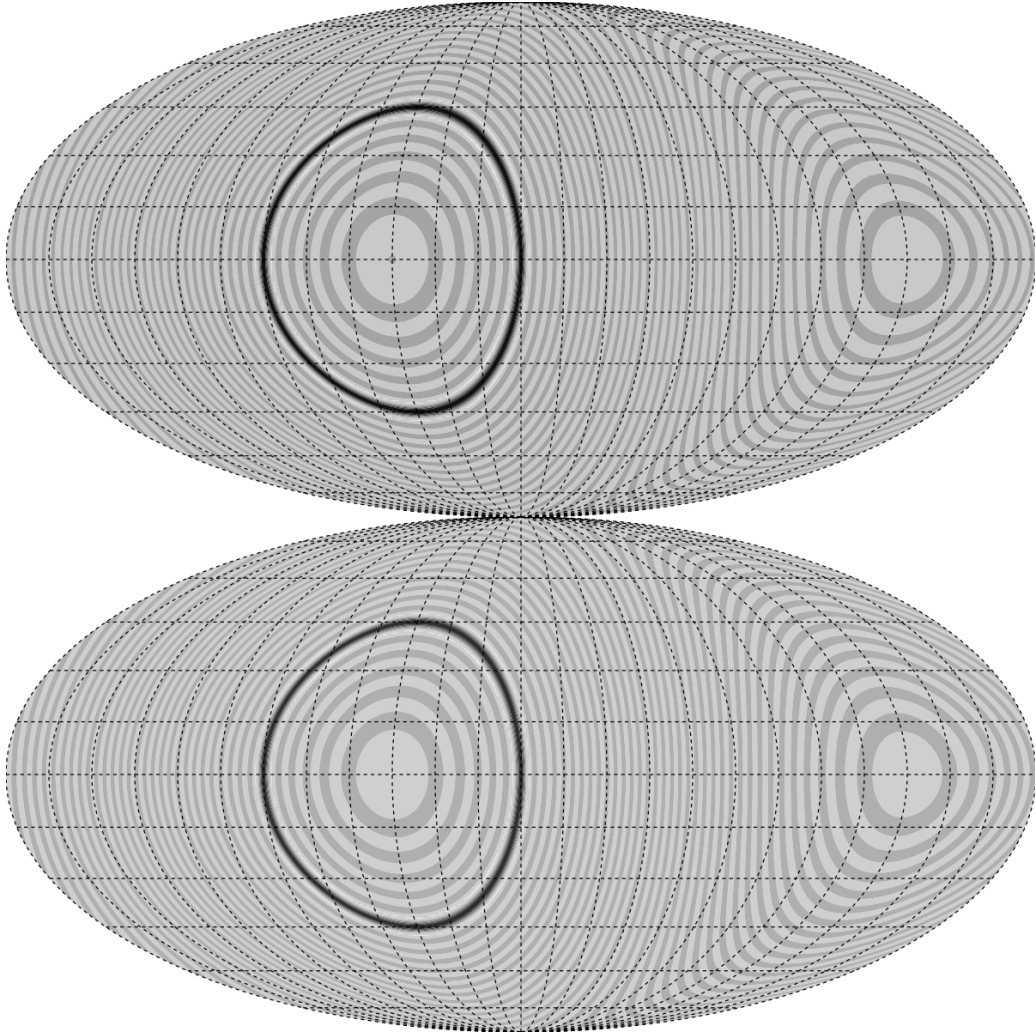
$\Delta t = 512^{-1} \, \mathrm{Hz}^{-1}$, $N_{\mathcal{T}} = 7$, $l_{\mathcal{T}} = 9$, $l_{\xi} = 17$, $\Delta \xi_{\mathrm{RMS}} = -11 \, \mathrm{dB}$
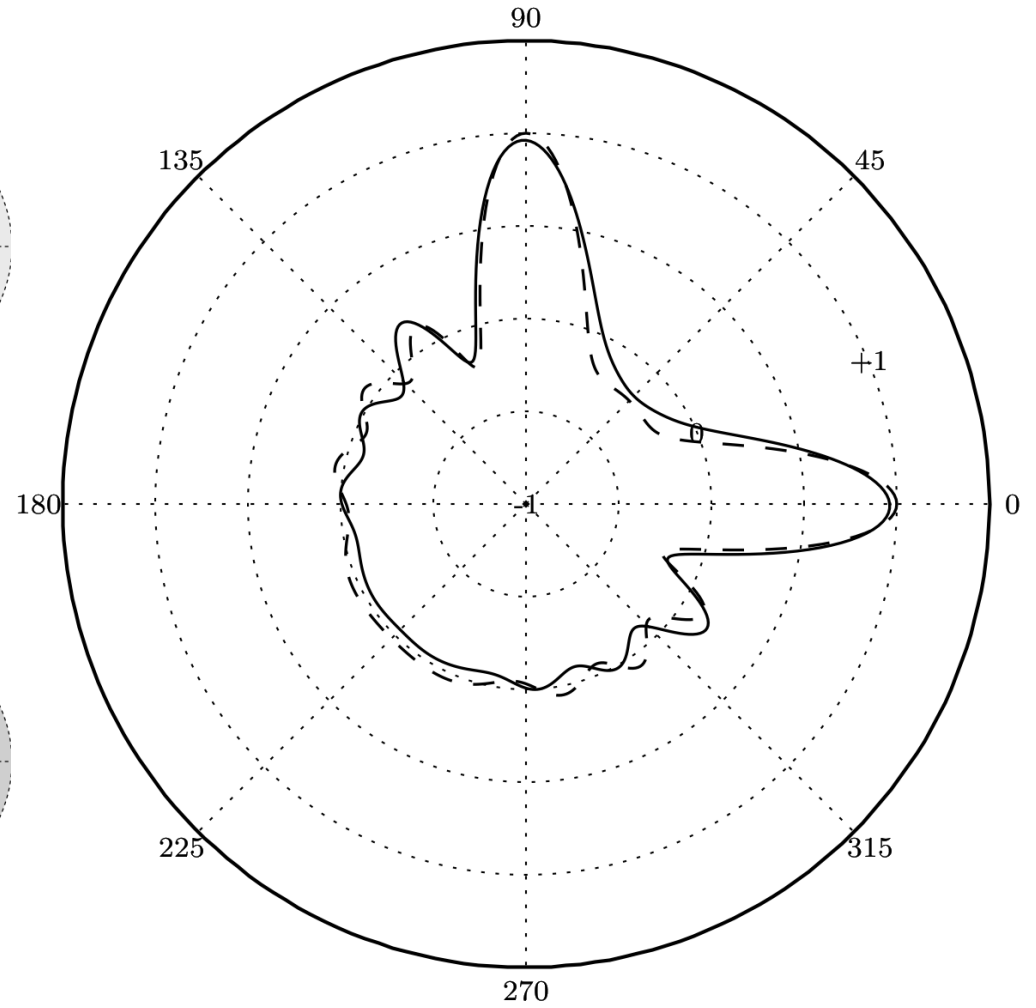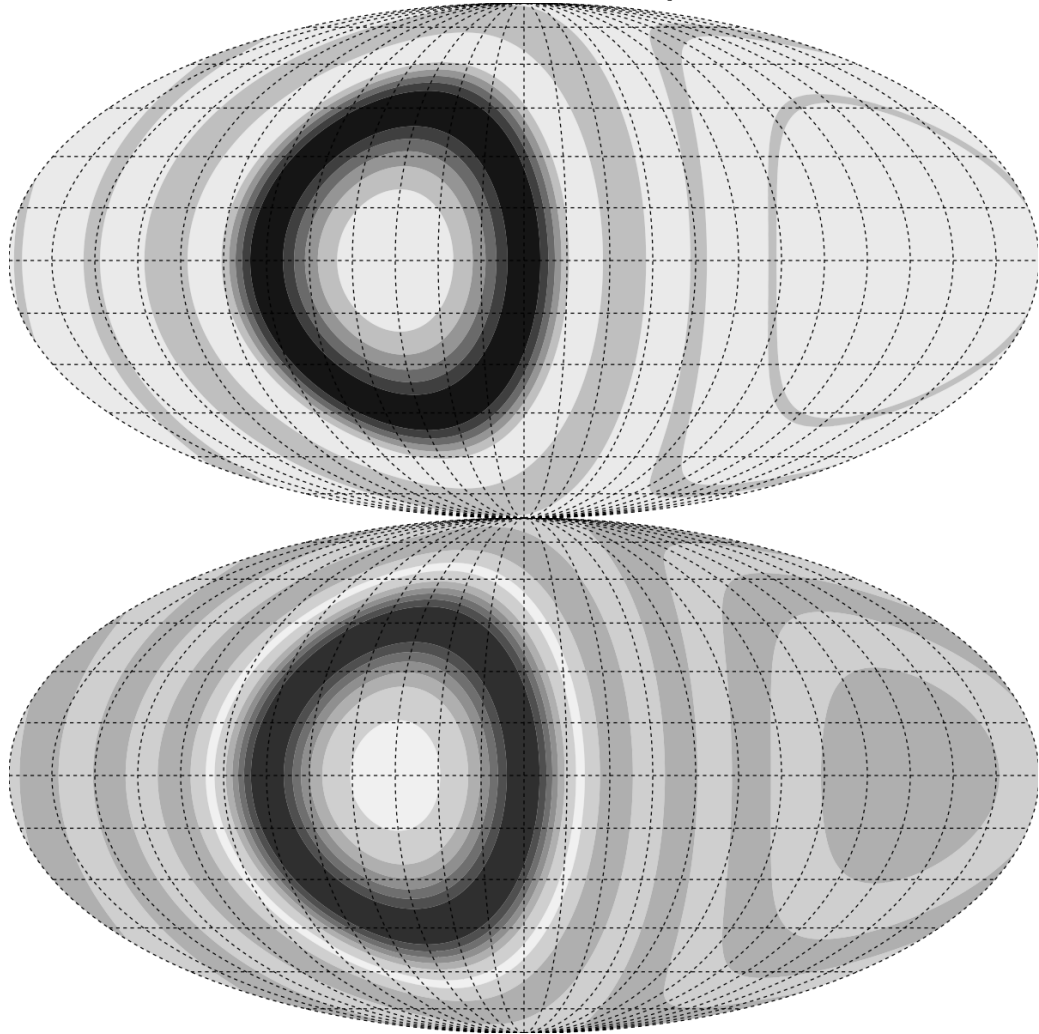
$\Delta t = 512^{-1}\,\mathrm{Hz}^{-1}$, $N_{\mathcal{T}} = 127$, $l_{\mathcal{T}} = 13$, $l_{\xi} = 19$, $\Delta\xi_{\mathrm{RMS}} = -26\,\mathrm{dB}$

$\Delta t = 4096^{-1}\,\mathrm{Hz}^{-1}$, $N_{\mathcal{T}} = 131$, $l_{\mathcal{T}} = 69$, $l_{\xi} = 163$, $\Delta\xi_{\mathrm{RMS}} = -27\,\mathrm{dB}$

# Speed

| Processor | Word Size | Frequency | $l_\xi$ | Correlator Type | Correlator Speed |
|-----------|-----------|-----------|---------|-----------------|------------------|
| Pentium M | 32 bits | 1.8 GHz | 19 | F.D. | 822251 samples/s |
| | | | | T.D. | 18094.5 samples/s |
| | | | 131 | F.D. | 128038 samples/s |
| | | | | T.D. | 149.5 samples/s |

- Defined by wall-clock time required for the correlators to process some number of samples of stationary, unit-variance, Gaussian white noise.

- Does not include:

  - time required to perform harmonic analysis of delay matrices,

  - I/O,

  - time required to generate random data.

- Does include:

  - rotation of harmonic expansion of result to sky-fixed co-ordinate system,

  - Fourier transforms of input data in the case of the frequency-domain correlator.

# Summary

- The recipe is:
  - work in the harmonic domain (instead of pixel domain),
  - work with detector outputs in the frequency domain,
  - cross-correlate in an Earth-fixed co-ordinate system using a short integration to accomodate the rotation of the sky.

- Adds surprisingly little additional cost over an equivalent incoherent single-instrument search.

- Provides several knobs allowing accuracy to be traded for speed.

- Result can be incrementally improved — computing a few harmonic co-efficients at first, then more at a later time if desired, is not more costly than computing all at once.

- Is easily parallelized.