Fluctuation-dissipation theorem for thermo-refractive noise

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1. Formulation
2. Some applications
3. Proof
4. Discussion

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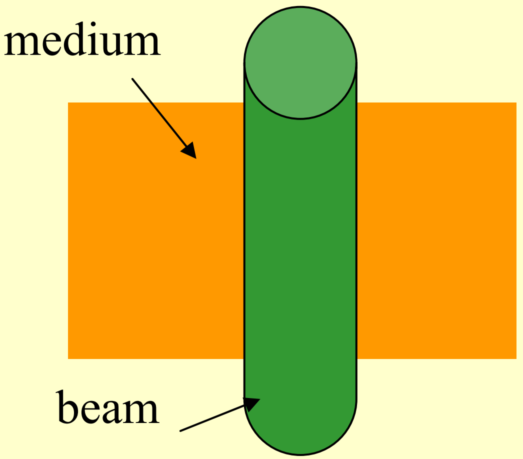
Thermo-refractive noise

Braginsky, Gorodetsky, & Vyatchanin 2000



Index of refraction:

$$\delta n(r,t) = k\,\delta T(r,t)$$

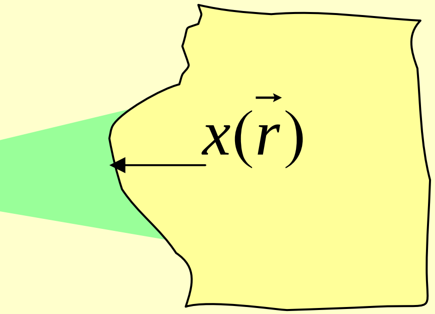
$$\delta\varphi(t) = \int I(r)\,k\,\delta T(r,t)\,d\,r$$

phaseshift — intensity — temperature fluctuation

readout variable

$$\delta \hat{T} = \int d^3\vec{r}\,\delta T(\vec{r},t)\,q(\vec{r}).$$

Thermo-mechanical noise



Callen and Welton 51, L. 98

Readout variable:

$$X = \int x(\vec{r}) \cdot (\text{Gaussian beam}) d^2 r$$

Interaction

$$H_{\text{int}} = -F_0 \cos(2\pi f t) \cdot X$$

1. oscillating pressure

$$P(r) = F_0 \cos(2\pi f t) \times (\text{Gaussian beam})$$

2. Compute/measure dissipated power W_{diss}

3. $$S_x(f) = \frac{8 k_B T}{\omega^2} \frac{W_{diss}}{F_0^2}$$

Thermo-refractive noise

Readout variable:

$$\delta\hat{T} = \int d^3\vec{r}\,\delta T(\vec{r},t)q(\vec{r}).$$

1. oscillating entropy injection

$$\frac{\delta s(\vec{r})}{dV} = F_0\cos(\omega t)q(\vec{r}),$$

2. Compute/measure dissipated power, e.g.

$$W_{\rm diss} = \int d^3r\frac{\kappa}{T}\langle(\nabla\delta T)^2\rangle,$$

3.

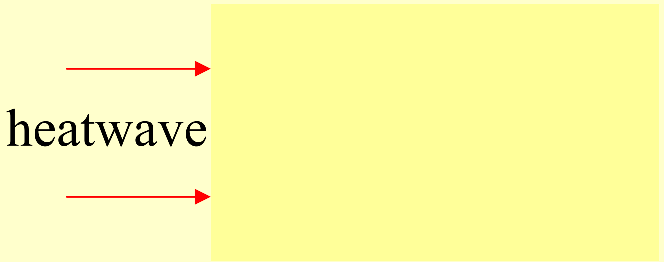
$$S_x(f) = \frac{8k_BT}{\omega^2}\frac{W_{diss}}{F_0^2}$$

Application 1: coating thermo-refractive noise.

readout:

$$\delta\hat{T} = \frac{1}{\pi r_0^2 l}\int_{-\infty}^{\infty} dx\,dy \int_0^{\infty} dz\,\delta T(\vec{r},t)\,e^{-(x^2+y^2)/r_0^2}e^{-z/l},$$

Mental experiment:



Answer:

$$S_{\delta\hat{T}}(f) = \frac{\sqrt{2}k_B T^2}{\pi r_0^2\sqrt{\omega C\rho\kappa}}.$$

Identical to Braginsky, Gorodetsky, & Vyatchanin 2000

Application 2: coating thermo-optical noise

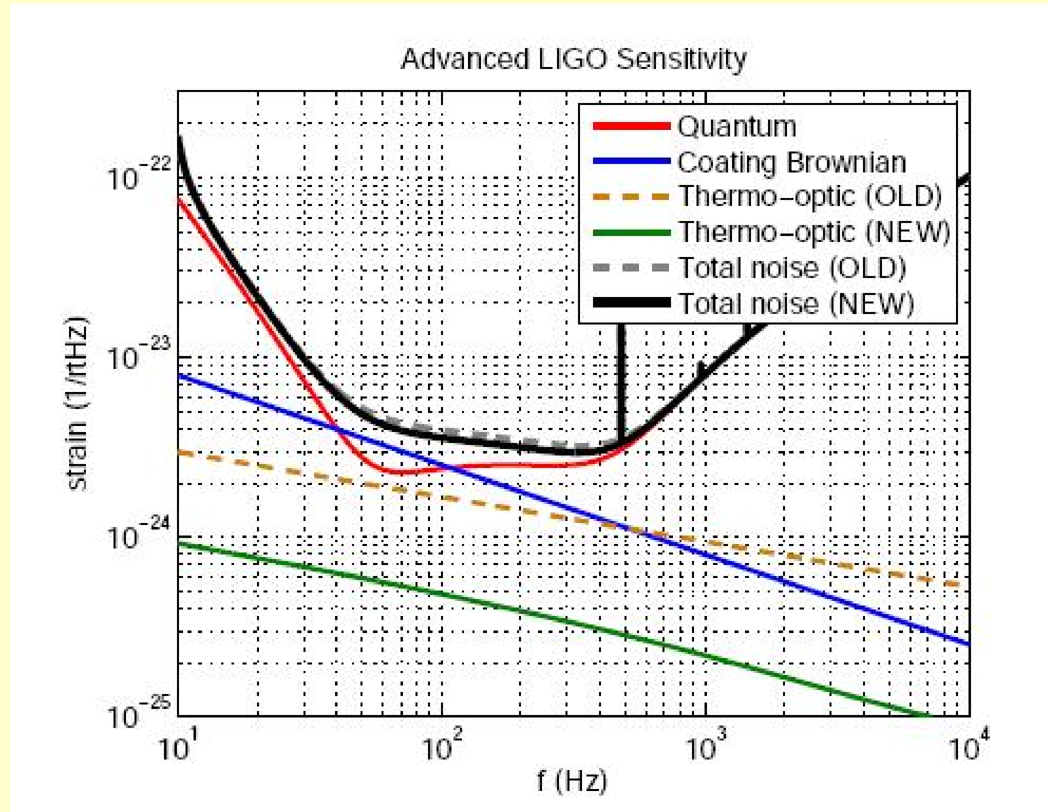
Evans, et al. 08

δT

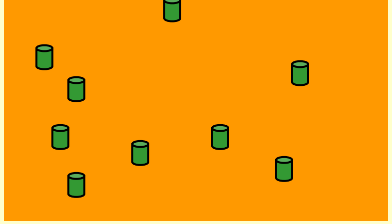
surface displacement

negative correlation!

refractive index



Proof:



- Thermometers-ensembles of identical harmonic oscillators, x_i, ω_0
- Choose $\omega_0 >> \omega$
- Assume fast ($<<1/\omega$) thermal coupling
- Consider dynamical coordinate:

$$\hat{T}_1 = \sum_i x_i^2 ,$$

for dense packing, this becomes

$$\hat{T}_1 = \int d^3 r \, n(\vec{r}) \langle x^2(\vec{r}) \rangle ,$$

density

but

$$\langle x^2(\vec{r}) \rangle = \frac{k_B}{m\omega_0^2} T(\vec{r}) ,$$

so by choosing

$$n(\vec{r}) = \frac{m\omega_0^2}{k_B} q(\vec{r}) ,$$

$$\hat{T}_1 = \sum_i x_i^2 ,$$

becomes

$$\delta\hat{T} = \int d^3\vec{r} \, \delta T(\vec{r}, t) q(\vec{r}) .$$

Proof (continued):

- Introduce interaction Hamiltonian

$$H_{\rm int} = -F_0 \cos(2\pi f t)\hat{T}_1,$$

- This amounts to slow sinusoidal change in spring constant. From adiabatic theorem,

$$\frac{\delta E_i}{E_i} = \frac{\delta\omega_0}{\omega_0} - \frac{\delta Q_i}{E_i},$$

- On average, oscillator energy stays constant=kT. Therefore,

$$\delta Q = \frac{F_0 \cos(2\pi f t)}{m\omega_0^2} \times k_B T n(\vec{r}) \times dV.$$

- Heat input can be reformulated as entropy input:

$$\frac{\delta s}{dV} = \frac{1}{T}\frac{\delta Q}{dV} = F_0 \cos(2\pi f t) q(\vec{r}).$$

QED

Conclusions:

FdT theorem is just as easy to use for thermal variable as for mechanical one.

$$\frac{\delta s(\vec{r})}{dV} = F_0 \cos(\omega t) q(\vec{r}),$$

$$\langle x^2(\vec{r})\rangle = \frac{k_B}{m\omega_0^2} T(\vec{r}),$$

$$S_{\delta\hat{T}}(f) = \frac{8k_B T}{\omega^2} \frac{W_{\rm diss}}{F_0^2};$$

$$\delta\hat{T} = \frac{1}{\pi r_0^2 l} \int_{-\infty}^{\infty} dx\, dy \int_0^{\infty} dz\, \delta T(\vec{r}, t) e^{-(x^2+y^2)/r_0^2} e^{-z/l},$$

$$H_{\rm int} = -F_0 \cos(2\pi f t)\hat{T}_1,$$

$$\hat{T}_1 = \sum_i x_i^2,$$

$$\frac{\delta E_i}{E_i} = \frac{\delta\omega_0}{\omega_0} - \frac{\delta Q_i}{E_i},$$

$$S_{\delta\hat{T}}(f) = \frac{\sqrt{2} k_B T^2}{\pi r_0^2 \sqrt{\omega C \rho \kappa}}.$$

$$W_{\rm diss} = \int d^3 r \frac{\kappa}{T} \langle (\nabla \delta T)^2 \rangle,$$

$$\frac{\delta\omega_0}{\omega_0} = \frac{F_0 \cos(2\pi f t)}{m\omega_0^2}.$$

$$n(\vec{r}) = \frac{m\omega_0^2}{k_B} q(\vec{r}),$$

$$\frac{\delta s}{dV} = \frac{1}{T}\frac{\delta Q}{dV} = F_0 \cos(2\pi f t) q(\vec{r}).$$

$$W_{\rm diss} = \int d^3 r \frac{\kappa}{T} \langle (\nabla \delta T)^2 \rangle,$$

$$\delta Q = \frac{F_0 \cos(2\pi f t)}{m\omega_0^2} \times k_B T n(\vec{r}) \times dV.$$

$$\hat{T}_1 = \int d^3 r\, n(\vec{r}) \langle x^2(\vec{r}) \rangle,$$