Network Analysis of Gravitational Waves

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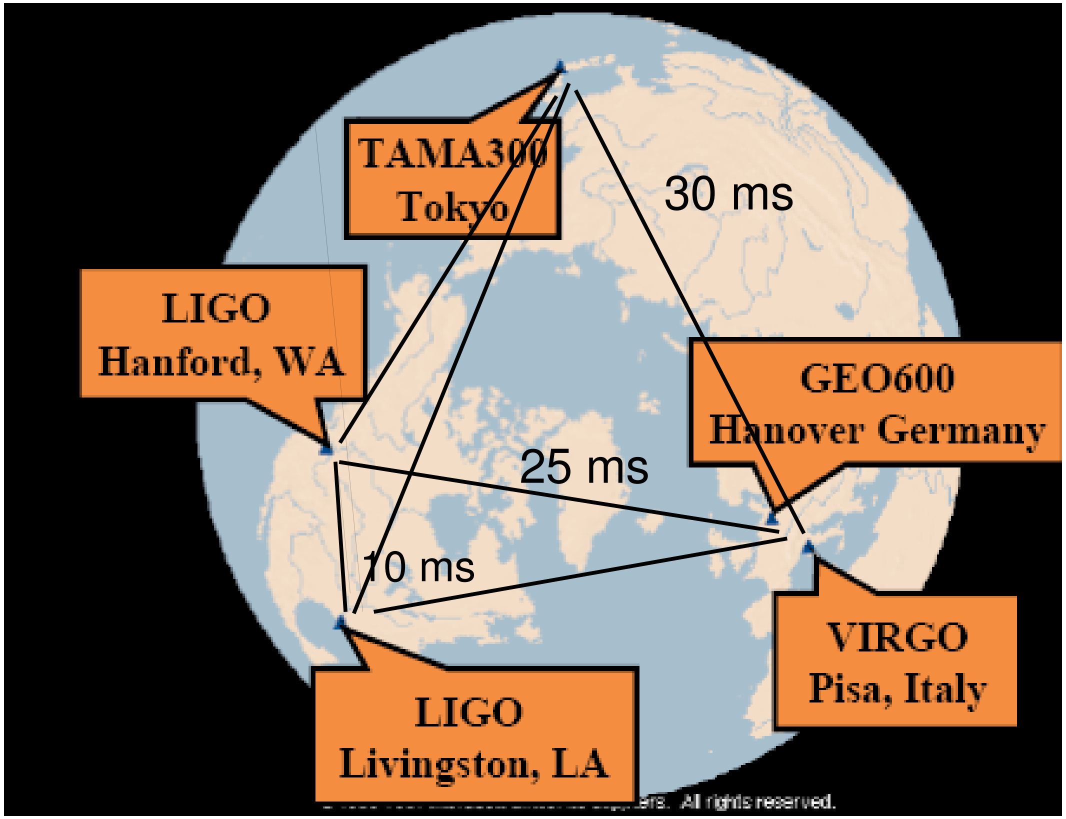
Outline

- Motivation
- Principle

- Network
  - Angular resolution
  - Detection/veto method
  - Waveform extraction
- Conclusion

Network Analysis

- Detection Confidence
- Source Localization
- Waveform Extraction
- Test GR



L1-AIGO : 40 ms

FAQ

- Data analysts' questions
  - How to combine data for best detection
    - Different noise level in different frequencies and in different detectors. Different directional sensitivity
  - How to distinguish between detector glitches and real GW events ? (veto/consistency check)

FAQ

- Astronomers/astrophysicists' questions
  - How well can we localize the source
  - How much information can we extract for the waveform?

(especially when we have no prior information on GW waveforms)

Principle

General Principle

- Detector response is linear to two wave polarizations

$$d_1(t) = f_1^+ h_+(t) + f^x{}_1 h_x(t) + n_1(t)$$

$$d_2(t+\tau_{12}) = f_2^+ h_+(t) + f^x{}_2 h_x(t) + n_2(t+\tau_{12})$$

$$\ldots\ldots$$

$$d_{N_d}(t+\tau_{1N_d}) = f_{N_d}^+ h_+(t) + f^x{}_{N_d} h_x(t) + n_{N_d}(t+\tau_{1N_d})$$

- assuming stationary Gaussian noise
- constant antenna beam pattern
  - short duration data segments

For each frequency bin k:

$$\vec{d}_k = A_k \vec{h}_k + \vec{n}_k$$

$$\vec{d}_k = \begin{pmatrix} d_{1\mathrm{k}}/\sigma_{1\mathrm{k}} \\ d_{2\mathrm{k}}/\sigma_{2\mathrm{k}} \\ \ldots \\ d_{N_d k}/\sigma_{N_d k} \end{pmatrix}, \quad \vec{h}_k = \begin{pmatrix} h_{+k} \\ h_{xk} \end{pmatrix}, \quad \vec{n}_k = \begin{pmatrix} n_{1\mathrm{k}}/\sigma_{1\mathrm{k}} \\ n_{2\mathrm{k}}/\sigma_{2\mathrm{k}} \\ \ldots \\ n_{N_I k}/\sigma_{N_d k} \end{pmatrix}$$

$$A_k = A^{\tau}_k A^0_k = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & e^{i2\pi f_k \tau_{12}} & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & e^{i2\pi f_k \tau_{1\mathrm{N}_d}} \end{pmatrix} \begin{pmatrix} f^+_1/\sigma_{1\mathrm{k}} & f^x_1/\sigma_{1\mathrm{k}} \\ f^+_2/\sigma_{2\mathrm{k}} & f^x_2/\sigma_{2\mathrm{k}} \\ \ldots & \ldots \\ f^+_{N_d}/\sigma_{N_d k} & f^x_{N_d}/\sigma_{N_d k} \end{pmatrix}$$

- Matrix A consists of time-delay info and directional sensitivity weighted by noise

- For all data

$$\vec{d} = A\vec{h}$$

$$\vec{d} = \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \\ \ldots \\ \vec{d}_{N_k} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & A_{N_k} \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} \vec{h}_1 \\ \vec{h}_2 \\ \ldots \\ \vec{h}_{N_k} \end{pmatrix}$$

* A: block diagonal.

Each frequency component can be treated independently

Tricks for Network Analysis of Burst Sources

- Consider two sets of unknown parameters:
  - source direction, wave polarizations at each frequency

$$\{\alpha, \delta; \quad \vec{h}_k, (k=1,\ldots,N_k)\}$$

- Apply Fisher's information matrix

$$I_{ij} = 2\Re[\partial_i (A\vec{h})^T \partial_j (A\vec{h})], \qquad \sigma_i^2 \geq (I^{-1})_{ii}$$

$$I_{ij} = 2\Re \begin{vmatrix} G_1^T G_1 & G_1^T G_2 \\ G_2^T G_1 & G_2^T G_2 \end{vmatrix}$$

$$\textit{where}, \quad G_1 = \partial_{\vec{\theta}}(A\vec{h}), G_2 = \partial_{\vec{h}}(A\vec{h}), \qquad T \text{ - complex conjugate}$$

Principle for Network Analysis
(burst search)

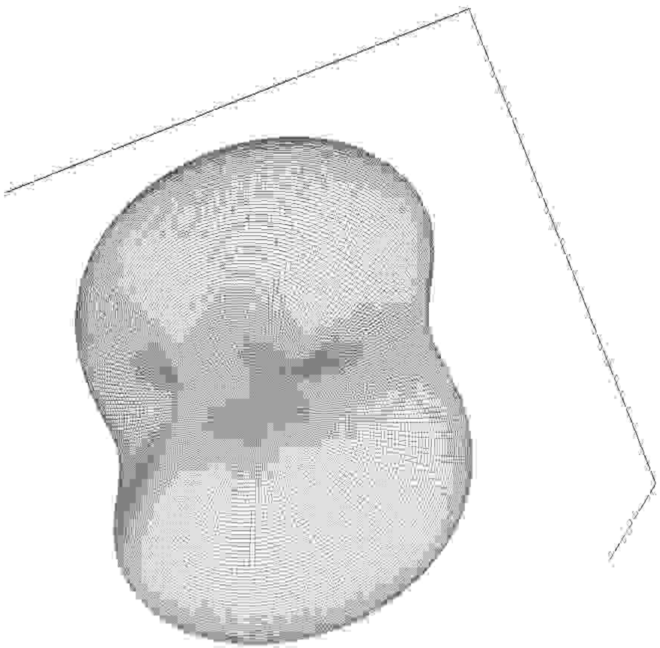
- Based on (1) Fisher's information matrix (2) Singular Value Decomposition method
  - -> determine best possible angular resolution $\{(\alpha, \delta)\}$
  - -> principle axes for waveform estimation $\{\vec{h}_k, (k=1,\ldots, N_k)\}$
  - -> optimal detection statistic and null-stream for consistency check/veto
    - MLR principle
    - Maximum SNR principle

1. Network Angular Resolution (Revisited)

Single-detector Angular Resolution

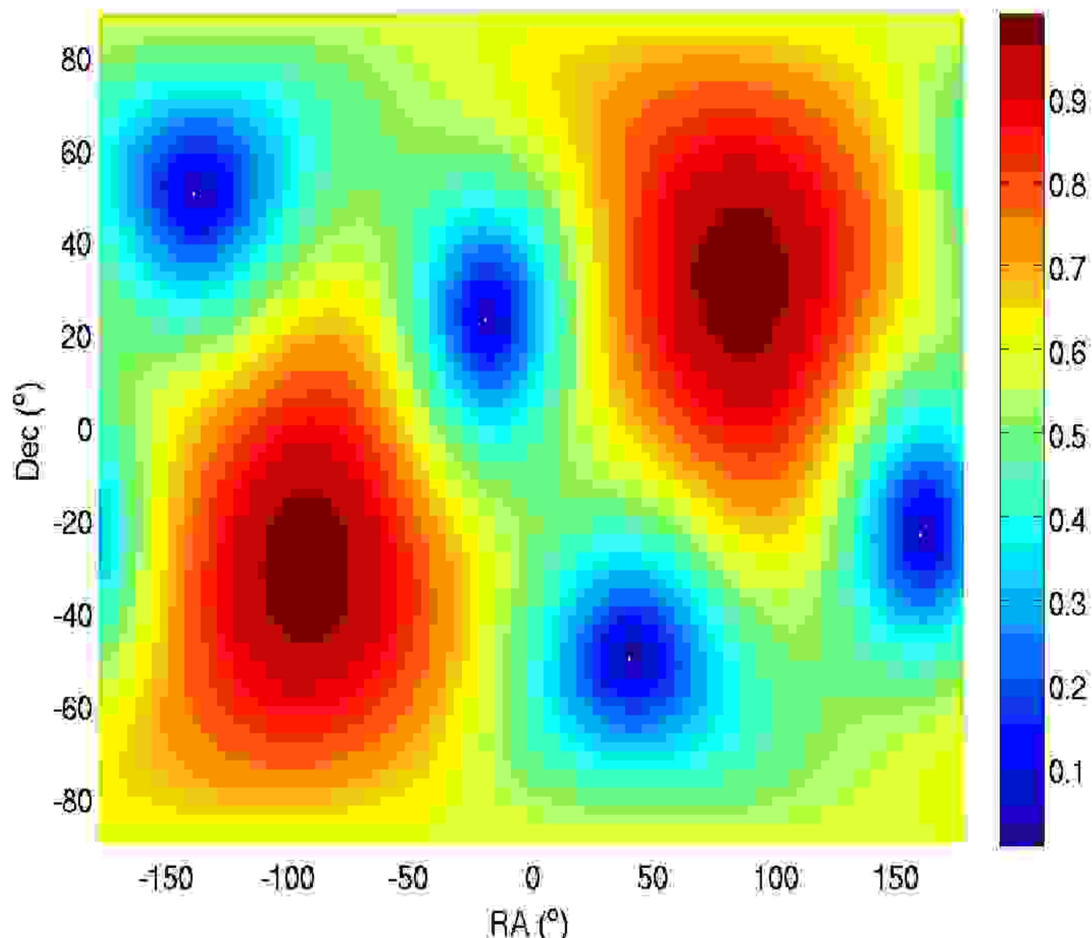
- Single GW detector is a good all sky monitor, but not a pointing instrument

Angular resolution (FWHM) ~100 degrees



antenna beam pattern

$$\sqrt{f_+^2 + f_x^2}$$



Network Angular Resolution: Previous Studies

- Order of magnitude estimation from diffraction limit for single -f source

$$\delta\omega \sim \frac{1}{\rho^2}\frac{\lambda^2}{A} \sim 1\,deg^2 \quad for\ \rho_{LHV} \sim 10, f \sim 500\,Hz$$

- Previous studies
  - numerical study/Kip's pvt comm. for 3-detector network
    - Guersel & Tinto 1989, Sylvestre 2003
  - Inspiral coherent search using Fisher's information matrix/numerical result: Pai et al. 2001,
  - LISA as a 2-detector network: Fisher's information matrix/MF/numerical result: Cutler 1998

Network Angular Resolution

(best case scenario)

- We give explicit analytic expression for multi-detectors
  - Fisher's information matrix~ I(single) + I(time-delay)

$$I^{\tau}_{ij}(\alpha,\delta)=(2\pi)^2\sum_{I=1}^{N_d}\sum_{k=1}^{N_k} f_k^2\rho_{Ik}^2\partial_i\tau_{1\mathrm{I}}\partial_j\tau_{1\mathrm{I}}$$

$$\tau_{1\mathrm{I}}=-(\hat{k}\cdot\hat{n}_{1\mathrm{I}})/c$$

  - Better directional resolution for
    - higher SNR, higher freq., more detectors
    - larger time-delay derivative against sky direction
      - Larger base-line

Network Angular Resolution
(best case)

- Worse directional resolution if wave direction is along
  - line connecting 2-detectors
  - plane formed by 3-detectors
    - only one direction can be determined (better than single-detector sensitivity) if wave travels in the plane of a 3-detector network

Network Angular Resolution
(burst search)

- when no information about waveform exist
  - angular resolution degraded by a projection factor

$$I^{-1}(\alpha,\delta)=(2\Re[G_1^T(I-P_2)G_1])^{-1}$$

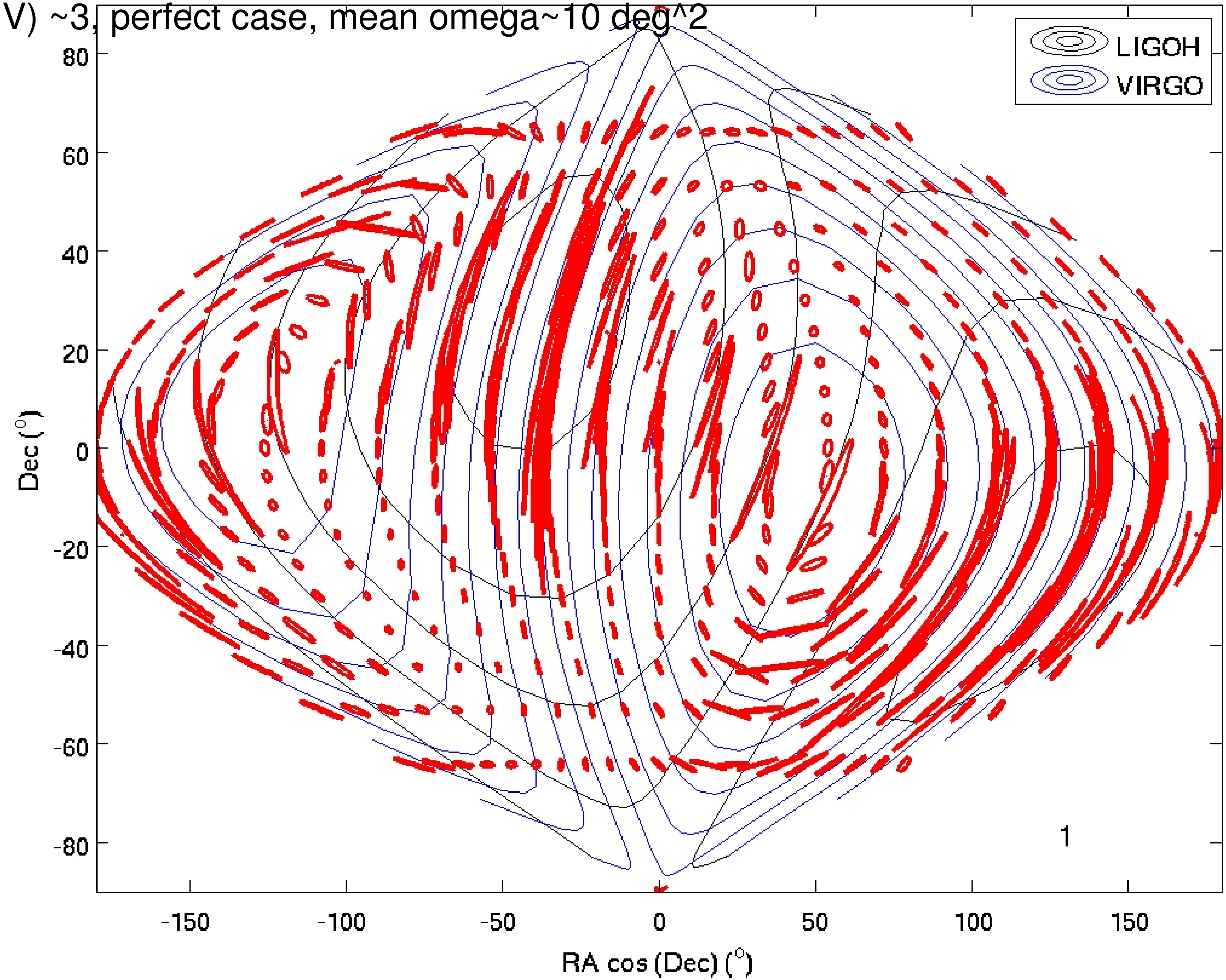
$I-P_2$ *is the projection matrix projecting* A *onto its null space*

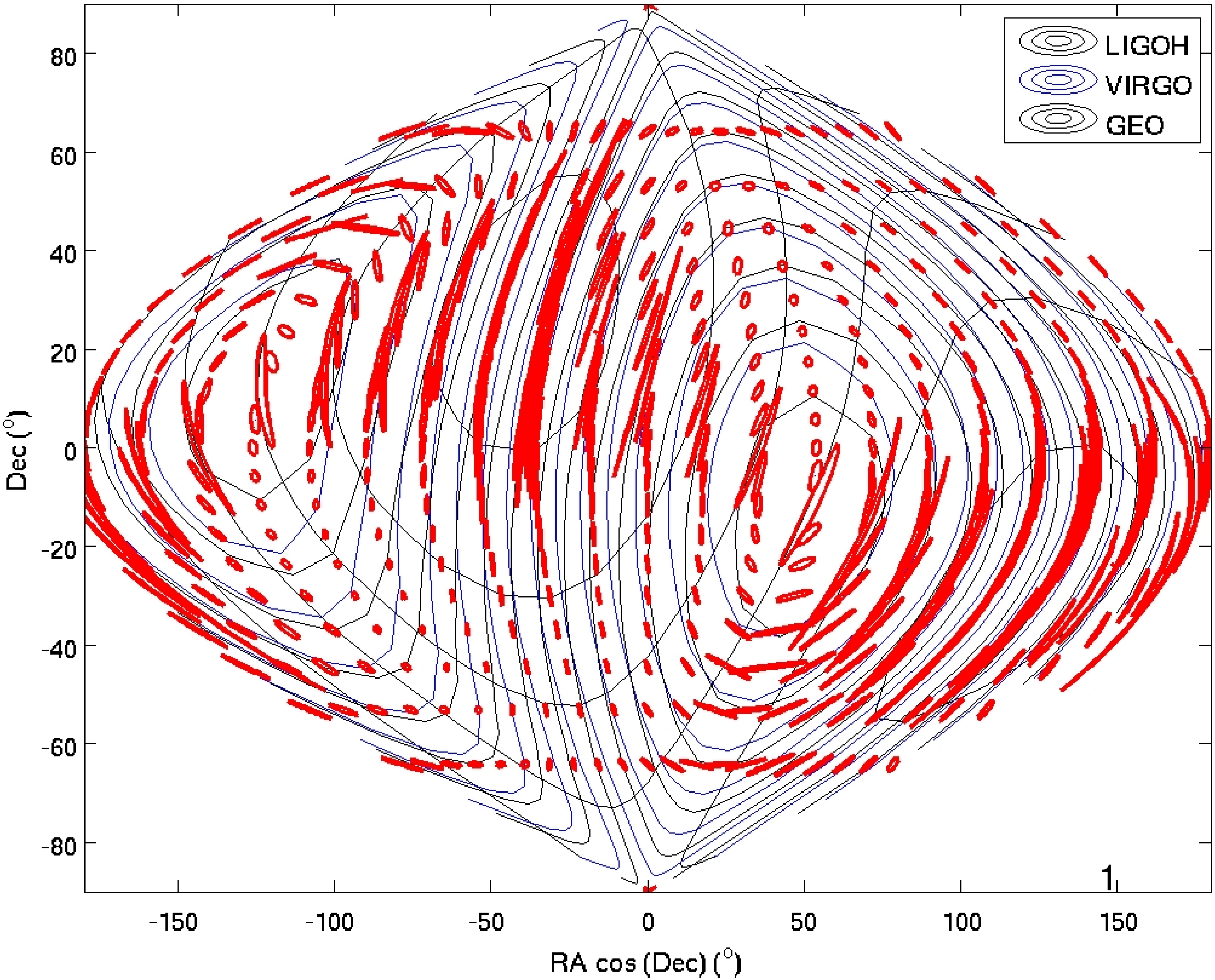
  - recall perfect case

$$I^{-1}(\alpha,\delta)=(2\Re[G_1^T G_1])^{-1}$$

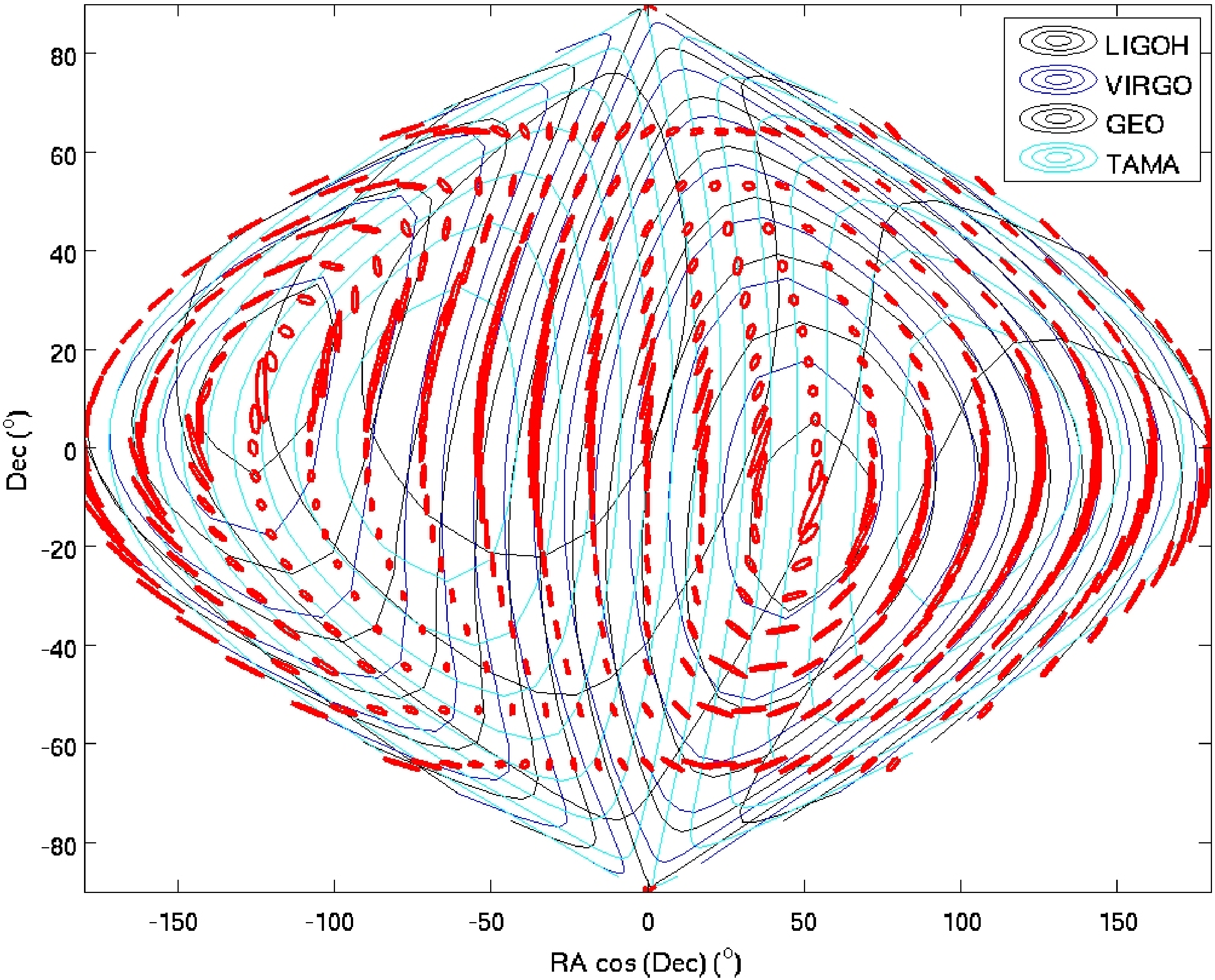
    - if A is full-ranking, i.e., with no redundant degree of freedom, directional information cannot be determined

BH-BH merger waveform. f~ 500 Hz, Mean SNR (LHV) ~3, perfect case, mean omega~10 deg^2



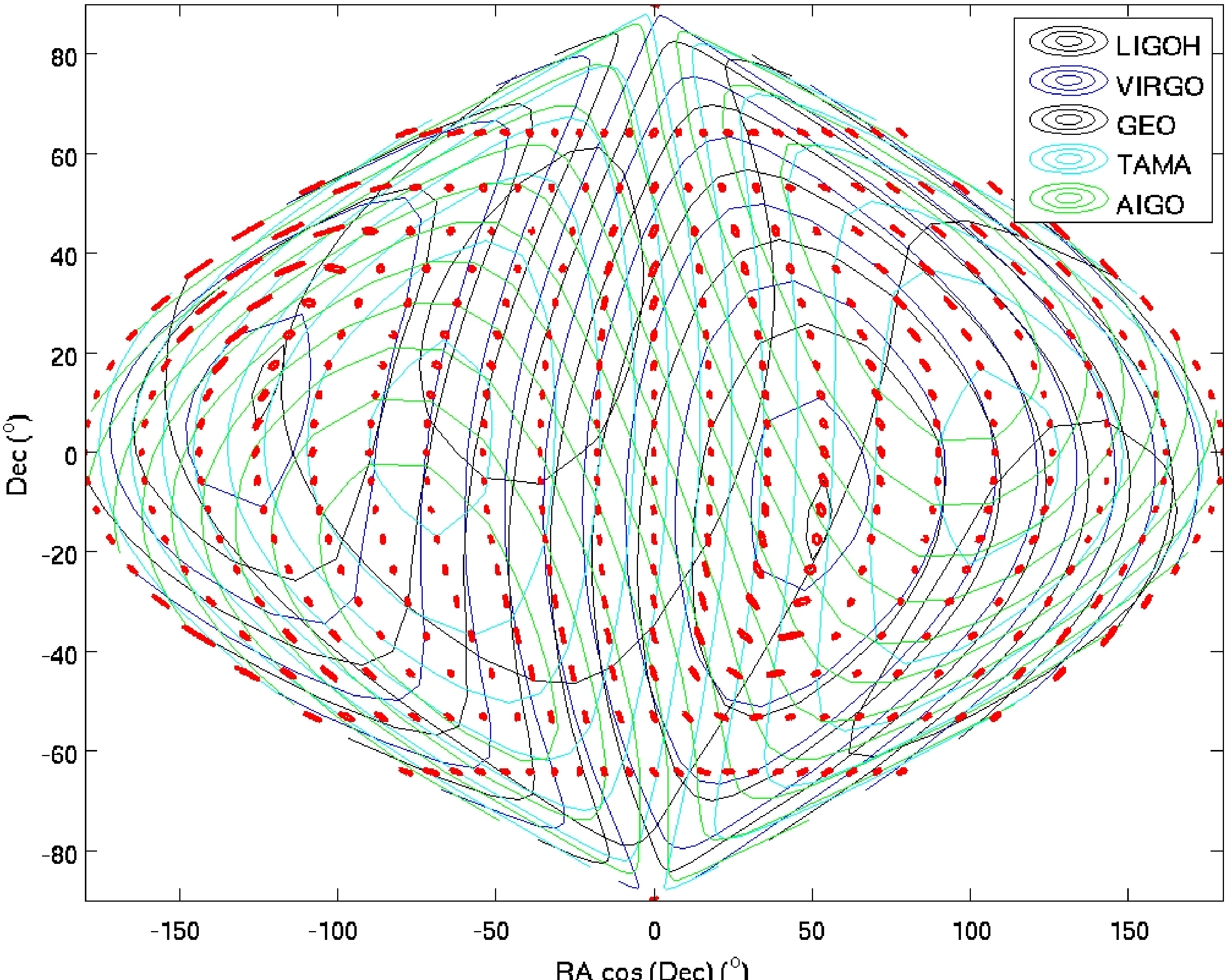


LIGOH
VIRGO
GEO
Dec (°)
RA cos (Dec) (°)
80
60
40
20
0
-20
-40
-60
-80
-150
-100
-50
0
50
100
150
1



LIGOH
VIRGO
GEO
TAMA
Dec (°)
RA cos (Dec) (°)
80
60
40
20
0
-20
-40
-60
-80
-150
-100
-50
0
50
100
150

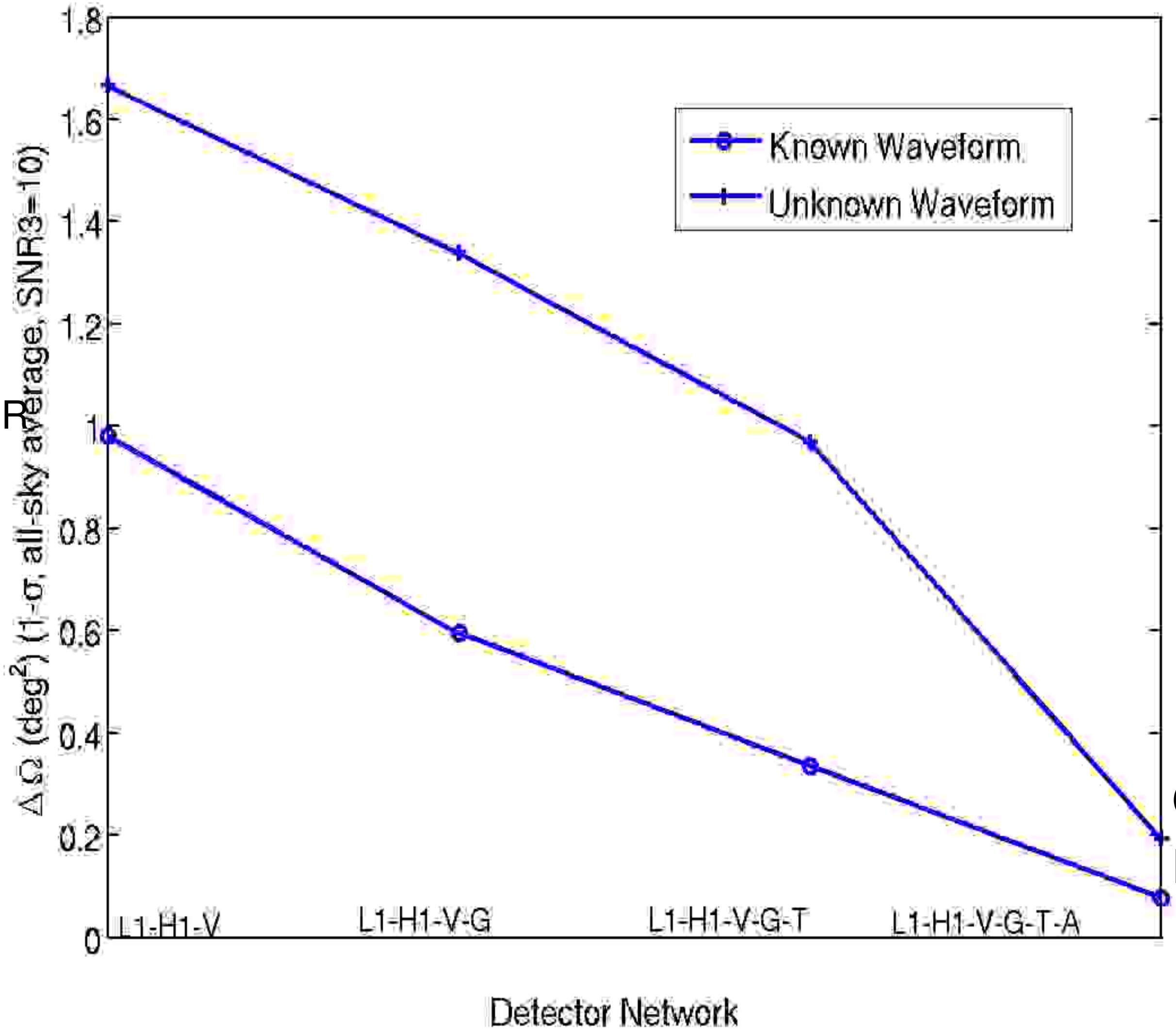
perfect case, mean SNR ~3.3, mean~ 1 square deg



imperfect case degraded by a factor of 3

BH-BH (10-10 Msun) merger waveform, fc~ 500 Hz

T~ 7 ms, (Lazarus project). Detector noise = designed sensitivity



Network Angular Resolution

- Angular resolution ~ 0.2 square-degree can be reached for burst-search
  - At SNR~10, by L1-H1-V-G-T-A
  - Advanced detectors, with ~10 times better sensitivity, all networks are possible for the same sources
- Direct astronomical follow-up observation of BH-BH coalescence is possible
  - for a large detector network and for advanced detectors.
- GEO/TAMA improve the angular resolution
  - long baseline + complimentary antenna beam pattern

2. Detection Statistics, Null Stream Construction, Waveform Extraction

-Using Singular Value Decomposition Method

- For all data

$$\vec{d} = A\vec{h}$$

$$\vec{d} = \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \\ \dots \\ \vec{d}_{N_k} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & A_{N_k} \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} \vec{h}_1 \\ \vec{h}_2 \\ \dots \\ \vec{h}_{N_k} \end{pmatrix}$$

* A: block diagonal.

Each frequency component can be treated independently

Principle for Network Analysis
(burst search)

- Our approach
  - perform singular value decomposition on A
    - For each frequency:

$$A_k = u_k s_k v_k^T, \qquad u_k^T u_k = I, \qquad v_k^T v_k = I, \qquad s_k = \begin{pmatrix} s_{k1} & 0 \\ 0 & s_{k2} \\ 0 & 0 \\ \ldots & \ldots \\ 0. & 0 \end{pmatrix}$$

Principle for Network Analysis
(burst search)

- Construct new data stream

$$\vec{d}' = u^T \vec{d}$$

  - projection of data onto basis with ranking based on directional sensitivity and noise level
- d' is the MLR estimator for new wave parameter h' – diagonal elements of Fisher's Info. Matrix
- $$\vec{h}' = s\, v^T \vec{h} = \begin{pmatrix} h_1' & h_1' & \dots & h_{N'}' & 0 & \dots & 0 \end{pmatrix}^T,$$
- $$N' \leq 2 \times N_k$$
- Reconstructed noise ~ N(0,1)

$$\vec{d}' = u^T \vec{d} = s(v^T \vec{h})$$

Signal Information

Ill-conditioned? (unstable sol. to h+, hx)

Null (no-signal) region

$$\begin{pmatrix} d'_1 \\ d'_2 \\ \dots \\ d'_{N'} \\ \dots \\ \dots \\ d'_{N_k N_d} \end{pmatrix} = \begin{pmatrix} s_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & s_2 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & s_{2N_k} & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} h_{v1} \\ h_{v2} \\ \dots \\ \dots \\ \dots \\ h_{v2N_k} \end{pmatrix}$$

$$singular\ values: s_1 \geq s_2 \geq \dots s_{N'} > 0, \qquad N' \leq 2N_k$$

Detection Statistic
(burst search)

- treat non-null component d' as new data and apply analysis
  - MF or burst search related method
    - Similar to excess power method, we search for maximum SNR by windowing the axes

$$\rho_N = \sum_{I=I_0+1}^{I_0+N-1} |u^T \vec{d}|_I^2$$

$$SNR = max_{I_{0,} N}(\rho_N / \sqrt{N})$$

If add all non-zero component, recover standard treatment of Anderson (2001), Hughes & Flanagan (1998)

Identifying Null-stream
(burst search)

- zero component of h' corresponds to null streams

$$(u^T \vec{d})_{I=N'+1,\ldots,N_k N_d} = 0$$

- there are $N \geq (N_d - 2) \times N_k$ "absolute" null-streams
- See also Wen & Schutz, CQG (2005), Ajith et al (2006), Chatterji et al (2006) gr-qc

Waveform Extraction and regularize "ill-conditioned" Streams

(burst search)

$$\vec{h} = s^{-1} v \vec{d}\,'$$

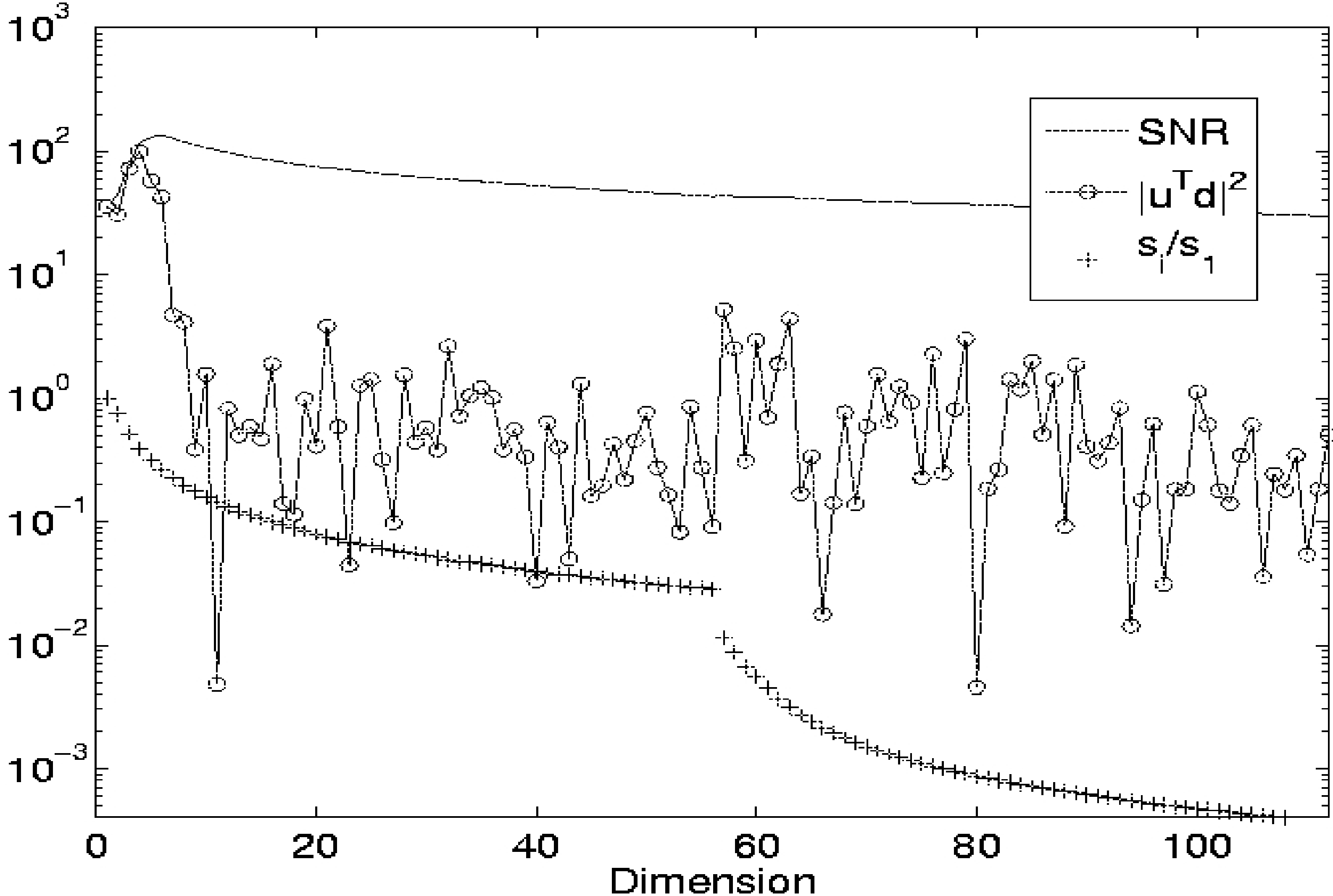
- For non-zero singular values only
  - Need treatments for ill-posed region
  - Recover Hughes & Flanagan (1998) estimator if no treatment is used
    - Which is equivalent to Guesel & Tinto (1998) for 3-detector case

Waveform Extraction and regularizing ill-conditioned Streams

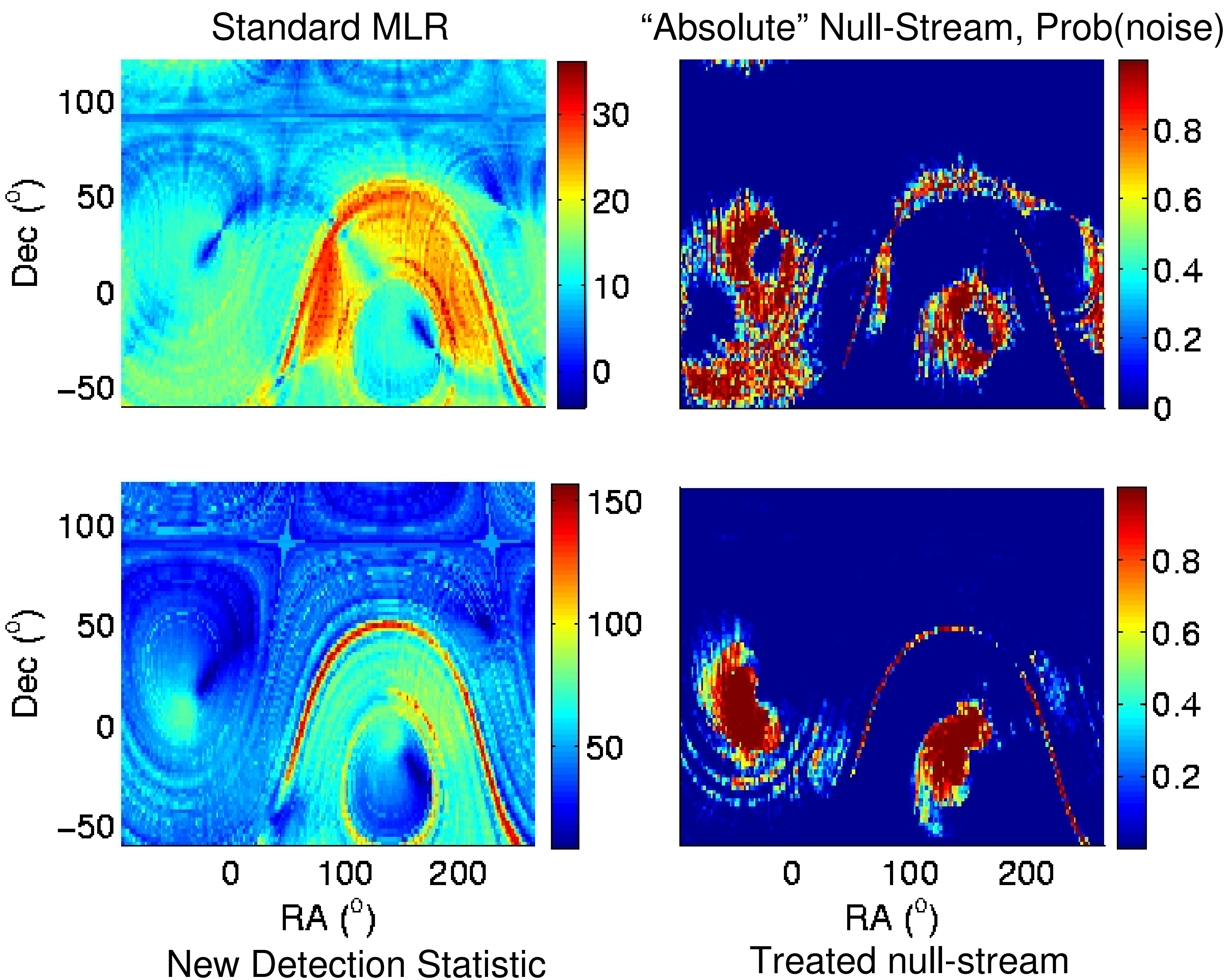
(burst search)

- Regularize ill-posed region: $s_i \ll 1$
  - for nearly parallel antenna beam pattern (e.g., L1-H1)
    - See also Rakhmanov, M. (2006), Klimenko et al (2005)
  - or noiser detectors
- Otherwise, unstable solution to h+, hx
- Help improving angular resolution & detection statistic

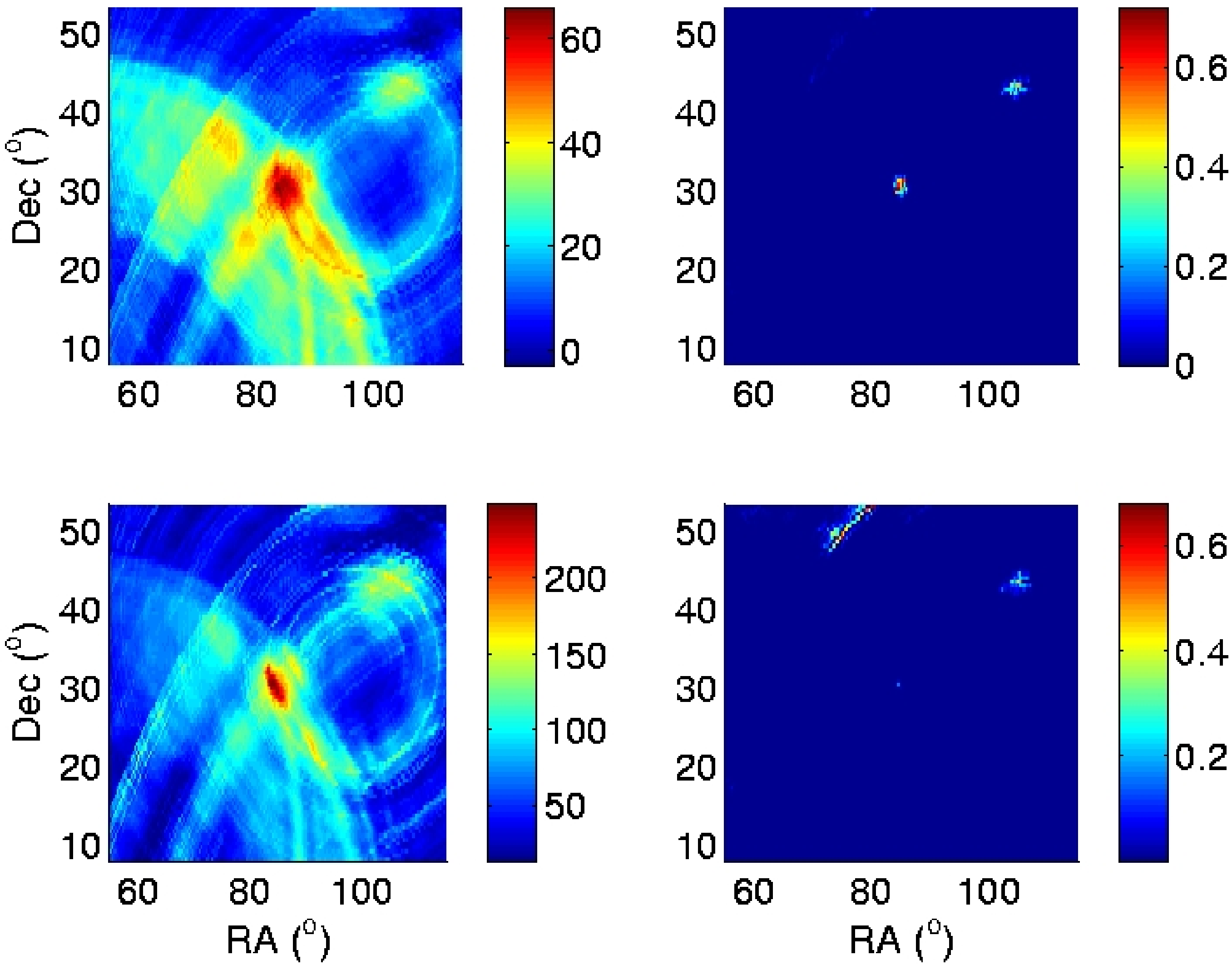
All Roads Lead to Rome



SNR
|u^T d|^2
s_i/s_1
Dimension
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
0
20
40
60
80
100



LHG, SNR ~ 20, SNR(GEO)~2, src location in center

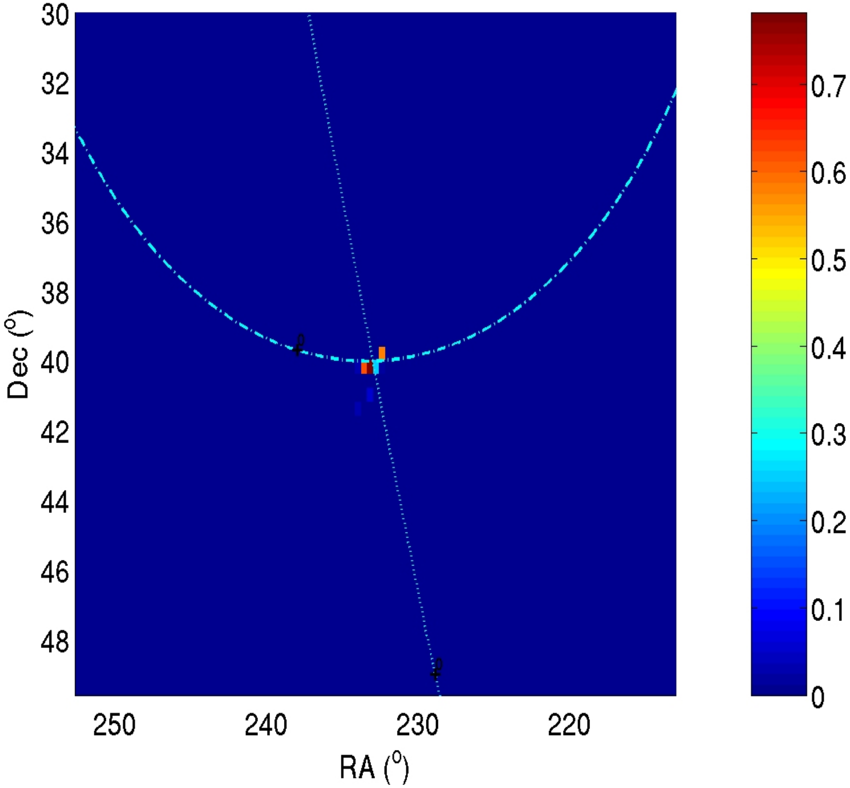


LHGTV, SNR=25

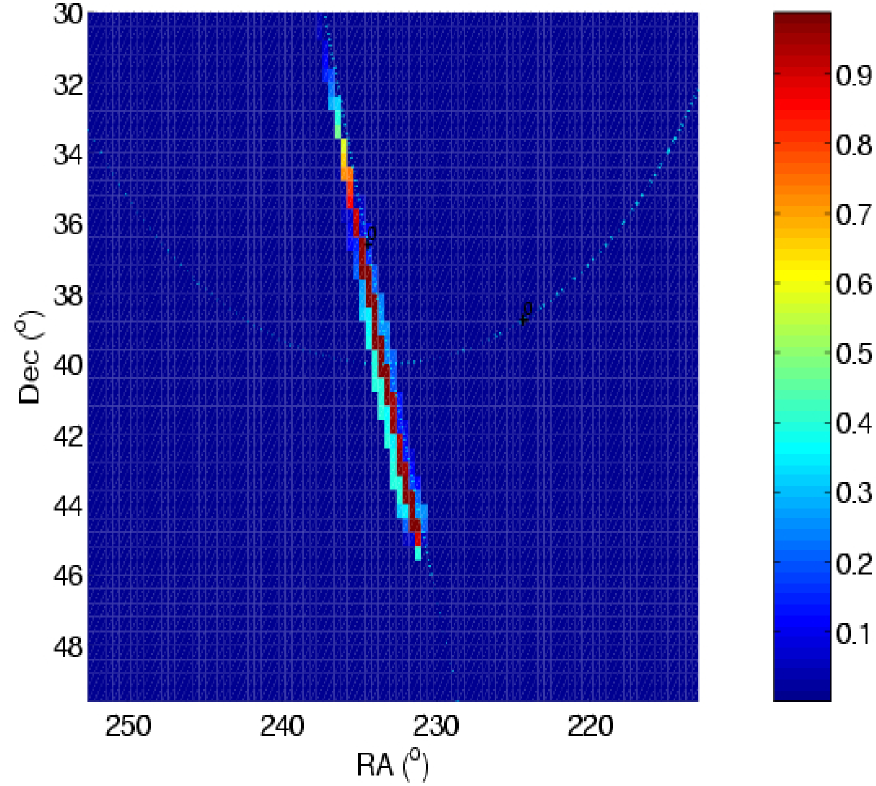
Real GW Events

localization and consistency check

Probability of $P(\alpha,\delta)$ is consistent with noise vs sky directions



1. LLO-LHO-GEO: SNR ~ 55, d = 1 Mpc

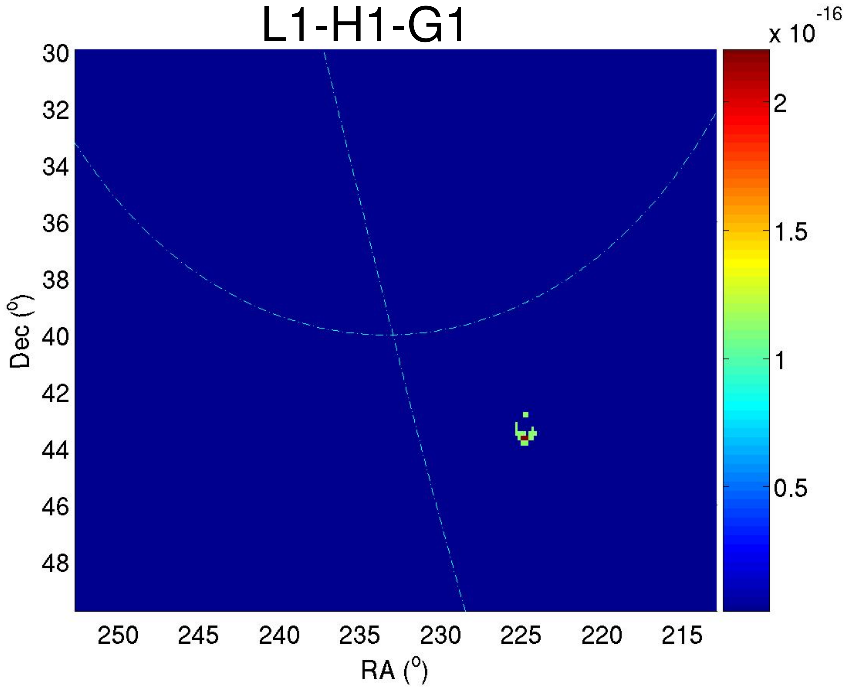


2. LLO-LHO: SNR = 35, same data

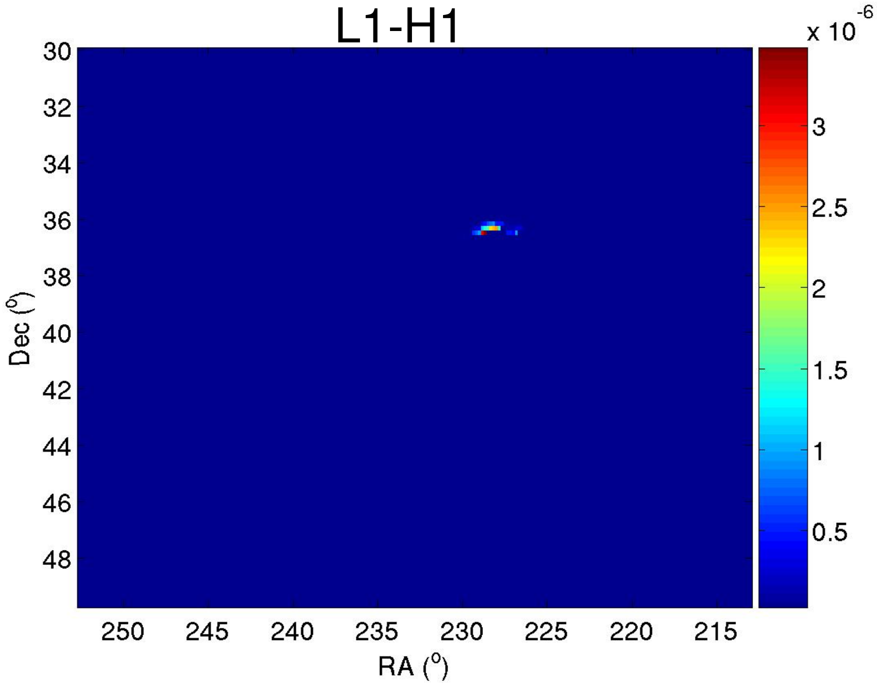
Wen L. and Schutz, B. 2005, CQG

Glitches

with "inconsistent" amplitudes h(H1) *= 0.5, h(G1) *=2

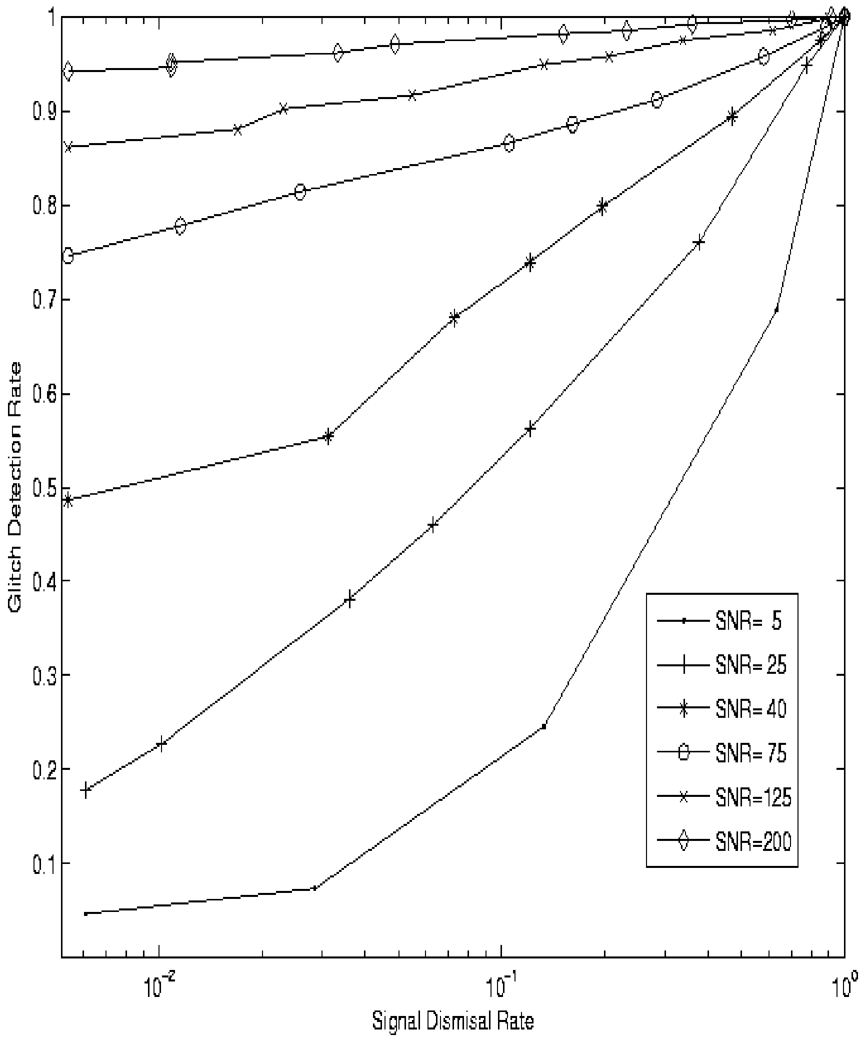


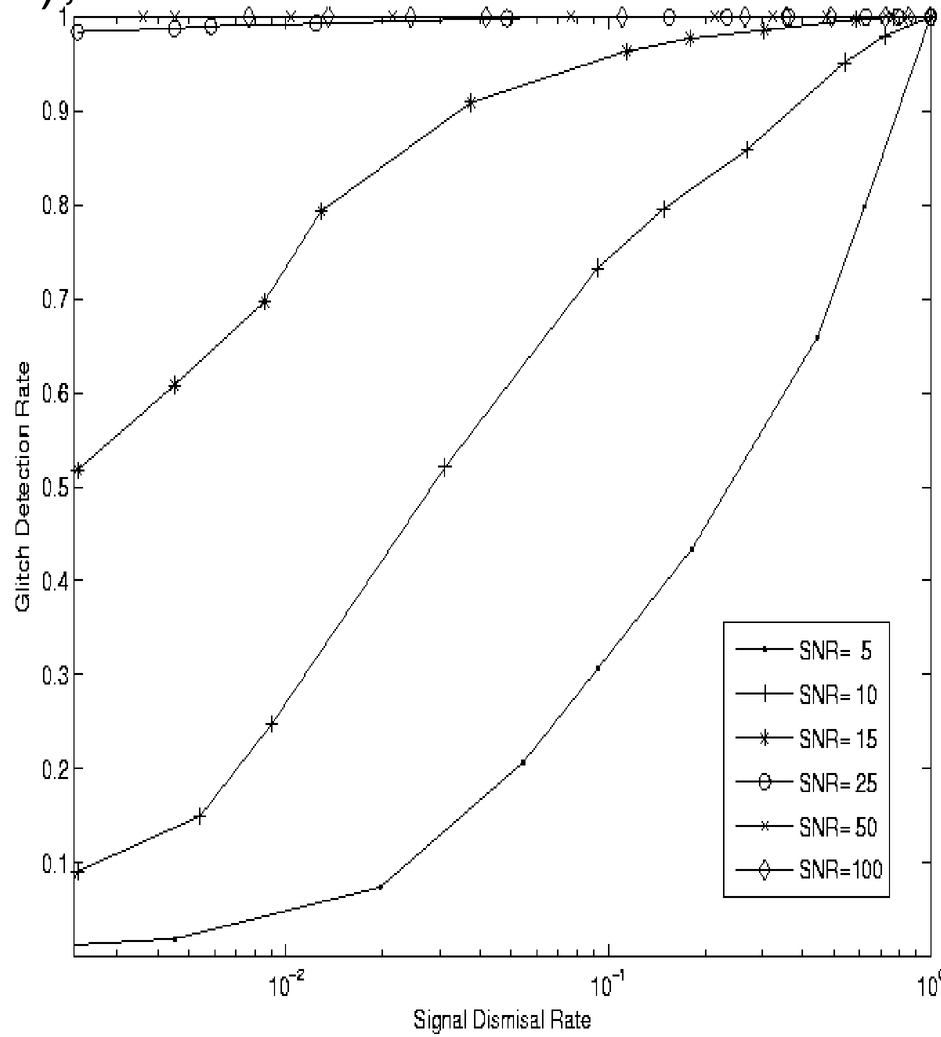
$$Prob < 2\times 10^{-16}$$



$$Prob < 4\times 10^{-6}$$

Wen & Schutz (2006), to be submitted

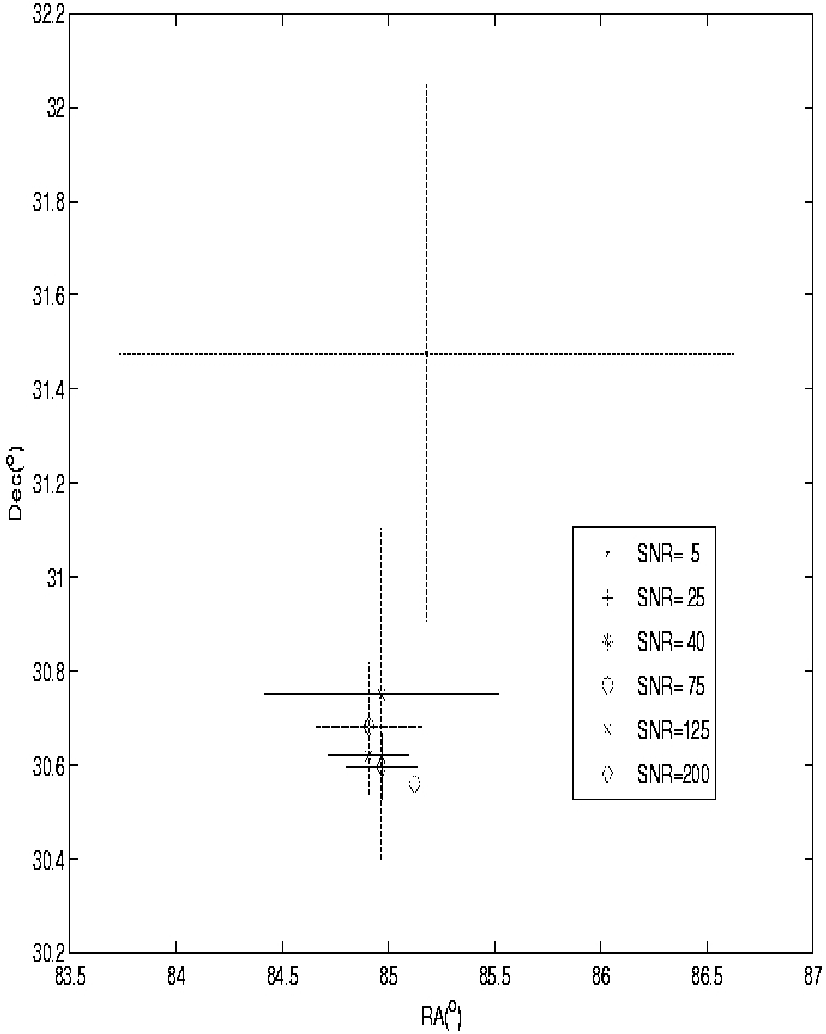


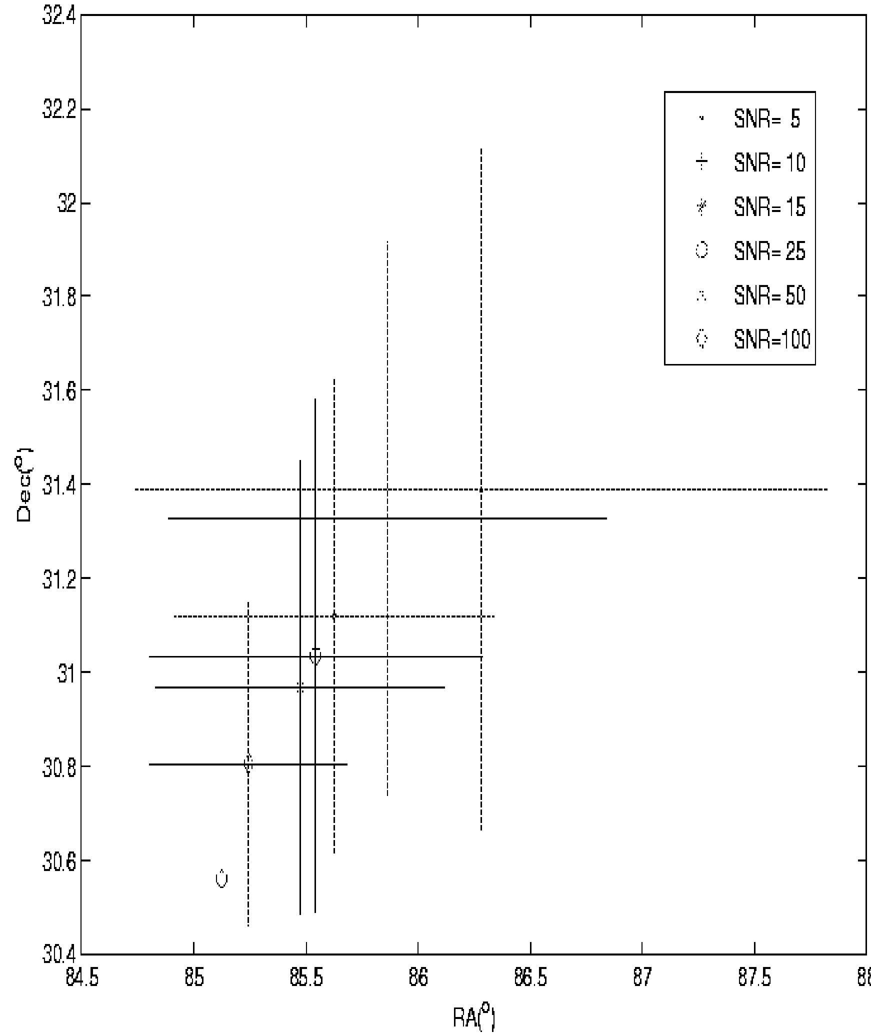


LHG

Efficiency of Null-Streams

LHV





Localization of Null-Streams, Wen & Schutz (2006), TBS

Conclusion

- We have studied angular resolution of the GW network
  - Analytic solution is given for both known and unknown waveforms
  - Apply to realistic network
- We have proposed constructing new data stream based on singular value decomposition method
  - optimal detection statistic/null stream/waveform extraction/localization run in parallel
  - simple to implement
  - MLR -> a GW is detected only if both detection and null stream are consistent in sky direction