Talk given at LIGO
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1

DIFFRACTION EFFECTS AT SUSI

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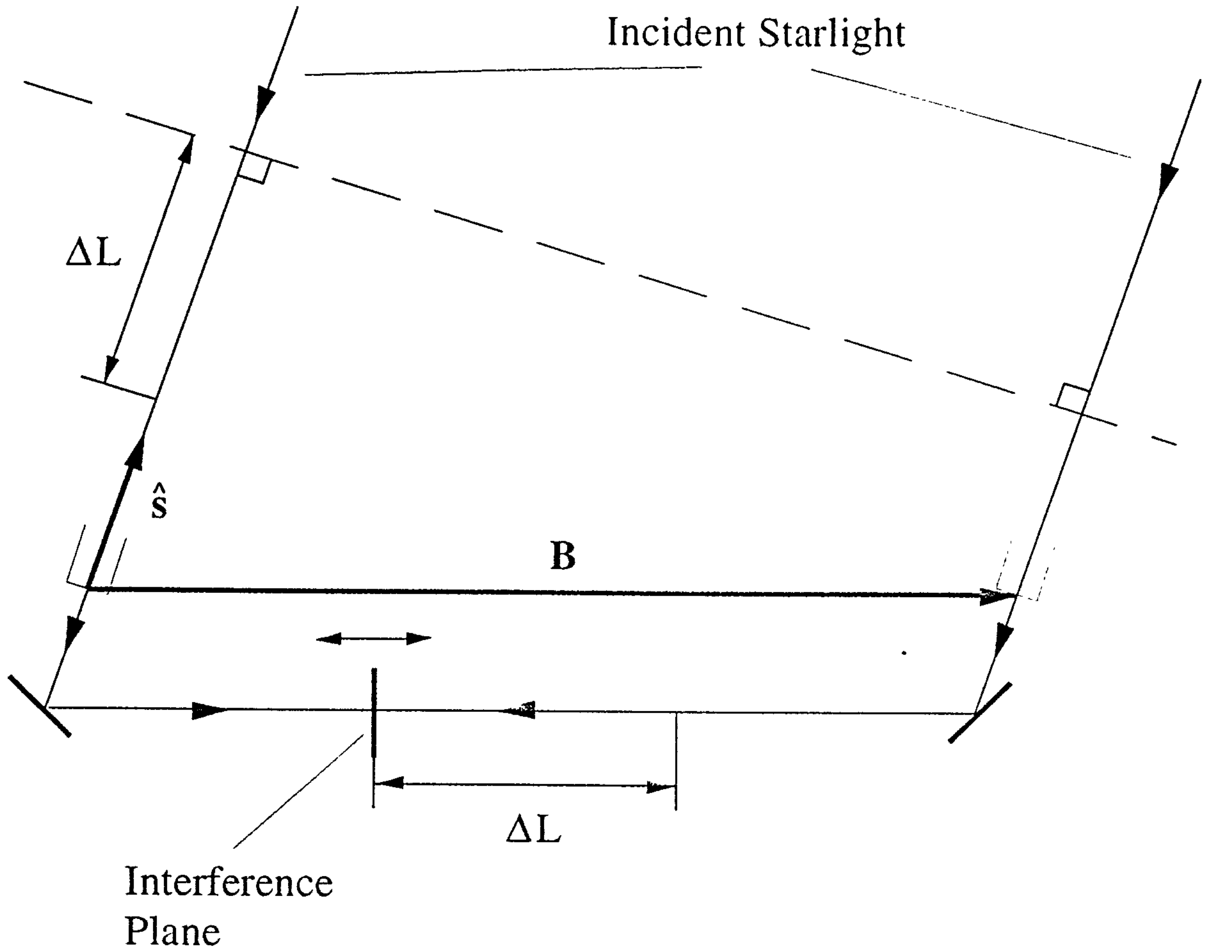
Talk Outline

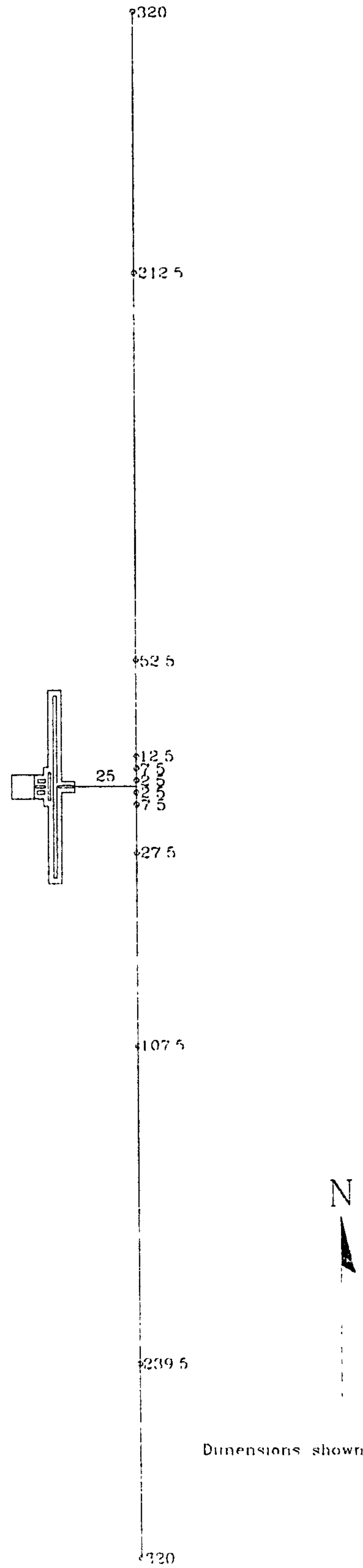
- Overview of SUSI
- Diffraction Theories
- The Fresnel Approximation
- Visibility Loss Calculation



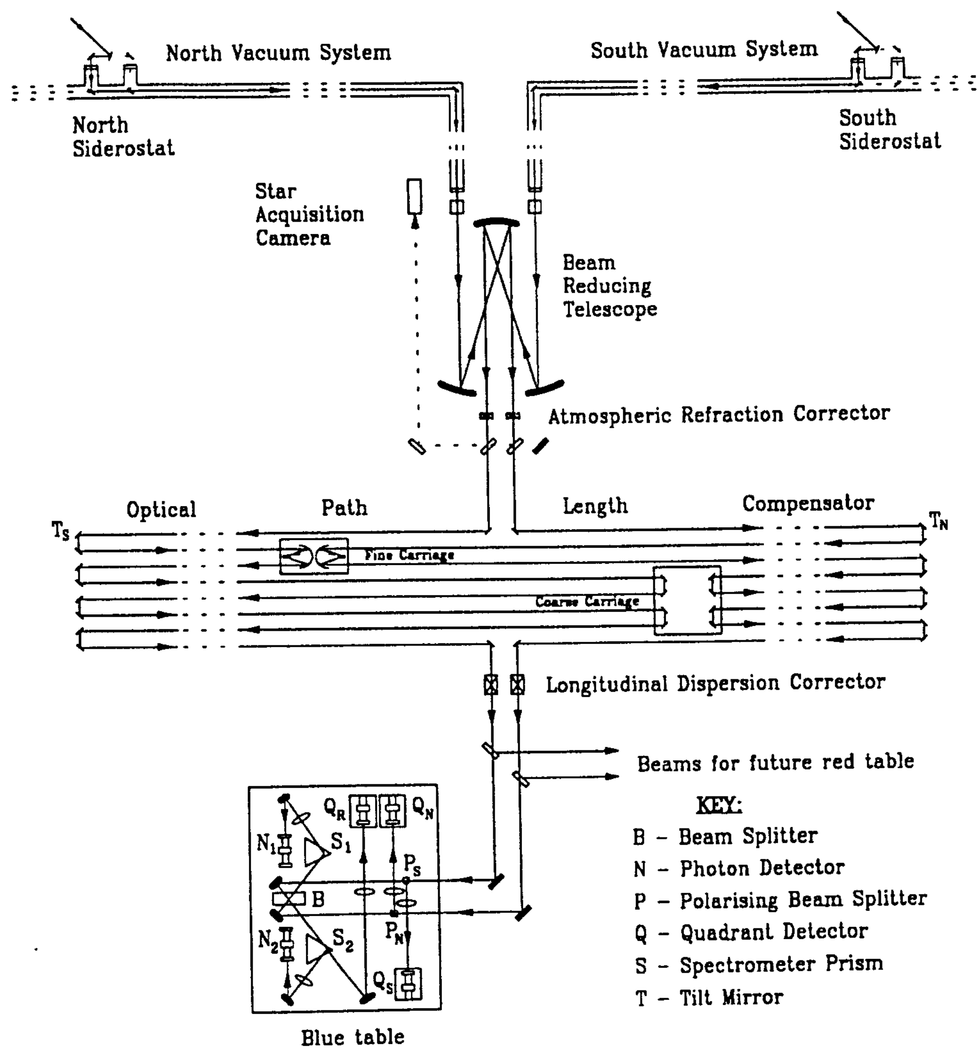
$$I(\mathbf{x}) = I_1(\mathbf{x}) + I_2(\mathbf{x}) + 2\sqrt{I_1(\mathbf{x})I_2(\mathbf{x})} \times |\gamma(\tau)| \cos\left((\phi_1(\mathbf{x}) - \phi_2(\mathbf{x})) + \delta\right)$$

$$\mathcal{V} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

$$\mathcal{V} = \frac{2\int_0^\infty \sqrt{I_1(\mathbf{x})I_2(\mathbf{x})} \cos(\phi_1(\mathbf{x}) - \phi_2(\mathbf{x}))d\mathbf{x}}{\int_0^\infty (I_1(\mathbf{x}) + I_2(\mathbf{x}))\, d\mathbf{x}} |\gamma(\tau)|$$



Dimensions shown in metres


North Vacuum System
South Vacuum System
North
Siderostat
South
Siderostat
Star
Acquisition
Camera
Beam
Reducing
Telescope
Atmospheric Refraction Corrector
Optical
Path
Length
Compensator
T_S
T_N
Fine Carriage
Coarse Carriage
Longitudinal Dispersion Corrector
Beams for future red table
Q_R
Q_N
N_1
S_1
P_S
B
N_2
S_2
P_N
Q_S
Blue table
KEY:
B - Beam Splitter
N - Photon Detector
P - Polarising Beam Splitter
Q - Quadrant Detector
S - Spectrometer Prism
T - Tilt Mirror

Diffraction Theories

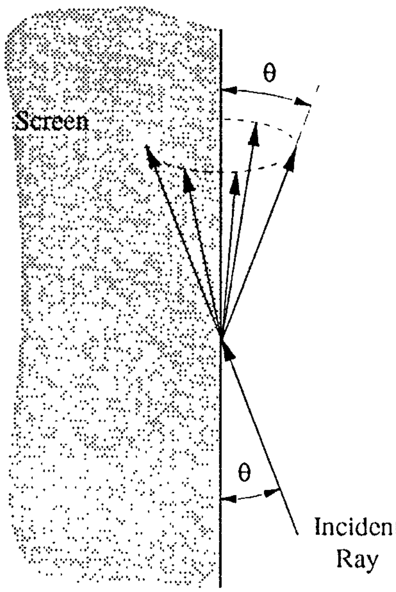
- Electromagnetic (Vector)
- Scalar
  - Kirchhoff
  - Rayleigh-Sommerfeld

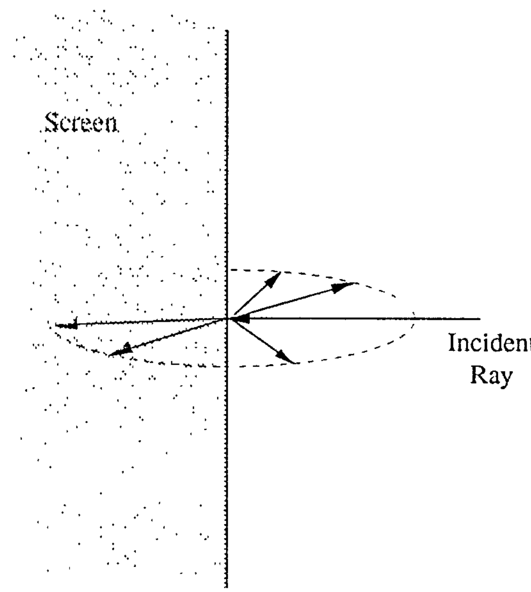
- Geometric Theory of Diffraction (GTD)

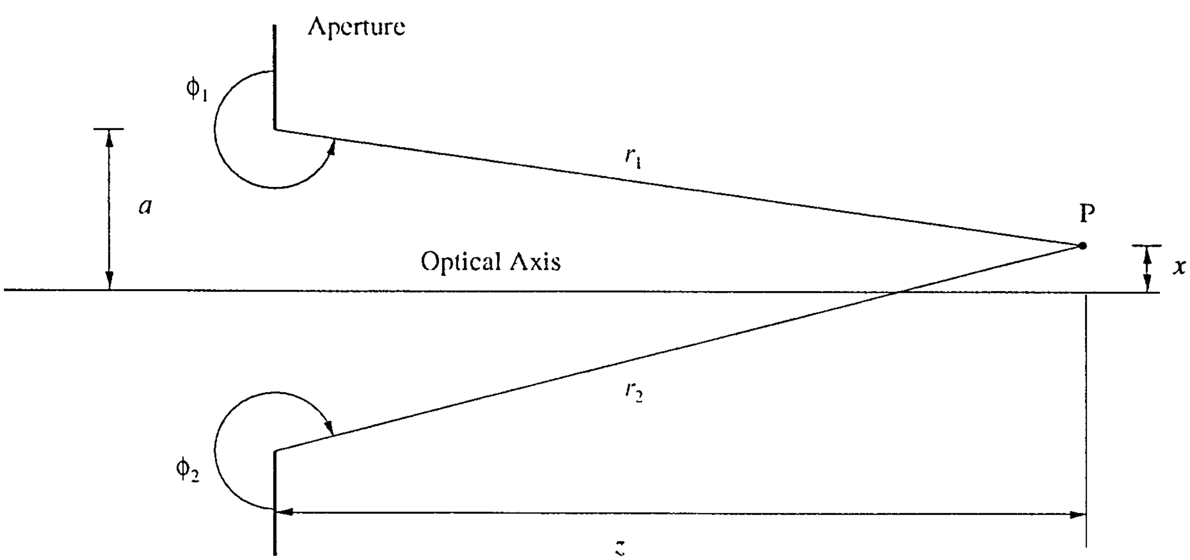
Practical Considerations

- Description of relevant features of wave propagation and diffraction effect.
- Relative ease of calculation.
- Possible need to incorporate vignetting from cascaded apertures.
- Would it be possible to incorporate an atmospherically distorted wavefront?

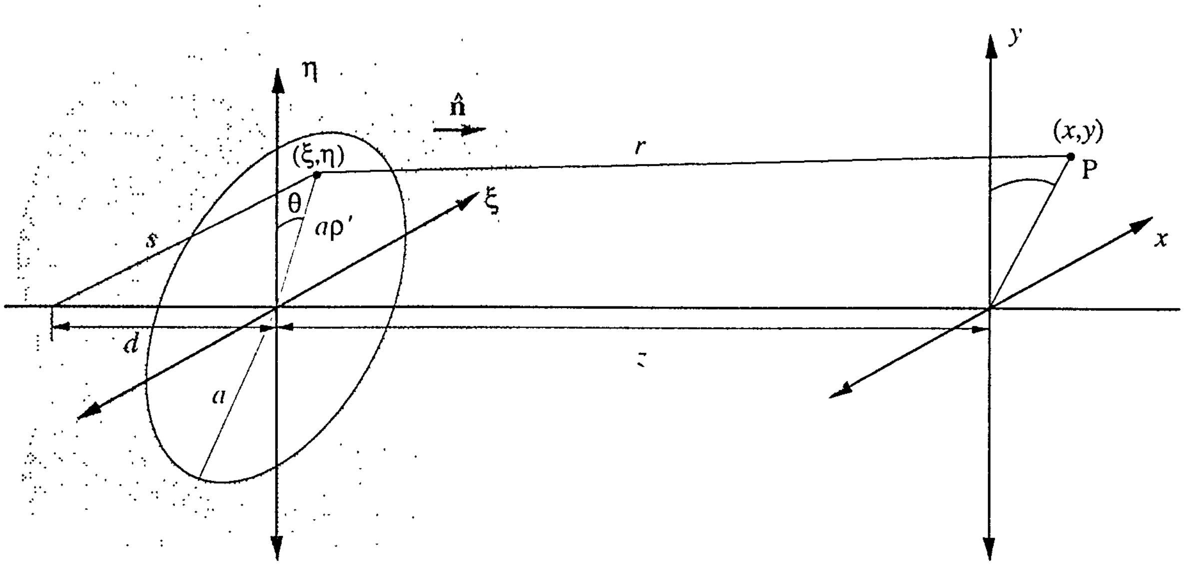
Fermat's Principle for Edge Diffracted Rays







The Fresnel Approximation



$$r = \sqrt{z^2 + (x - a\rho\sin\theta)^2 + (y - a\rho\cos\theta)^2}$$

$$U(P) = \frac{-A}{2\lambda}\int \frac{e^{ikr}}{r}\left(1 + \frac{z}{r}\right) dS$$

Apply Fresnel Approximation

$$r \approx z(1 + \frac{(x - a\rho\sin\theta)^2 + (a\rho\cos\theta)^2}{2z^2} - \frac{((x - a\rho\sin\theta)^2 + (a\rho\cos\theta)^2)^2}{8z^4} + \cdots)$$

Introduce the optical coordinates u, v

$$u = \frac{2\pi a^2}{\lambda z}$$
$$v = \frac{2\pi a x}{\lambda z}$$
$$\rho = \frac{v}{u}$$

$$U(P) = -iuAe^{i\left(kz + \frac{v^2}{2u}\right)} \int_0^1 e^{i\frac{u}{2}\rho'^2} J_0(v\rho')\rho' d\rho'$$

$$U(P) = iAe^{i\left(kz+\frac{u}{2}+\frac{v^2}{2u}\right)}\left[U_1(u,v) - iU_2(u,v)\right]$$

where the Lommel functions are defined as

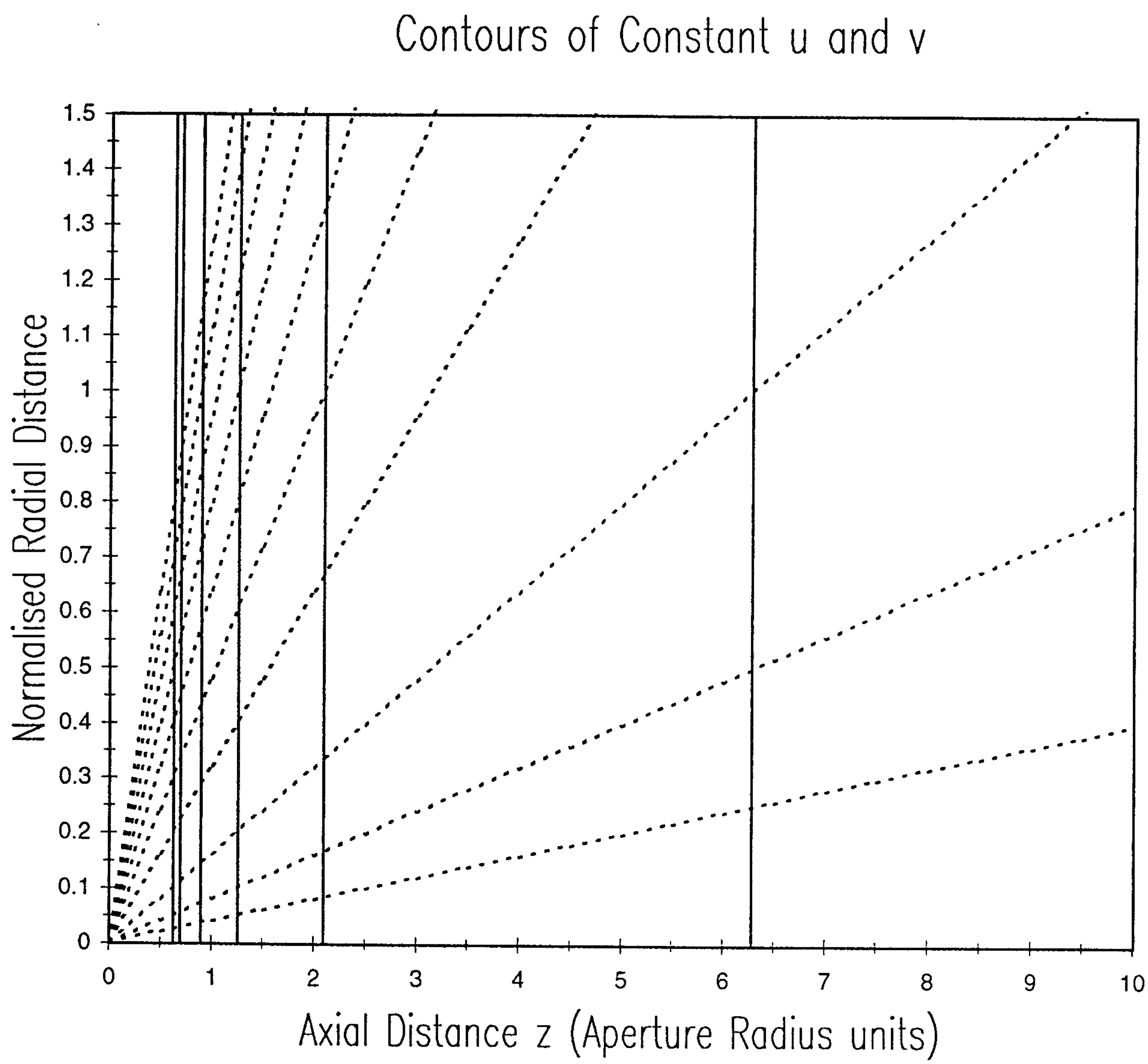
$$U_n(u,v) = \sum_{p=0}^{\infty}(-1)^p\left(\frac{u}{v}\right)^{n+2p} J_{n+2p}(v)$$

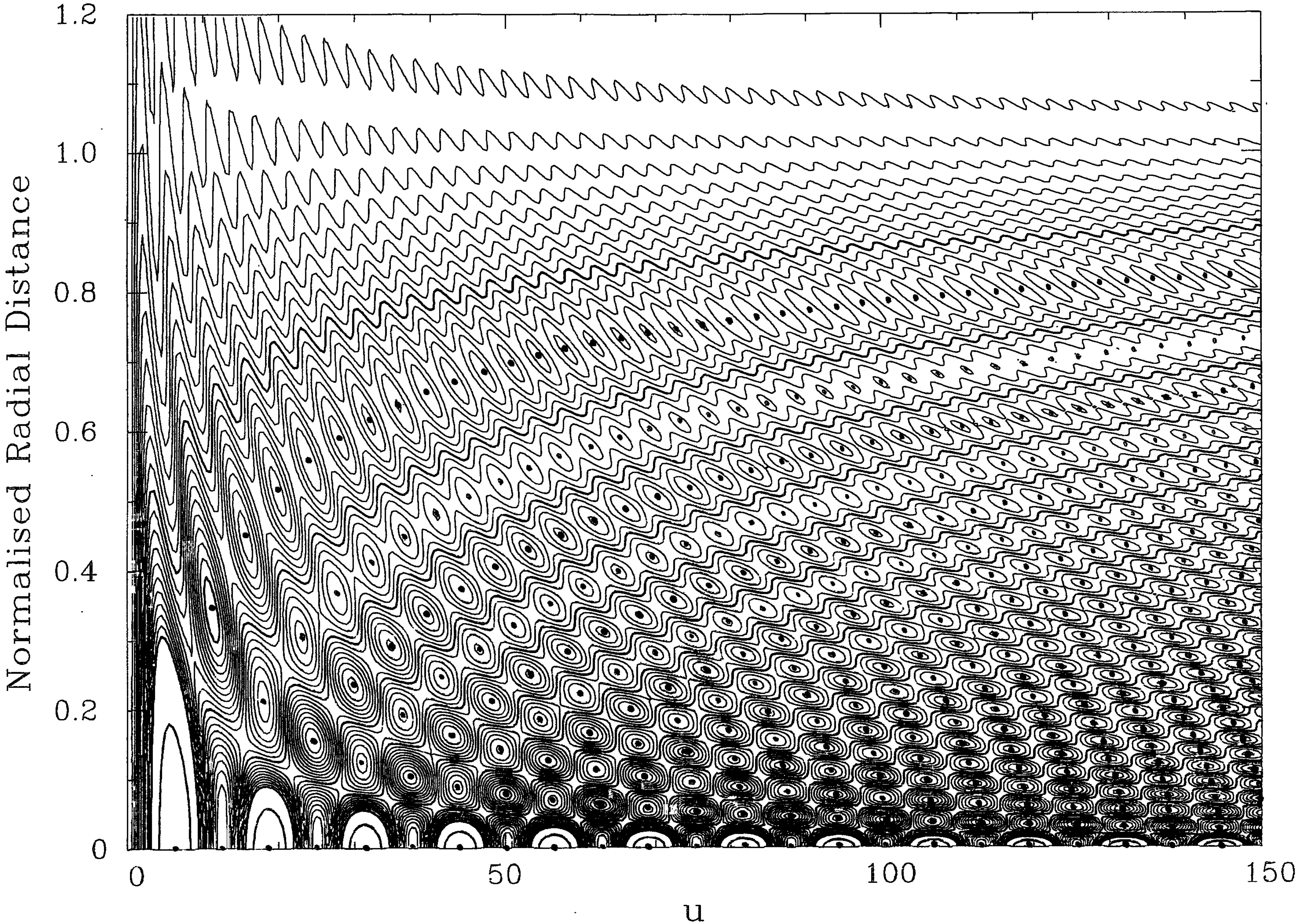
Intensity is simply expressed as

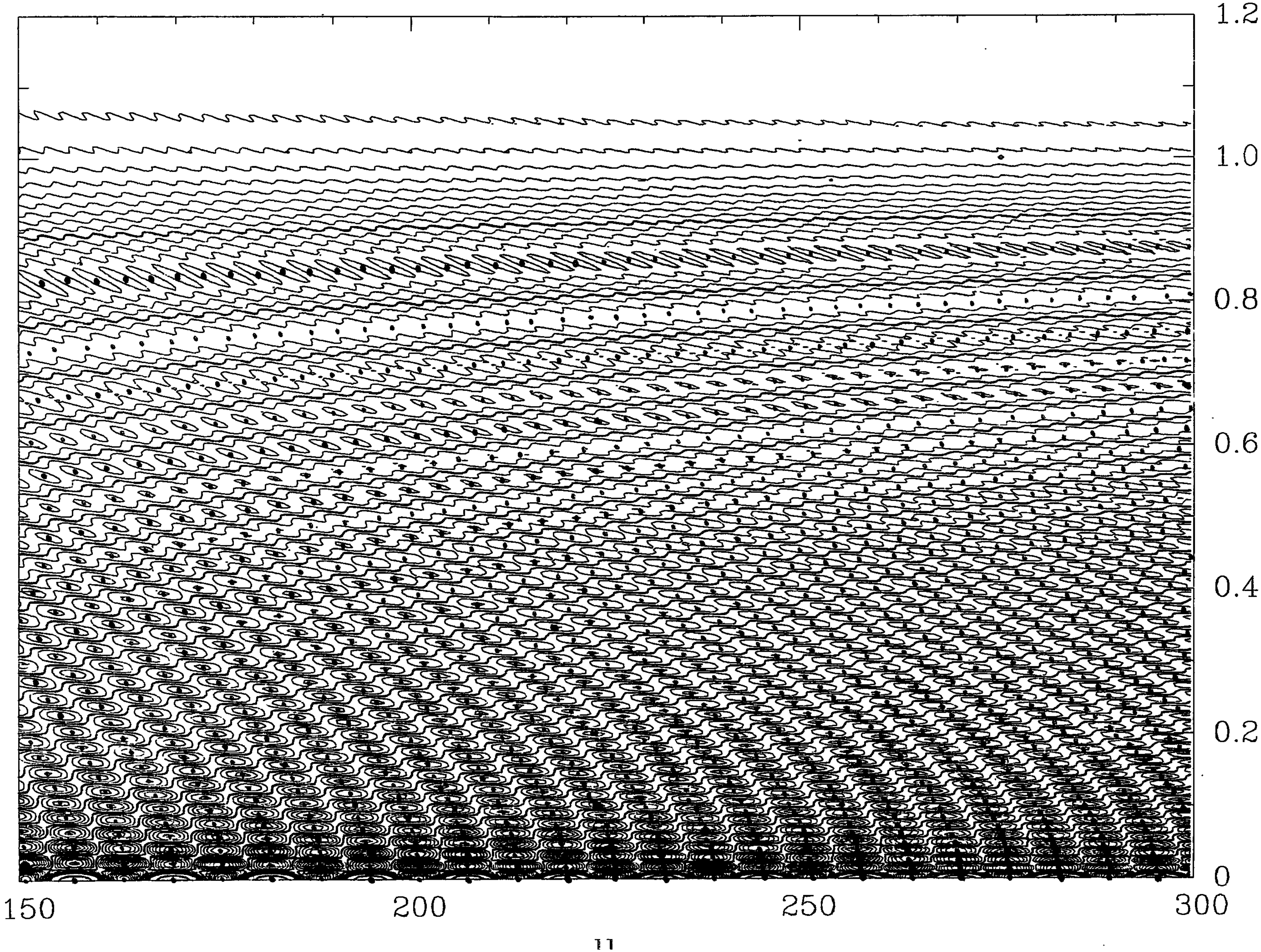
$$I(u,v) = \left(U_1^2(u,v) + U_2^2(u,v)\right) I_0$$

and Phase by

$$\phi(u,v) = kz + \frac{u}{2} + \frac{v^2}{2u} - \tan^{-1}\left(\frac{U_2(u,v)}{U_1(u,v)}\right) + \frac{\pi}{2}$$


Contours of Constant u and v
Normalised Radial Distance
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
8
9
10
Axial Distance z (Aperture Radius units)


Normalised Radial Distance
1.2
1.0
0.8
0.6
0.4
0.2
0
0
50
100
150
u


1.2
1.0
0.8
0.6
0.4
0.2
0
150
200
250
300

Lommel Functions

$$U_n(u,v) = \sum_{p=0}^{\infty}(-1)^p\left(\frac{u}{v}\right)^{n+2p} J_{n+2p}(v)$$

$$V_n(u,v) = \sum_{p=0}^{\infty}(-1)^p\left(\frac{v}{u}\right)^{n+2p} J_{n+2p}(v)$$

$$U_1(u,v) = \sin\left(\frac{u}{2}+\frac{v^2}{2u}\right) - V_1(u,v)$$

$$U_2(u,v) = -\cos\left(\frac{u}{2}+\frac{v^2}{2u}\right) + V_0(u,v)$$

At the shadow boundary (where $u = v$)

$$U_1(u,u) = \frac{1}{2}\sin u$$

$$U_2(u,u) = \frac{1}{2}\left(J_0(u) - \cos u\right)$$

Along the shadow boundary

$$I(u,u) = \frac{I_0}{4}\left(1 - 2J_0(u)\cos u + J_0^2(u)\right)$$

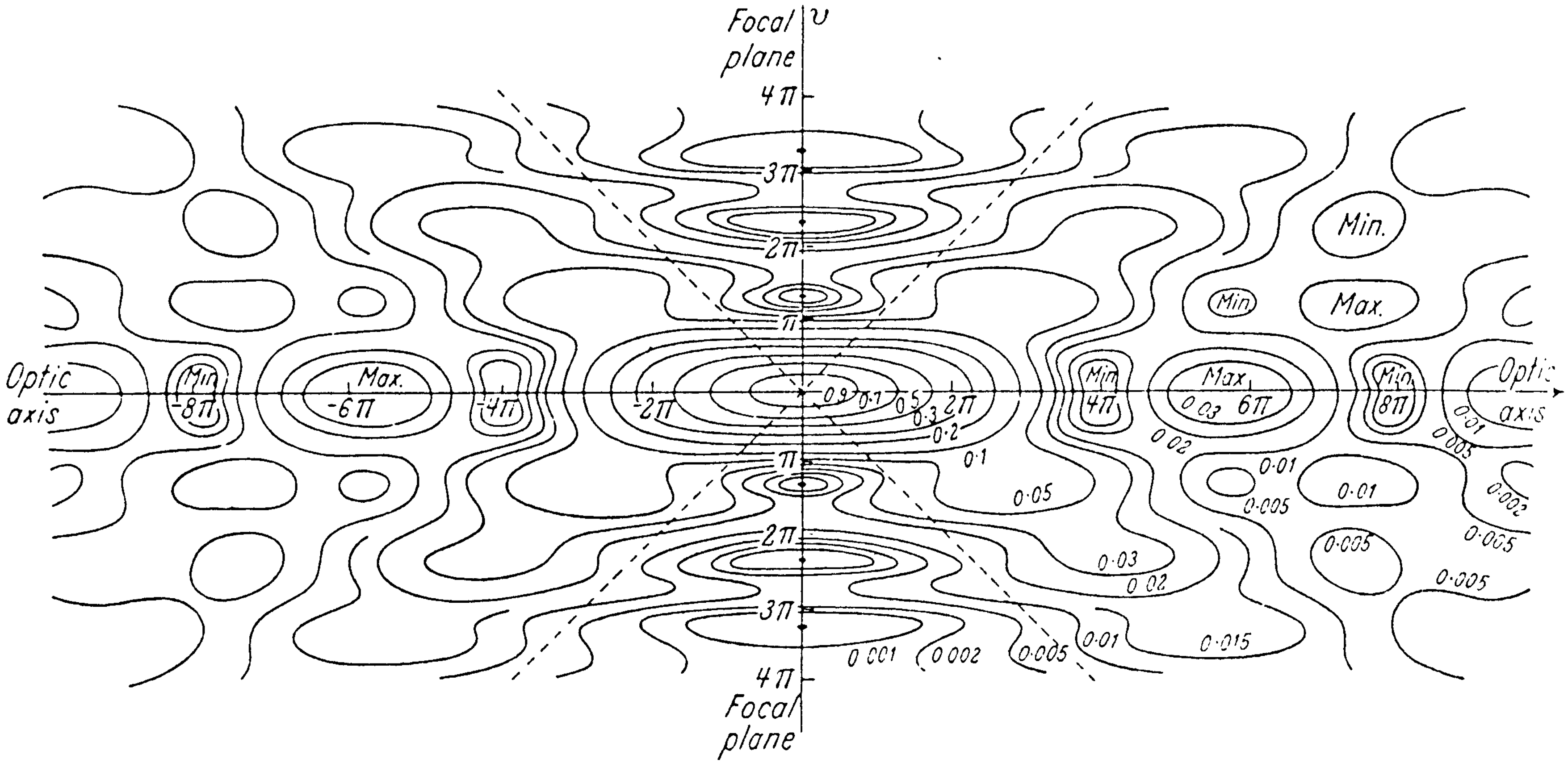
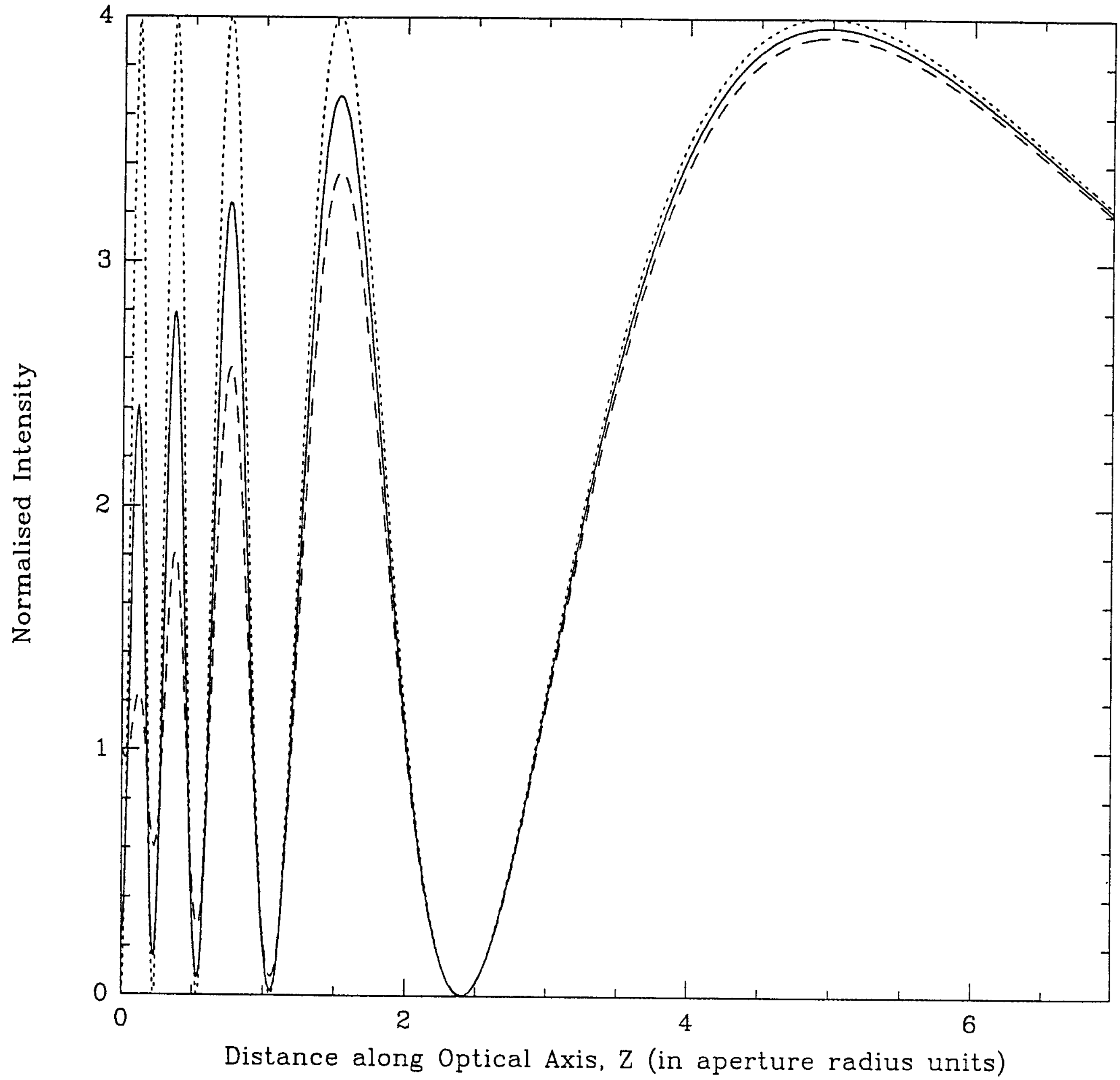


Fig. 8.41. Isophotes [contour lines of the intensity $I(u, v)$] in a meridional plane near focus of a converging spherical wave diffracted at a circular aperture. The intensity is normalized to unity at focus. The dotted lines represent the boundary of the geometrical shadow. When the figure is rotated about the u-axis, the minima on the v-axis generate the AIRY dark rings.

(Adapted from E. H. LINFOOT and E. WOLF, *Proc. Phys. Soc.*, B, **69** (1956), 823.)



4
3
2
1
0
Normalised Intensity
0
2
4
6
Distance along Optical Axis, Z (in aperture radius units)

Axial Comparisons

Exact Axial Solutions

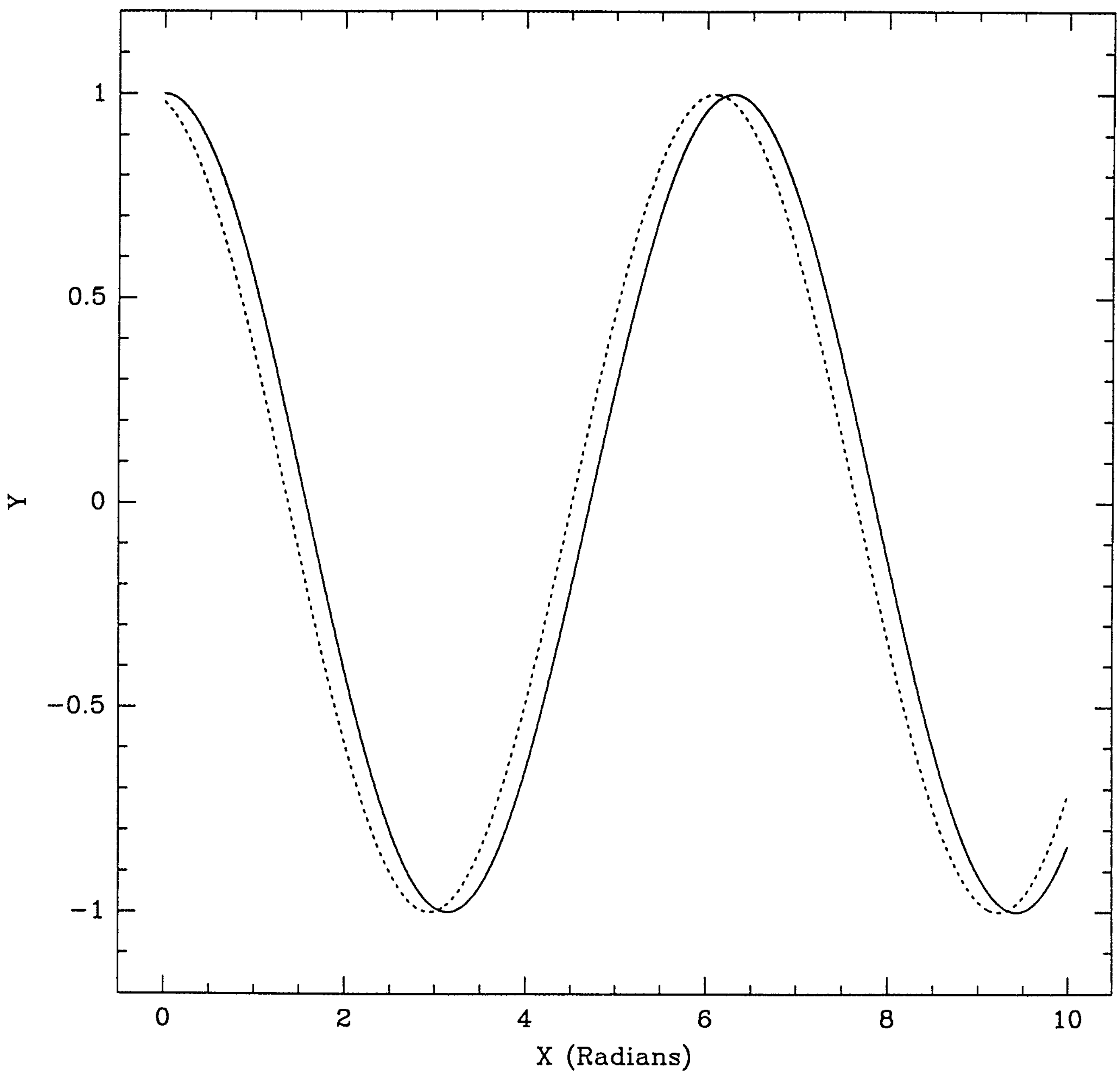
$$I_K(z) = 1 - \left(\frac{z}{\sqrt{z^2+a^2}} + 1\right)\cos\left(k(\sqrt{z^2+a^2} - z)\right) + \frac{1}{4}\left(\frac{z}{\sqrt{z^2+a^2}} + 1\right)^2$$

Number of axial maxima is given by a/λ

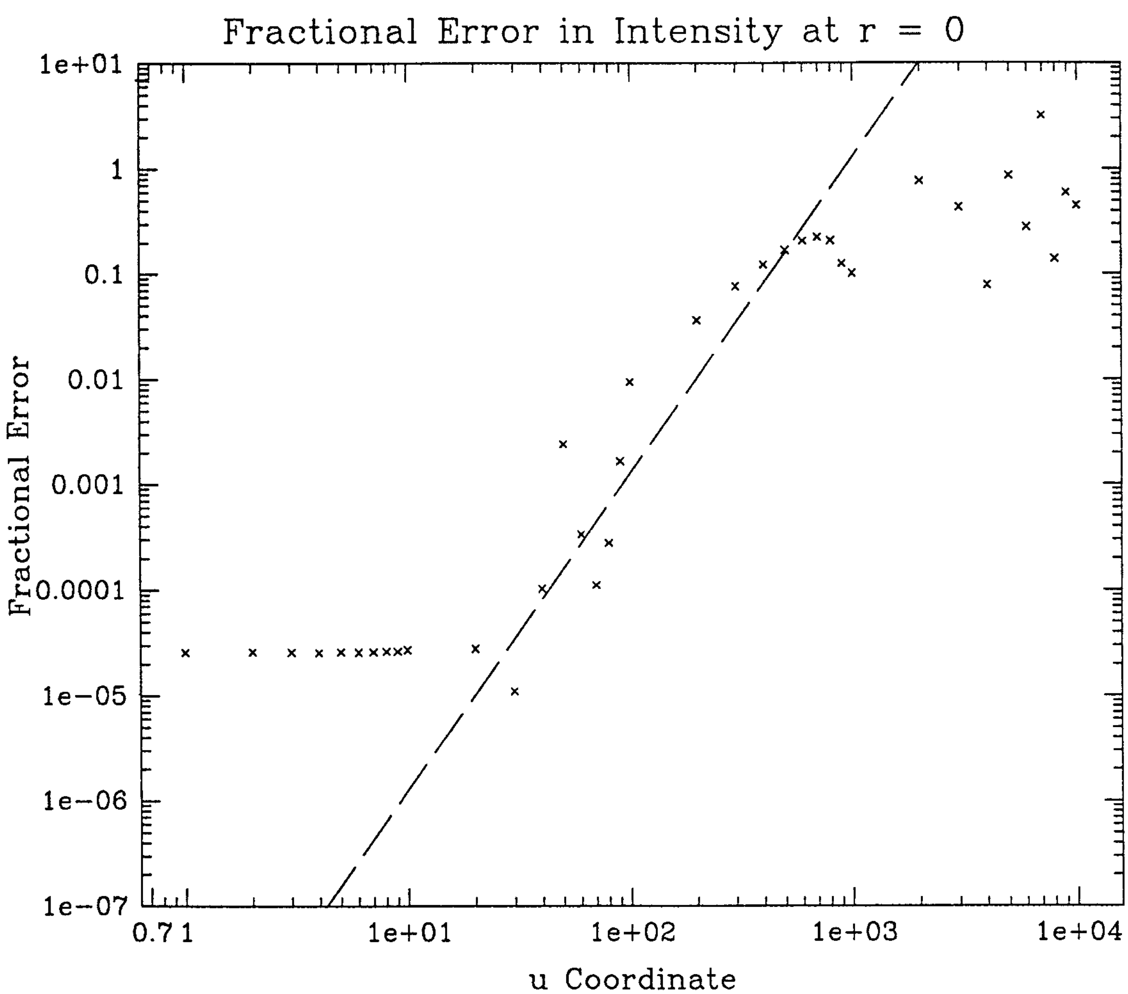
Fresnel Axial Solution

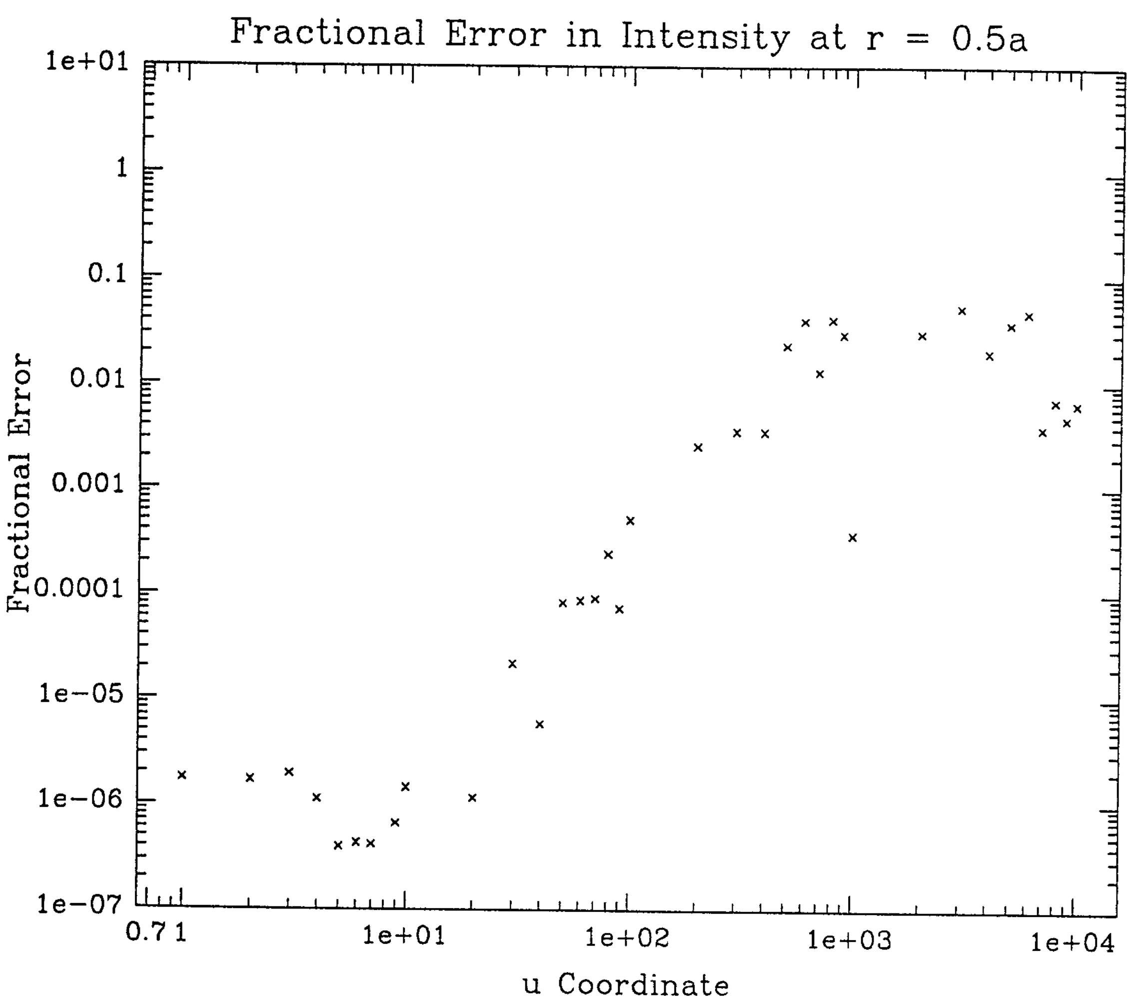
$$I(u,0) = 4\sin^2\left(\frac{u}{4}\right) = 4\sin^2\left(\frac{\pi a^2}{2\lambda z}\right)$$

$$\epsilon \approx \frac{ka^4}{z^3}$$

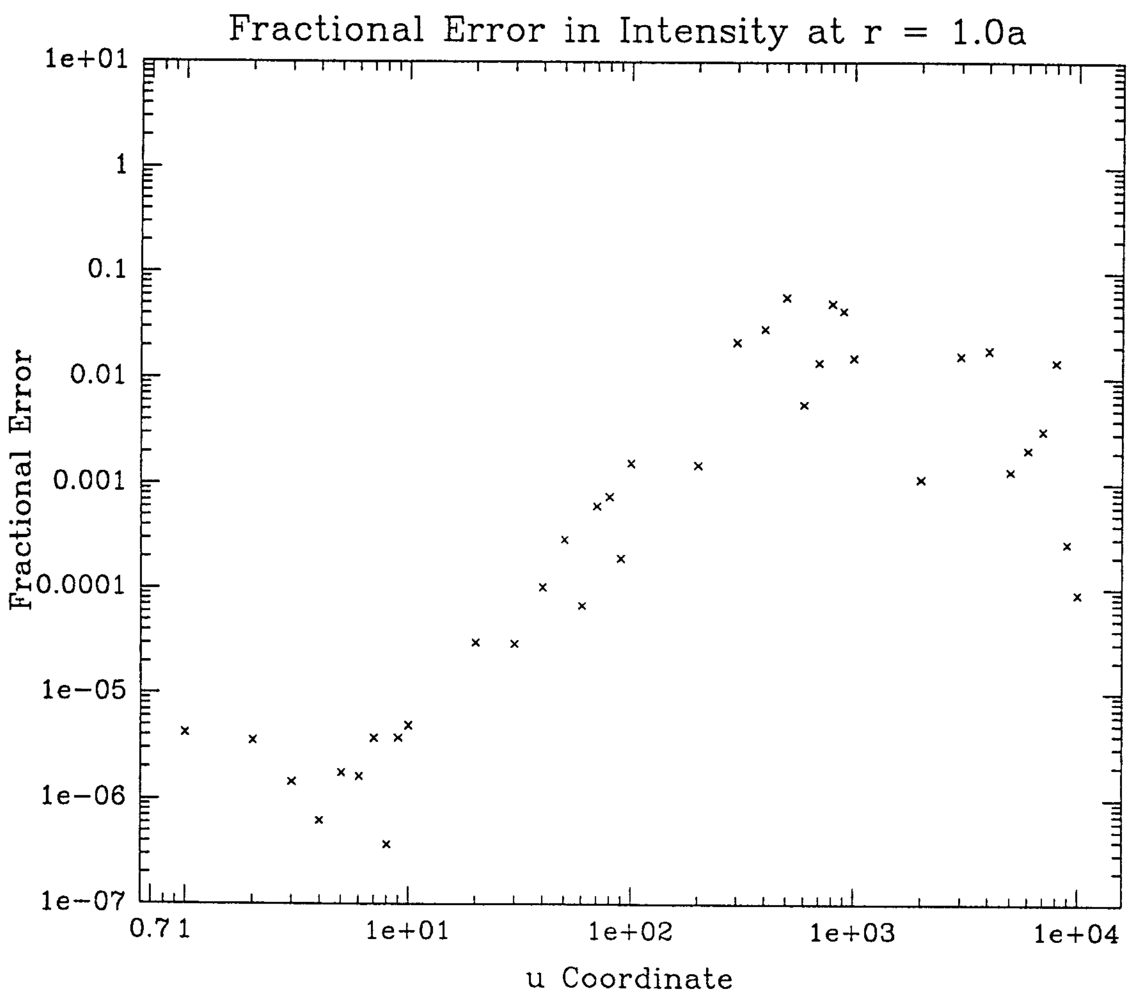


1
0.5
0
−0.5
−1
Y
0
2
4
6
8
10
X (Radians)


Fractional Error in Intensity at r = 0
Fractional Error
u Coordinate
1e+01
1
0.1
0.01
0.001
0.0001
1e−05
1e−06
1e−07
0.7 1
1e+01
1e+02
1e+03
1e+04



Fractional Error in Intensity at r = 0.5a
Fractional Error
u Coordinate
1e+01
1
0.1
0.01
0.001
0.0001
1e−05
1e−06
1e−07
0.7 1
1e+01
1e+02
1e+03
1e+04


Fractional Error in Intensity at r = 1.0a
Fractional Error
1e+01
1
0.1
0.01
0.001
0.0001
1e−05
1e−06
1e−07
0.7 1
1e+01
1e+02
1e+03
1e+04
u Coordinate

Improving the Approximation

$$\begin{aligned} kr &= k\sqrt{z^2 + a^2\rho^2 + a^2\rho'^2 - 2a^2\rho\rho'\sin\theta} \\ &\approx \frac{1}{2}U\rho'^2 - V\rho'\sin\theta + W \end{aligned}$$

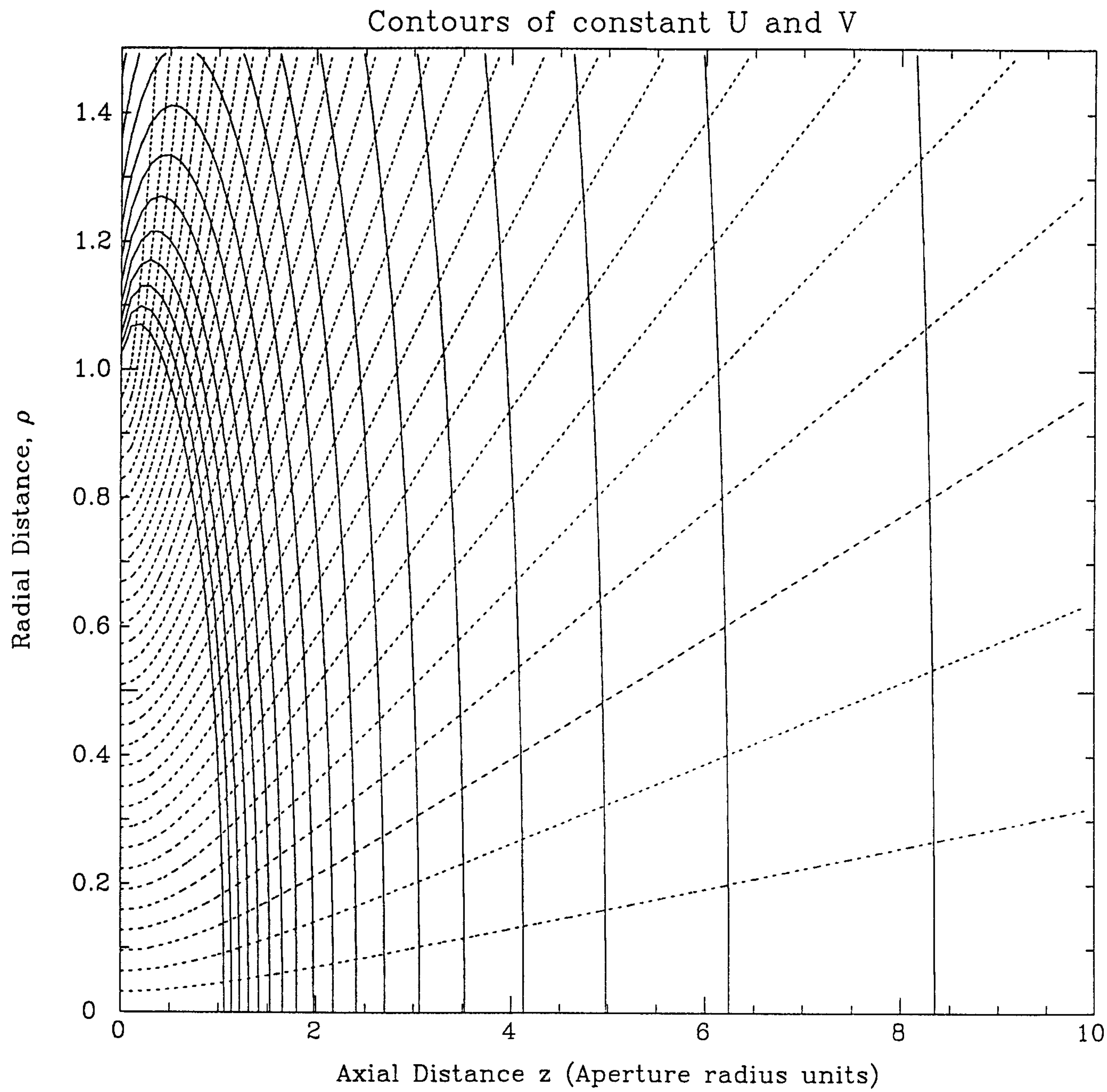
Exterior stationary points at $\rho' = 1, \theta = 0.$

Interior stationary point at $d(kr) = 0$

$$\begin{aligned} d(kr) &= \frac{\partial(kr)}{\partial\rho'}d\rho' + \frac{\partial(kr)}{\partial\theta}d\theta \\ &= (U\rho' - V\sin\theta)d\rho' - (V\cos\theta)d\theta \\ \rho' &= \rho = \frac{V}{U}, \theta = 0 \end{aligned}$$

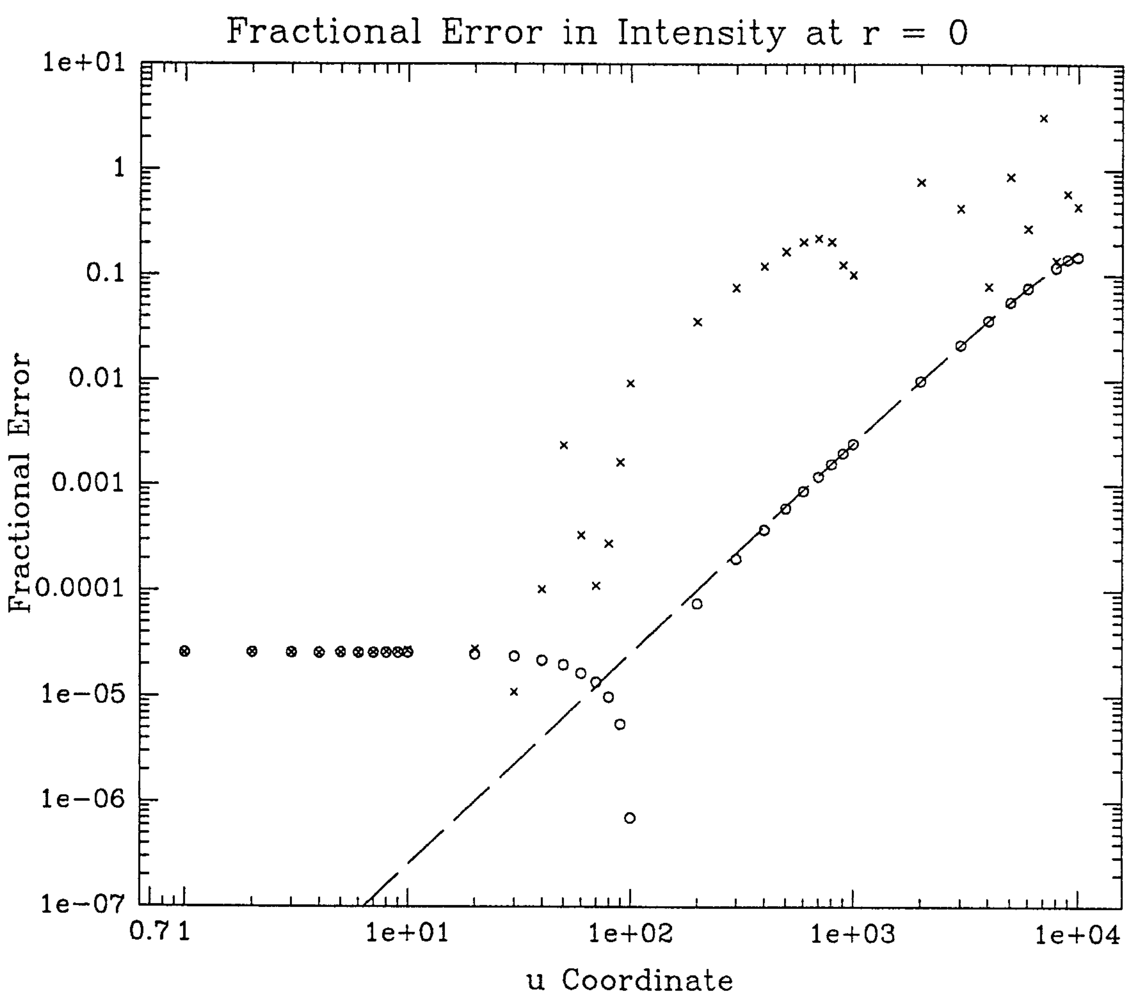
$$\begin{aligned} U &= \frac{k}{2}(p+q)^2, \\ V &= \frac{k}{2}(p^2 - q^2), \\ W &= \frac{k}{4}(p^2 - q^2) + kz \end{aligned}$$

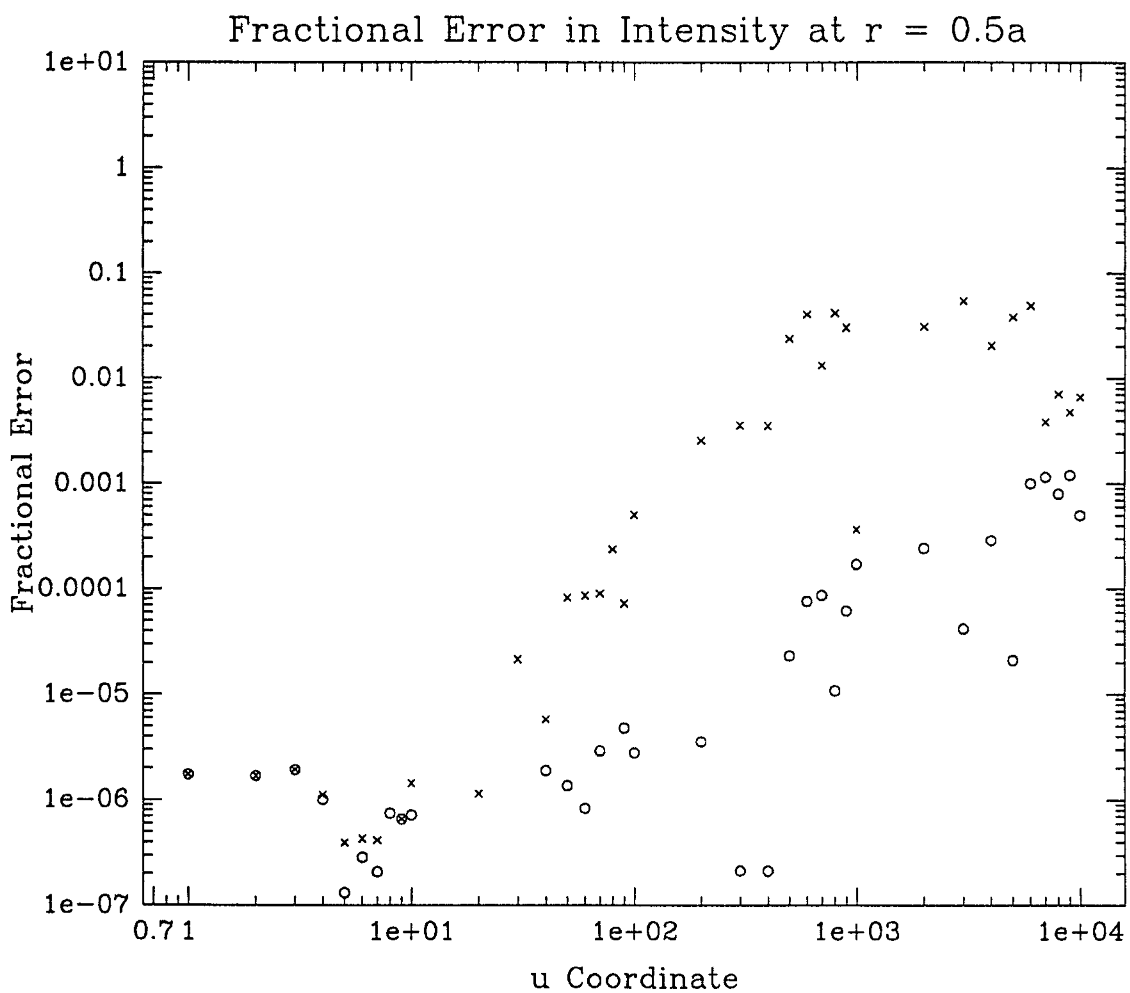
$$\begin{aligned} p &= \sqrt{\sqrt{z^2 + a^2(1+\rho)^2} - z} \\ q &= \sqrt{\sqrt{z^2 + a^2(1-\rho)^2} - z} \end{aligned}$$



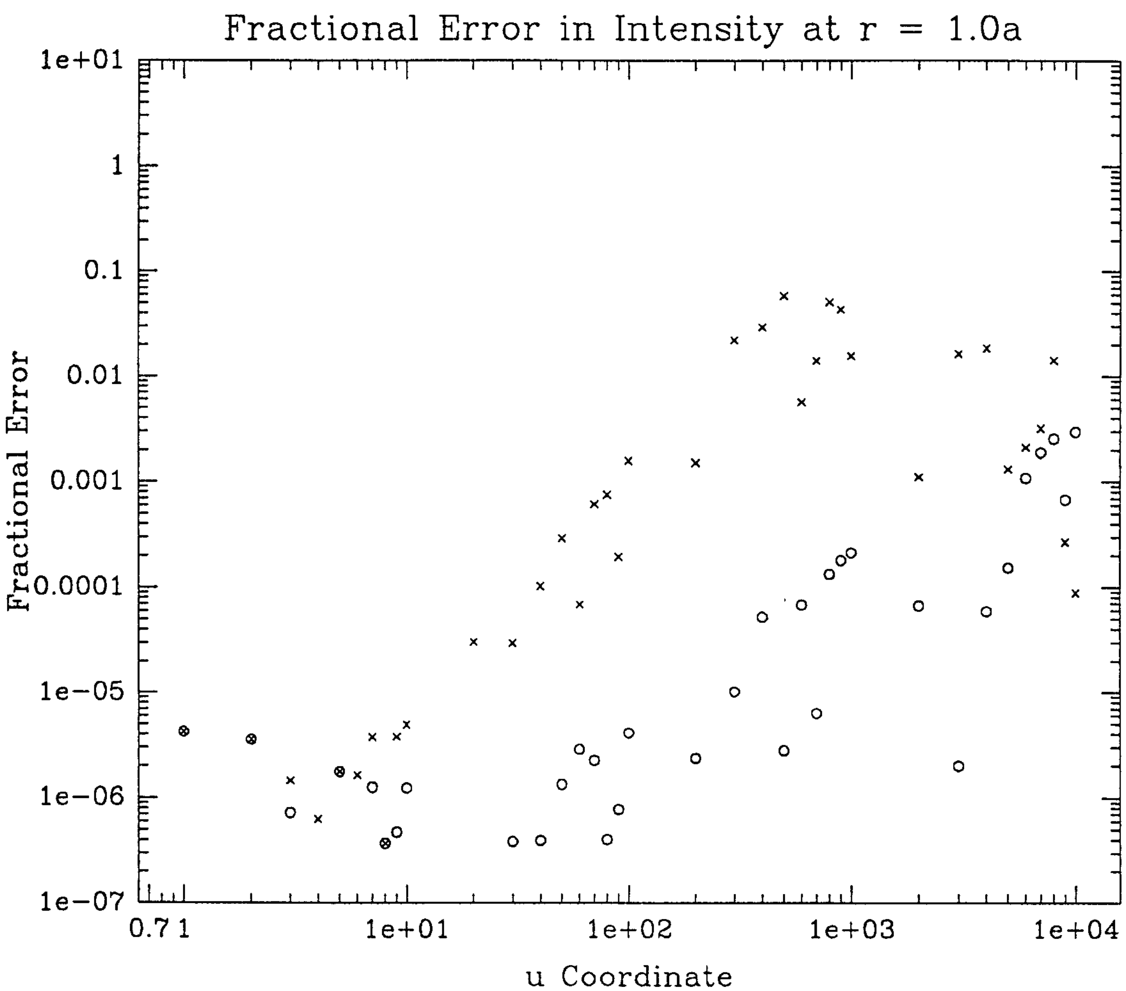
$$U \approx u\left(1 - \frac{a^2\left(1 + 3\rho^2 + \rho^4\right)}{4z^2} + \cdots\right)$$

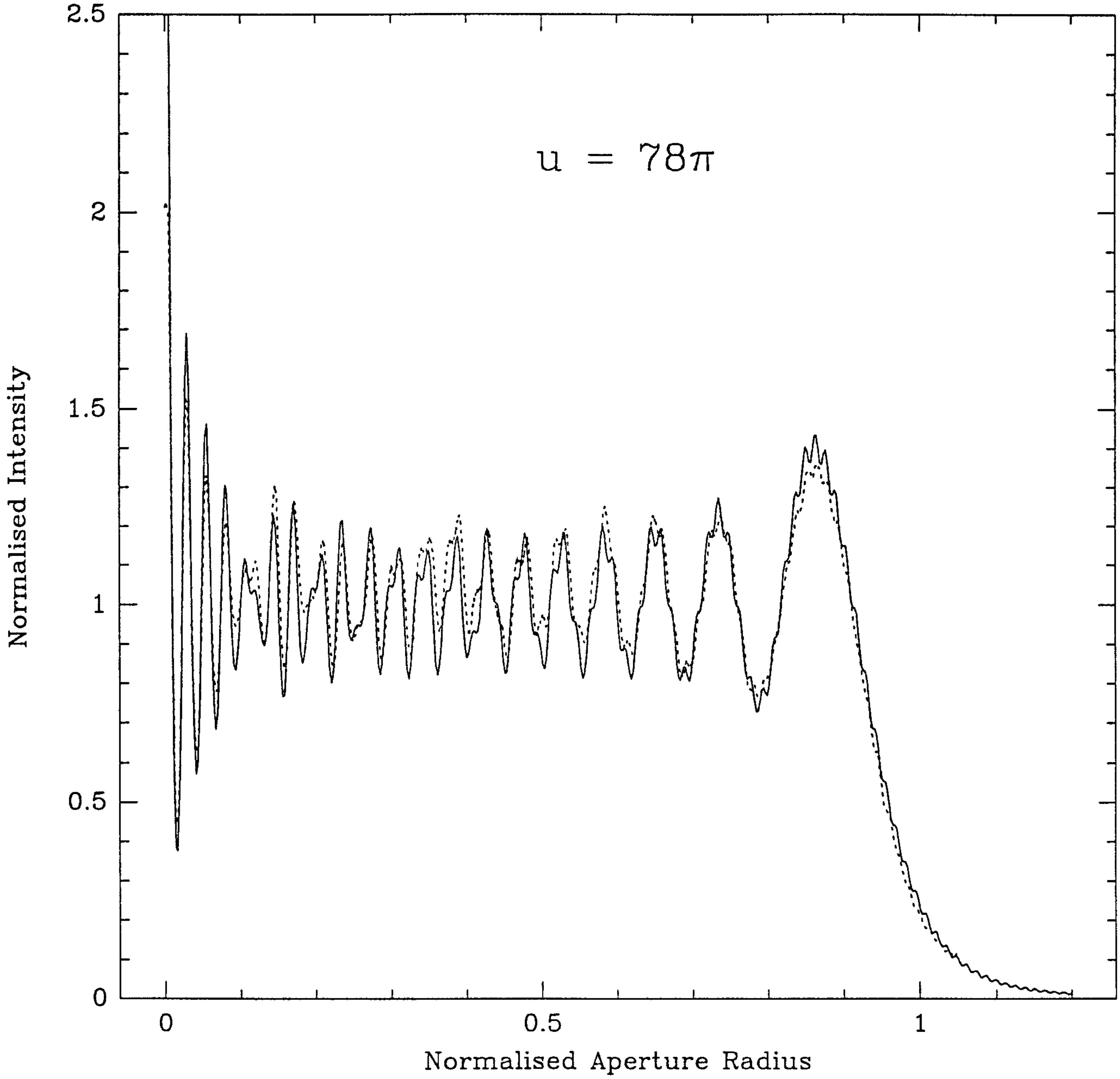
$$V \approx v\left(1 - \frac{a^2\left(1 + \rho^2\right)}{2z^2} + \cdots\right)$$


Fractional Error in Intensity at r = 0
1e+01
1
0.1
0.01
0.001
0.0001
1e−05
1e−06
1e−07
Fractional Error
0.7 1
1e+01
1e+02
1e+03
1e+04
u Coordinate

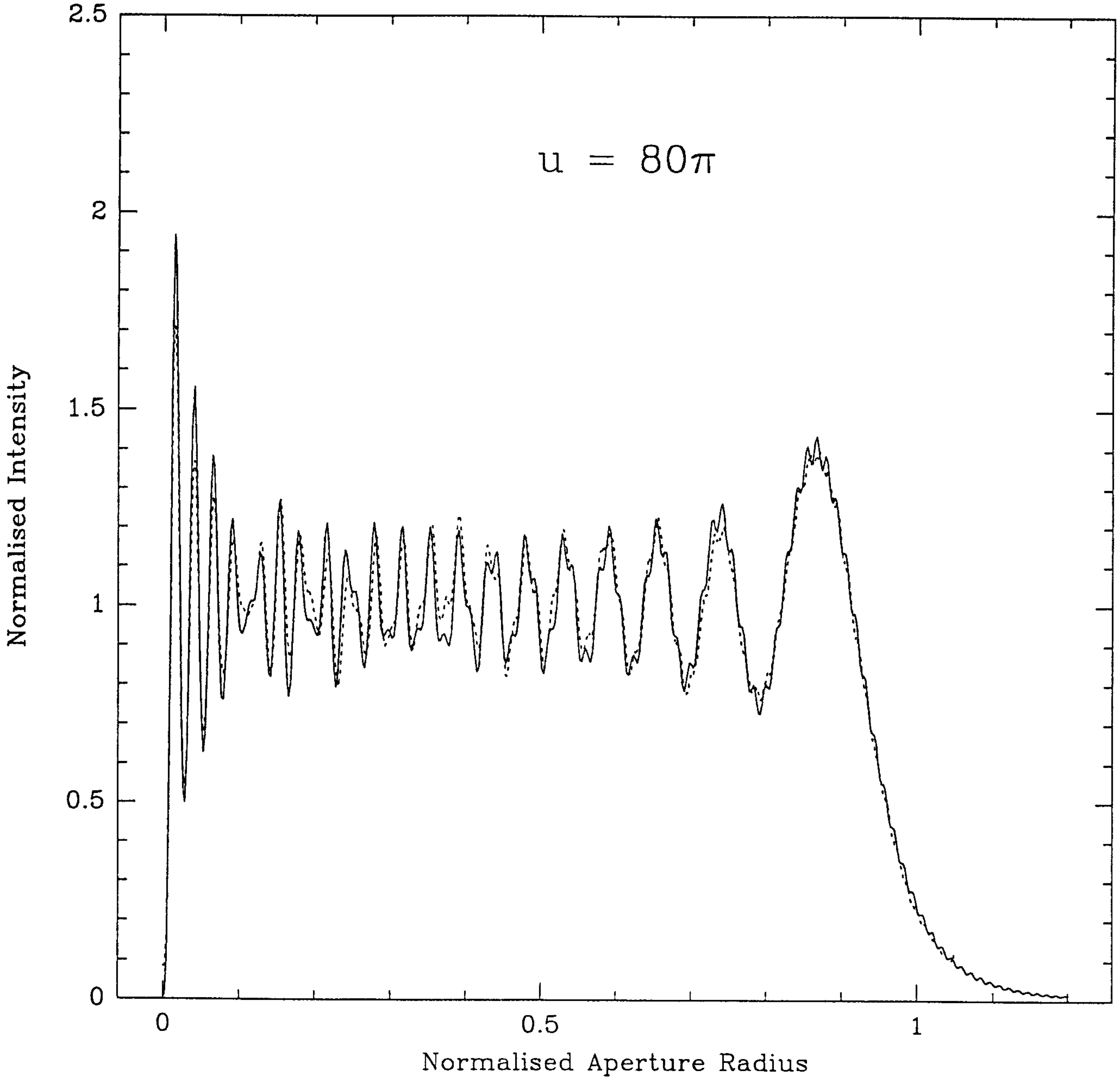


Fractional Error in Intensity at r = 0.5a
Fractional Error
1e+01
1
0.1
0.01
0.001
0.0001
1e−05
1e−06
1e−07
0.7 1
1e+01
1e+02
1e+03
1e+04
u Coordinate

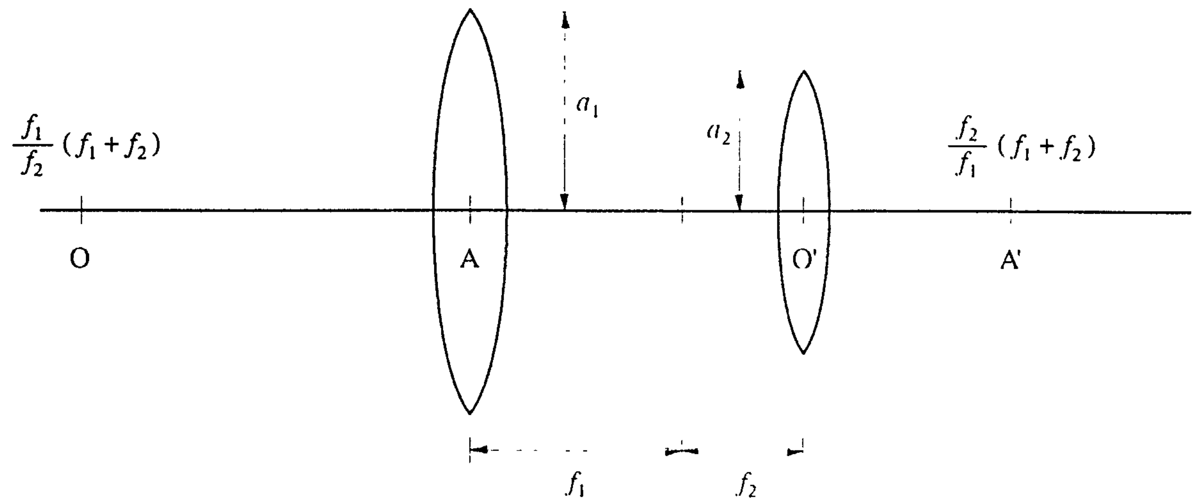

Fractional Error in Intensity at r = 1.0a
Fractional Error
1e+01
1
0.1
0.01
0.001
0.0001
1e−05
1e−06
1e−07
0.7 1
1e+01
1e+02
1e+03
1e+04
u Coordinate



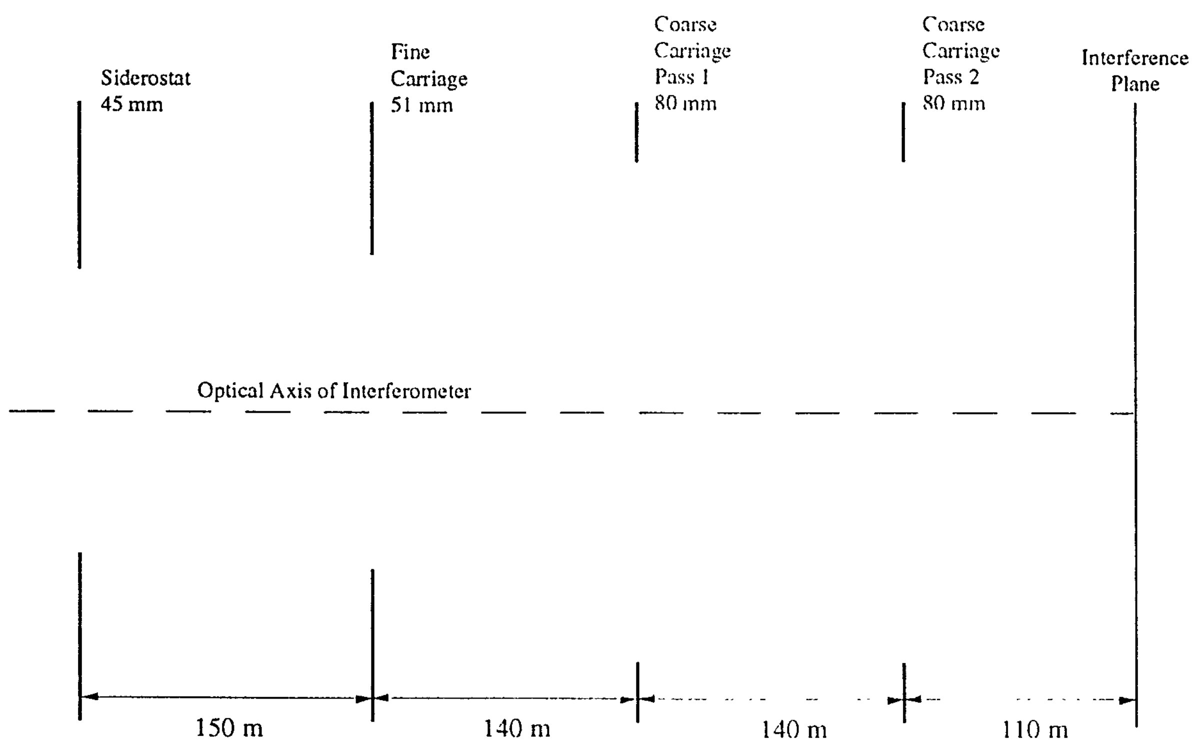
Measurement at $u = 78\pi$. Best fit was found at $u = 244.8509 = 77.938462\pi$

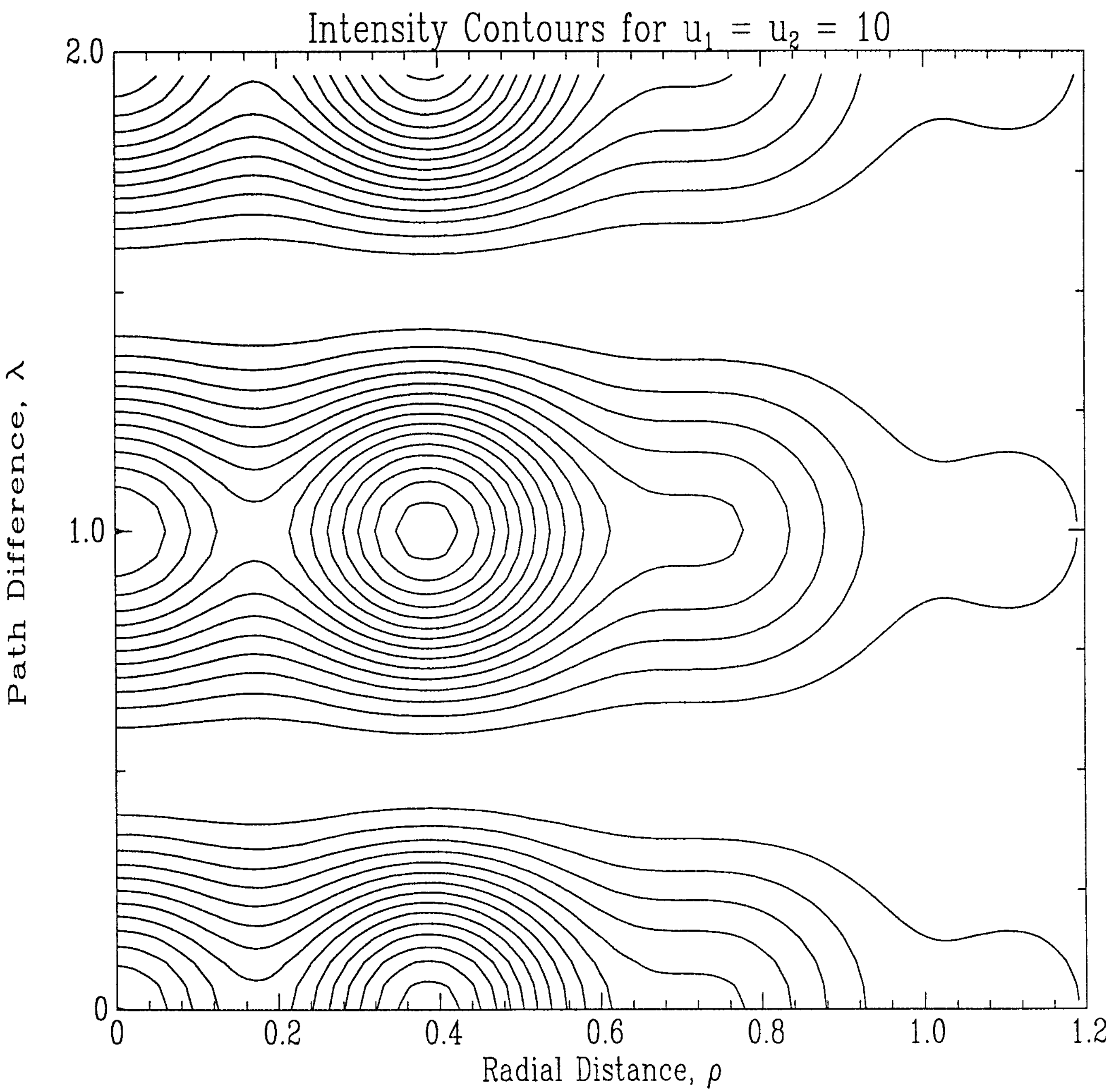


Measurement at $u = 80\pi$. Best fit was found at $u = 251.1794 = 79.952886\pi$

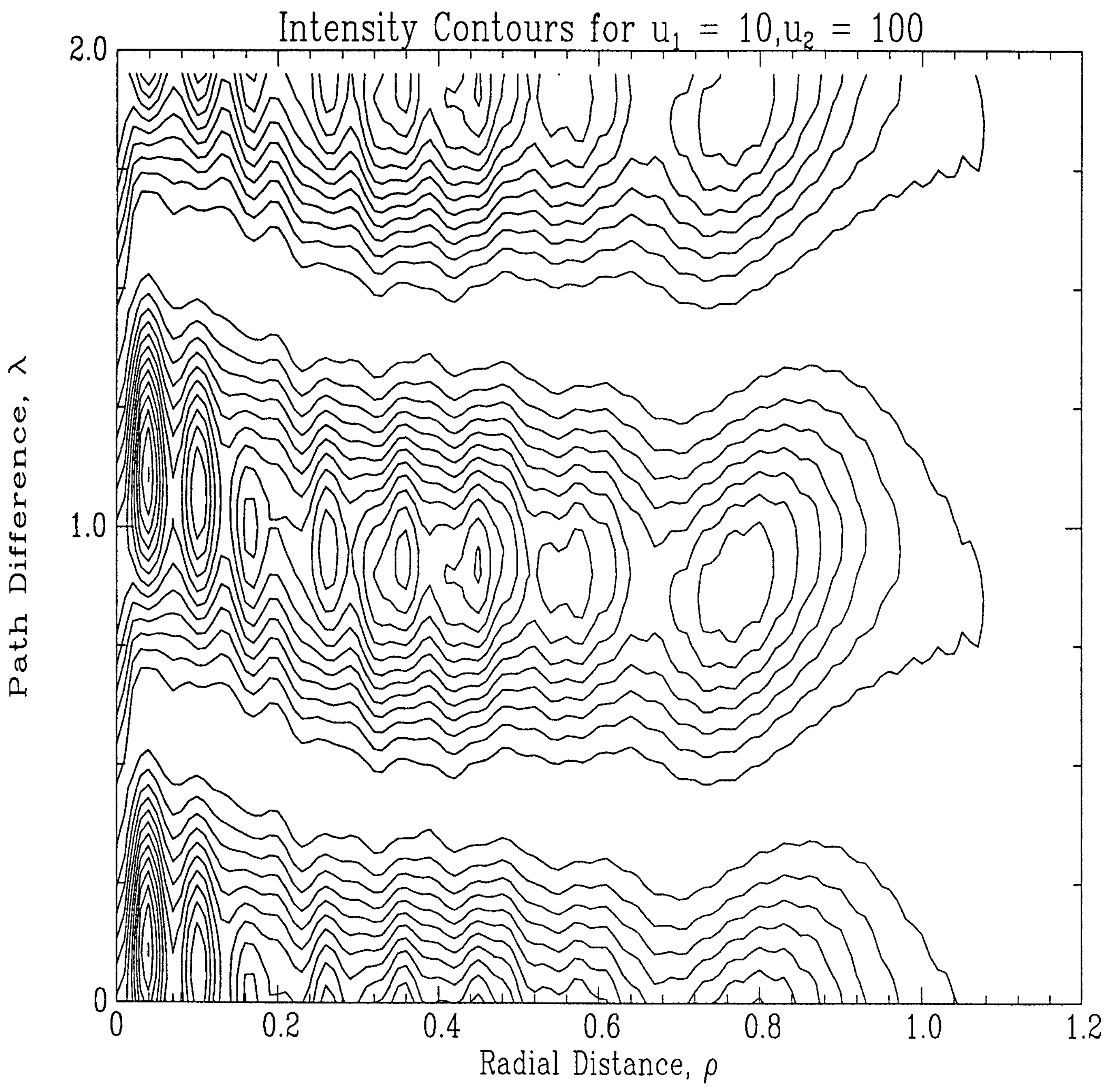


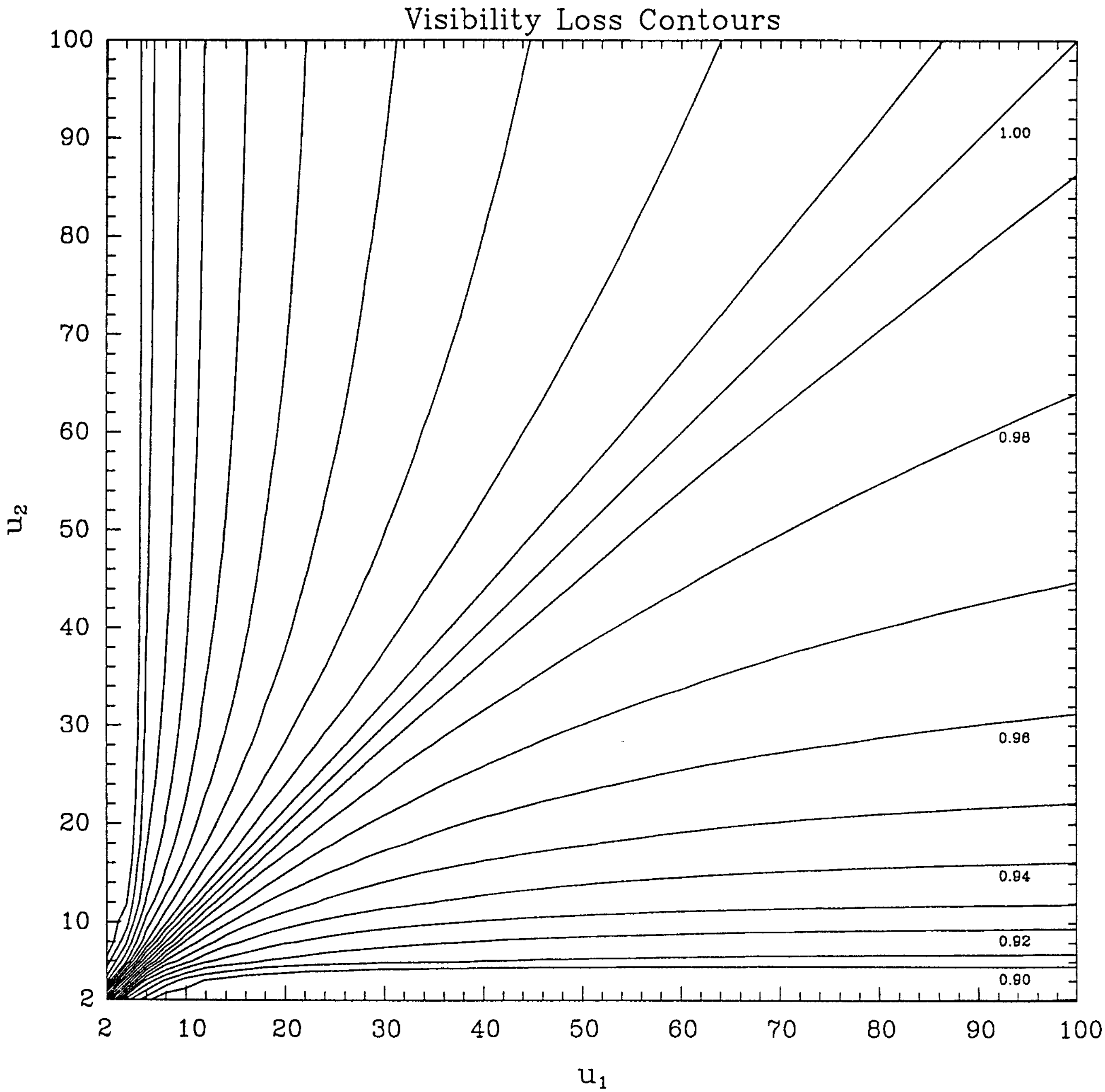
$$z_2 = \left(\frac{f_2}{f_1}\right)^2 z_1 - \frac{f_2}{f_1}(f_1 + f_2)$$



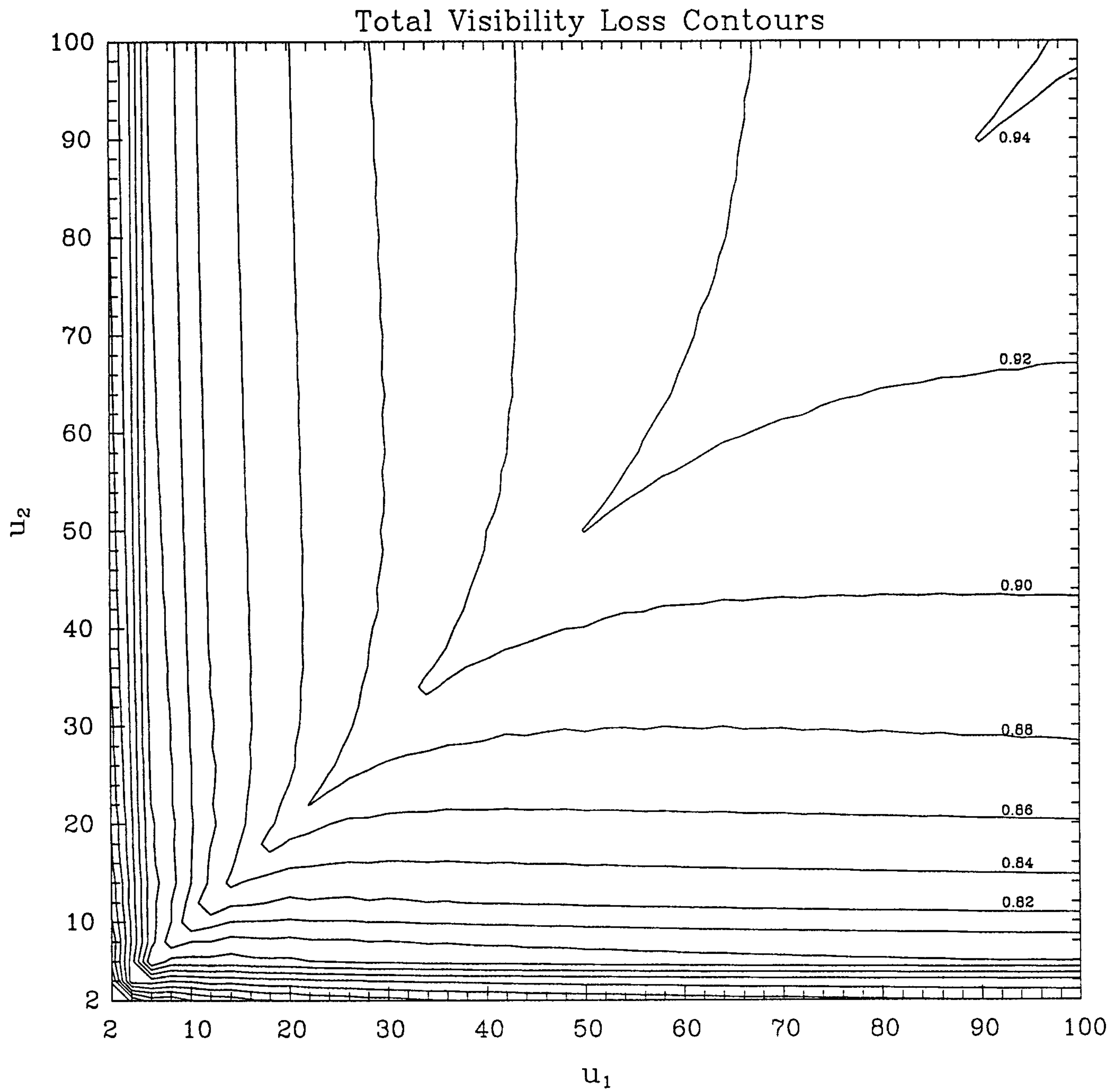


$$I(u_1, u_2, \rho, \delta) = I(u_1, \rho) + I(u_2, \rho) + 2\sqrt{I(u_1, \rho) I(u_2, \rho)} \cos(\phi(u_1, \rho) - \phi(u_2, \rho) + \frac{2\pi}{\lambda}\delta)$$


Intensity Contours for $u_1 = 10, u_2 = 100$
2.0
1.0
0
Path Difference, λ
0
0.2
0.4
0.6
0.8
1.0
1.2
Radial Distance, ρ



$$\mathcal{V} = \frac{2\int_0^{\rho_0} \sqrt{I(u_1,\rho) I(u_2,\rho)} \cos(\phi(u_1,\rho) - \phi(u_2,\rho) + \frac{2\pi}{\lambda}\delta_{max})\rho d\rho}{\int_0^{\rho_0} (I(u_1,\rho) + I(u_2,\rho))\rho d\rho}$$



$$\frac{\int_0^{\rho_0}(I(u_1,\rho)+I(u_2,\rho))\rho d\rho}{\int_0^{\infty}(I(u_1,\rho)+I(u_2,\rho))\rho d\rho}$$

$$\mathcal{V} = 2\int_0^{\rho_0}\sqrt{I(u_1,\rho)I(u_2,\rho)}\cos(\phi(u_1,\rho)-\phi(u_2,\rho)+\frac{2\pi}{\lambda}\delta_{max})\rho d\rho$$

Preferred Calculation Scheme

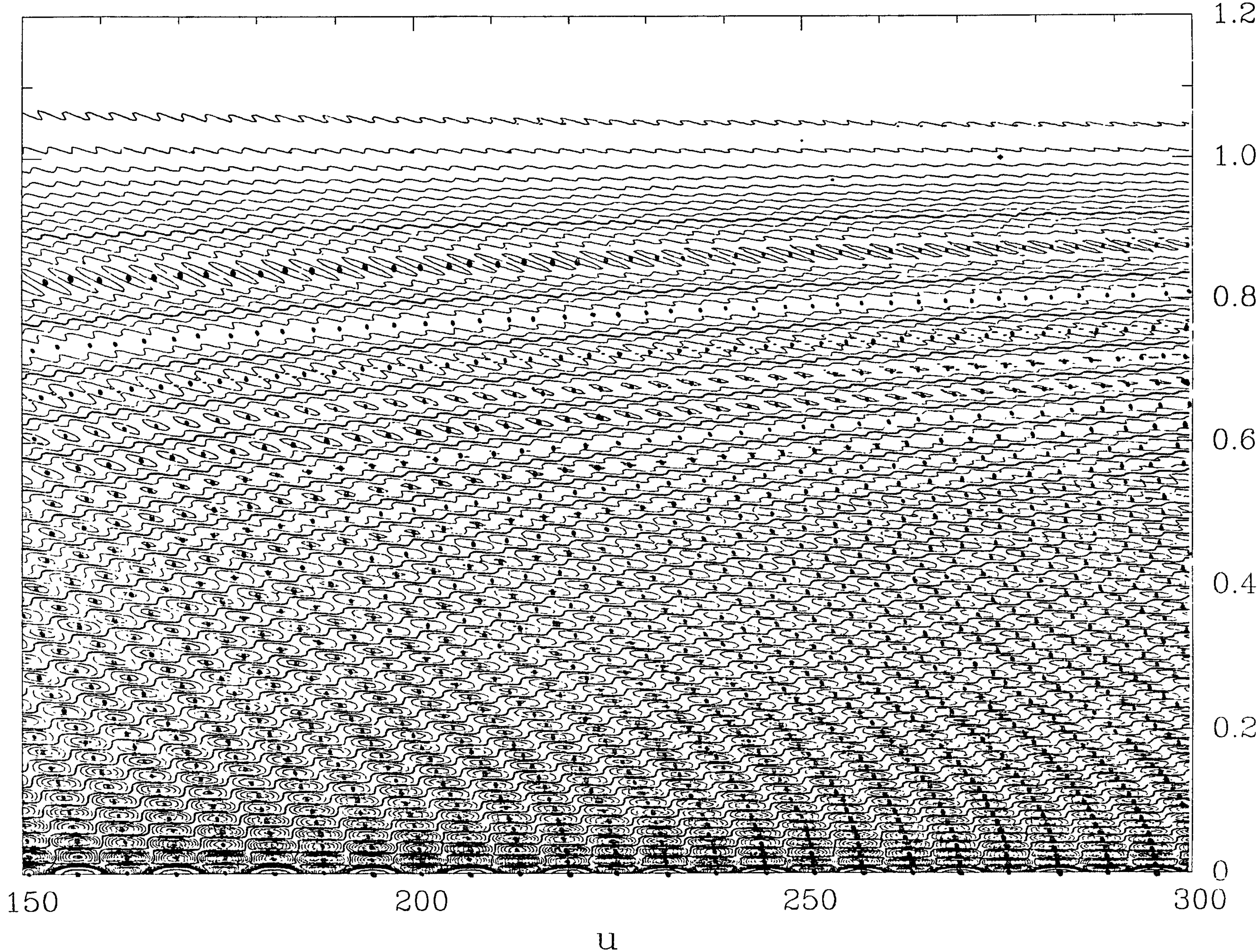
$$U(P) = \frac{k}{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\alpha, \beta, z)\, e^{-ik(\alpha x + \beta y)} d\alpha d\beta$$

$$\alpha = \sin\theta$$
$$\beta = \cos\psi$$

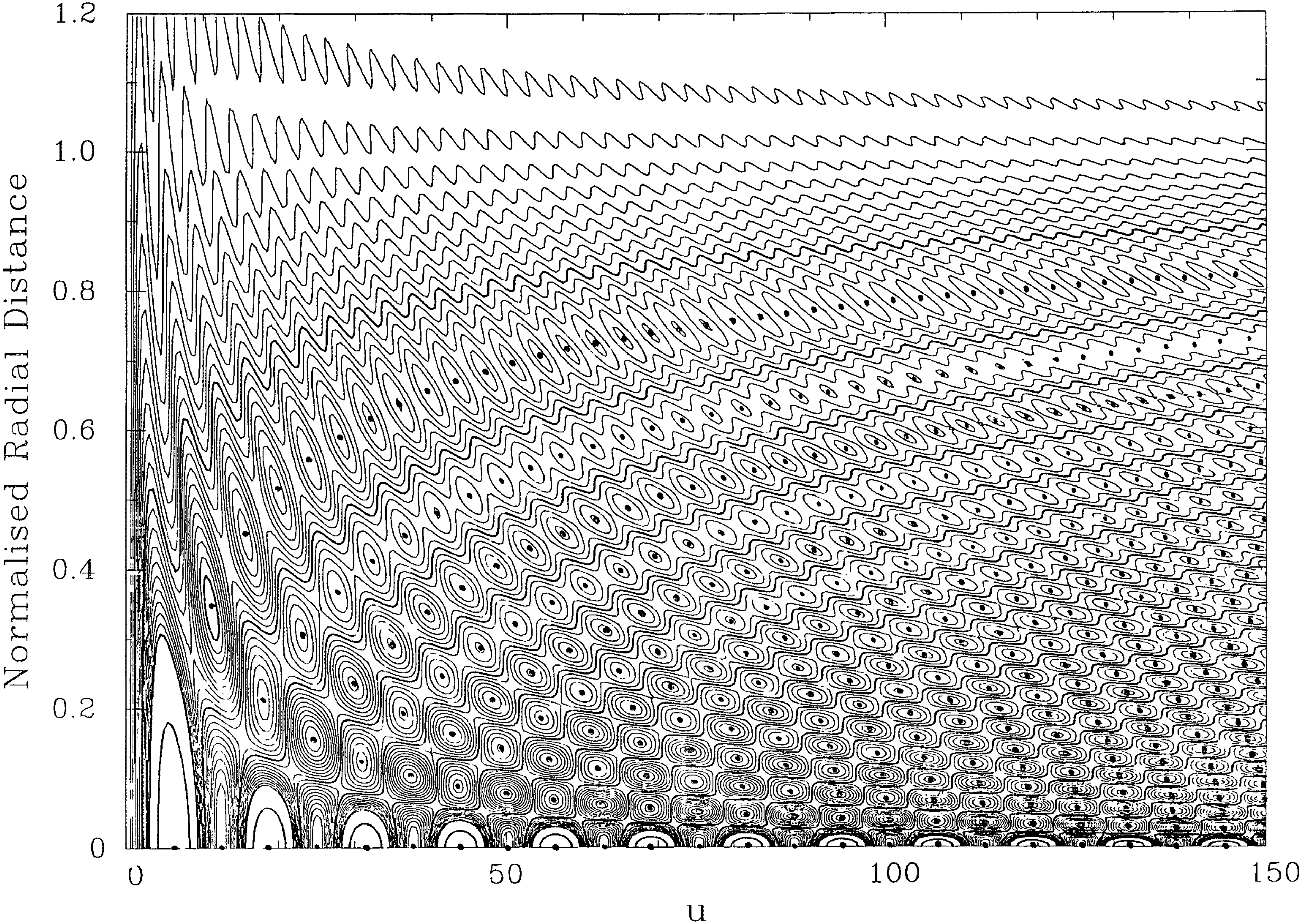
θ and ψ are angles with aperture normal

$$A(\alpha, \beta, z) = A_0(\alpha, \beta)\, e^{ikz\sqrt{1-(\alpha^2+\beta^2)}}$$

$$A_0(\alpha, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x, y, 0)\, e^{ik(\alpha x + \beta y)} dx dy$$



1.2
1.0
0.8
0.6
0.4
0.2
0
150
200
250
300
u



1.2
1.0
0.8
0.6
0.4
0.2
0
Normalised Radial Distance
0
50
100
150
u