R-modes and gravitational wave searches: frequencies and frequency derivatives LIGO DCC: T1100355

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Contents

1 Introduction

How exactly should an r-mode search be carried out—what intervals in the mode frequency and its time derivatives should be searched over? To leading order, the mode frequency (which is the same as the gravitational wave frequency) is given by

$$|f_{\text{mode}}| = \frac{4}{3} f_{\text{star}}. \tag{1}$$

However, this comes from modelling the star as a slowly-rotating barotropic Newtonian perfect fluid. There are a number of additional pieces of physics that might shift this frequency, with the shift depending upon the stellar mass and equation of state. Given we won't know the exact size of this shift, we will need to identify bands in frequency (and presumably its time derivatives) to search over. At the very least, we need to worry about the following:

- The effects of General Relativity
- The effect of the crust
- The effects of rapid rotation
- The effects of stratification (i.e. going beyond the barotropic approximation)
- The effect of the magnetic field
- The effects of a possible superfluid component

We should eventually look at each of these in tern to see which has the largest effect on the mode frequency, and what the corresponding uncertainties say about the frequency band to be searched over.

2 Frequency conventions

We will need to be careful in converting from the rotating frame mode frequency σ_{R} (given in several references) to the inertial frame mode frequency σ_{I} (the relevant one for gravitational wave searches). Following the standard convention, if perturbations are proportional to $e^{i(\sigma_{\rm I} t + m\phi)}$, the conversion formula is

$$\sigma_{\rm I} = \sigma_{\rm R} - m\Omega. \tag{2}$$

In fact many authors give their results in terms of a dimensionless rotating frame frequency κ, defined by

$$\kappa = \frac{\sigma_R}{\Omega} = \frac{\sigma_{\rm I} + m\Omega}{\Omega}. \tag{3}$$

The gravitational wave frequency itself, in cycles per unit time, is

$$f_{\rm mode} = \frac{\sigma_{\rm I}}{2\pi}. \tag{4}$$

This implies

$$\frac{f_{\rm mode}}{f_{\rm star}} = \kappa - m \tag{5}$$

For r-modes, in the slow rotation limit, the inertial frame frequency is given by

$$\sigma_{\rm I}^{\rm Newt,\, slow} = -\frac{(m-1)(m+2)}{m+1}\Omega \qquad = -\frac{4}{3}\Omega \quad \text{for} \quad m = 2 \tag{6}$$

The corresponding rotating frame mode frequency is

$$\sigma_{\rm R}^{\rm Newt,\, slow} = \frac{2}{m+1}\Omega \qquad = \frac{2}{3}\Omega \quad \text{for} \quad m = 2 \tag{7}$$

In terms of the κ parameter this becomes

$$\kappa^{\rm Newt,\, slow} \equiv \kappa_0 = \frac{2}{m+1} \qquad = \frac{2}{3} \quad \text{for} \quad m = 2. \tag{8}$$

According to Greenspan (1990), page 52, the inertial modes have mode frequencies, as measured in the rotating frame, of $-2\Omega < \sigma_R < 2\Omega$, which translates into inertial frame frequencies in the interval $-4\Omega < \sigma_{\rm I} < 0$. (This result holds for an incompressible fluid in a container; I am not sure if the result extends to a self-gravitating compressible fluid; certainly, all the results in Lockitch & Friedman (1999) paper described below are consistent with this). Physically, this means the mode propagates backwards with respect to the inertial frame, but this sign is of no relevance to the gravitational wave search frequency, which only depends upon $|f_{\rm mode}|$.

3 Estimating numbers of templates

If we model the GW signal as a Taylor series:

$$\Phi_{\rm GW} = 2\pi \int_0^{T_{\rm obs}} f + f_1 t + \frac{1}{2!} f_2 t^2 + \ldots \frac{1}{n!} f_n t^n + \ldots, \tag{9}$$

then the resolution on the parameterf_n, corresponding to a phase error of one cycle, is given by:

$$\delta f_n = \frac{(n+1)!}{T_{\rm obs}^{n+1}}. \tag{10}$$

Parameterising in terms of a one year observation the resolution in f and its first two time derivatives are:

$$\delta f \;=\; 3.16 \times 10^{-8}\,{\rm Hz} \left(\frac{1\,{\rm yr}}{T_{\rm obs}}\right), \tag{11}$$

$$\delta f_1 \;=\; 2.00 \times 10^{-15}\,{\rm Hz/s} \left(\frac{1\,{\rm yr}}{T_{\rm obs}}\right)^2, \tag{12}$$

$$\delta f_2 \;=\; 1.90 \times 10^{-22}\,{\rm Hz/s}^2 \left(\frac{1\,{\rm yr}}{T_{\rm obs}}\right)^3. \tag{13}$$

The number of templates required for parameter f_n is then

$$N_{f_n} = \frac{\Delta f_n}{\delta f_n}, \tag{14}$$

where Δf_n is the interval in f_n to be searched over. Once we have estimates of the uncertainty in the r-mode frequency and its time derivatives we will use these formulae to estimates template numbers required for a search.

4 Stellar parameters

The uncertainties in the equation of state and in the mass of any particular star are crucial, as it is these uncertainties that lead to our uncertainty in the gravitational wave frequency. Typically, in the papers cited below, the frequency shifts depend upon some combination of mass M and radius R. In fact, the compactness M/R and average density $\bar{\rho}_0$ will be needed. We need to identify extreme values for the combinations in question (i.e. for M/R

and M/R^3). In particular, we can make use of constraints on the mass-radius $M(R)$ relationship that come from simple theoretical arguments, as described in Lattimer & Prakash (2007).

The most useful limit concerns the stellar compactness M/R. As described in Lattimer & Prakash (2007), there is an upper bound on this quantity that follows from insisting the equation of state is causal:

$$\frac{M}{R} \lesssim 0.35 \tag{15}$$

Parameterising:

$$\frac{M}{M_\odot} \lesssim 2.33 \left(\frac{R}{10^6\,\mathrm{cm}} \right) \tag{16}$$

The region in the M verses R plane excluded by this constraint is labelled as 'causality' in Figure 2 of Lattimer & Prakash (2007).

Some use can also be made of the spin frequency of the fastest observed pulsar, PSR J1748-2446ad, which has a spin frequency $f_{\mathrm{J1748}} = 716$ Hz. This spin frequency can be related to this pulsar's mass and radius by equation (12) of Lattimer & Prakash (2007):

$$\frac{716\,\mathrm{Hz}}{f_{\mathrm{J1748}}} \gtrsim 0.69 \left(\frac{M_\odot}{M_{\mathrm{J1748}}} \right)^{1/2} \left(\frac{R_{\mathrm{J1748}}}{10^6\,\mathrm{cm}} \right)^{3/2} \tag{17}$$

In terms of the minimum mass:

$$\frac{M_{\mathrm{J1748}}}{M_\odot} \gtrsim 0.472 \left(\frac{R_{\mathrm{J1748}}}{10^6\,\mathrm{cm}} \right)^3 \left(\frac{f_{\mathrm{J1748}}}{716\,\mathrm{Hz}} \right)^2 \tag{18}$$

or, in terms of minimum average density $\bar{\rho}_0$:

$$\bar{\rho}_{\mathrm{J1748}} \gtrsim 2.25 \times 10^{14}\,\mathrm{g\,cm^{-3}} \left(\frac{f_{\mathrm{J1748}}}{716\,\mathrm{Hz}} \right)^2 \tag{19}$$

These inequalities should be applied only to PSR J1748-2446ad; other more slowly spinning stars would lead to weaker lower bounds. However, if we make the assumption that *all* pulsars are sufficiently dense that they would, if given the angular momentum, be capable of spinning this fast, we can extend this bound to all stars:

$$\bar{\rho} \gtrsim 2.25 \times 10^{14}\,\mathrm{g\,cm^{-3}} \tag{20}$$

Making this assumption, the region excluded by this constraint in the M verses R plane is labelled as 'rotation' in Figure 2 of Lattimer & Prakash (2007). This is quite a weak bound, as this is less than nuclear density $\rho_{\rm nuc} = 2.8 \times 10^{14}$ g cm^{-3}; such a low density star would be all crust, no neutron fluid!

We are therefore restricting all neutron stars to lie in the grey shaded area of Figure 2 of Lattimer & Prakash (2007), bounded above by the causality bound on the compactness, and bounded below by the minimum density of the fastest observed pulsar. We have an upper limit on M/R, and a lower limit on $\bar{\rho}_0$. However, we still need a lower limit on M/R and an upper limit on $\bar{\rho}_0$.

To obtain these bounds we can also exploit the fact that neutron star masses are likely to all be greater than some minimum value $M_{\rm min}$. Perusal of Figure 1 of Lattimer & Prakash (2010) suggest taking the following value:

$$M_{\rm min} = 1 M_\odot \tag{21}$$

The assumed value for $M_{\rm min}$ is pretty conservative, as nature probably doesn't supply a means of making lighter neutron stars by this, as less massive stellar cores probably wouldn't collapse.

To obtain a lower bound on the compactness M/R, we can combine equation (18) with our guess as to the minimum allowed neutron star mass to give

$$\frac{M}{R} > 0.115 \left(\frac{M_{\rm min}}{1 M_\odot} \right)^{1/2} \left(\frac{f_{\rm fastest}}{716\,{\rm Hz}} \right)^{2/3} \tag{22}$$

Combining with equation (15) we therefore have

$$0.115 \left(\frac{M_{\rm min}}{1 M_\odot} \right)^{1/2} \left(\frac{f_{\rm fastest}}{716\,{\rm Hz}} \right)^{2/3} < \frac{M}{R} < 0.35 \tag{23}$$

To obtain an upper bound on the average density $\bar{\rho}_0$, we can combine the causality bound on M/R of equation (15) with our guess as to the minimum allowed neutron star mass to give

$$\bar{\rho}_0 \lesssim M_{\rm min} \left[\frac{4}{3}\pi \left(\frac{G M_{\rm min}}{0.35 c^2} \right)^3 \right]^{-1} \tag{24}$$

We therefore obtain the bounds

$$2.25 \times 10^{14}\,{\rm g\,cm^{-3}} \left(\frac{f_{\rm J1748}}{716\,{\rm Hz}} \right) \lesssim \frac{\bar{\rho}_0}{{\rm g\,cm^{-3}}} \lesssim 6.29 \times 10^{15} \left(\frac{1 M_\odot}{M_{\rm min}} \right)^2 \tag{25}$$

Quantity	Crab	J0537-6019	J1748-2446ad
$f_{\rm star}$/ Hz	30.2	62.0	716.4
$\dot{f}_{\rm star}$ / Hz s^{-1}	-3.86×10^{-10}	-1.99×10^{-10}	$< 3 \times 10^{-13}$
$\ddot{f}_{\rm star}$ / Hz s^{-2}	1.24×10^{-20}	6.1×10^{-21}	

Table 1: Useful pulsar data

Suppressing factors of $M_{\rm min}$ and $f_{\rm J1748}$, we therefore have

$$0.115 \lesssim \frac{M}{R} \lesssim 0.35 \tag{26}$$

$$2.25 \times 10^{14}\,{\rm g\,cm^{-3}} \lesssim \frac{\bar{\rho}_0}{\rm g\,cm^{-3}} \lesssim 6.29 \times 10^{15} \tag{27}$$

For the record, the spin frequency data for a few interesting pulsars is given in Table 1.

5 Inclusion of various physical factors

5.1 The effects of General relativity

A key reference is Lockitch et al. (2003), who look at relativistic barotropes within the slow rotation approximation. The relevant dimensionless number which parameterises the strength of relativistic effects is the compactness parameter defined above:

$$\frac{M}{R} = 0.21 \left(\frac{M}{1.4\,M_\odot}\right) \left(\frac{10\,{\rm km}}{R}\right). \tag{28}$$

Their key results are shown in their Figures 1 and 2. They find that relativity *increases* the inertial frame mode frequency, and that for small M/R, the change in the dimensionless mode frequency $f_{\rm mode}/f_{\rm star}$ is linear in M/R, but non-linear effects are important for $M/R \gtrsim 0.1$. The authors don't give results for realistic equations of state, but do look at different polytropic indices n. They find that the effect on the mode frequency of varying M/R at fixed n is much bigger than the effect of varying n at fixed M/R, so the compactness parameter really is the key factor at play. For this reason, we will use the right hand panel of their Figure 1 to extract the value of the frequency shift as a function of M/R, even though this is valid only for $n = 0$ (incompressible stars).

The frequency shift is given in Lockitch et al. (2003) in terms of the quantity κ/κ_0, with κ and κ_0 defined in equations (3) and (8). The connection with the mode frequency is

$$\frac{f_{\rm mode}}{f_{\rm star}} = -\frac{4}{3}\left[1 + \frac{1}{2}\left(1 - \frac{\kappa}{\kappa_0}\right)\right] \tag{29}$$

Using the right hand panel of Figure 1 of Lockitch et al. (2003) we can read-off the values of κ/κ_0 corresponding to our extremal values of M/R as given by equation (26):

$$\frac{M}{R} = 0.115 \Rightarrow \frac{\kappa}{\kappa_0} \approx 0.92 \tag{30}$$

$$\frac{M}{R} = 0.35 \Rightarrow \frac{\kappa}{\kappa_0} \approx 0.62 \tag{31}$$

Inserting these into equation (29) gives

$$\frac{4}{3}(1 + 0.04) < \left|\frac{f_{\rm mode}}{f_{\rm star}}\right| < \frac{4}{3}(1 + 0.19) \tag{32}$$

This shows that relativistic effects increase the r-mode GW frequency, probably by somewhere between 4% and 19%. For the Crab, this corresponds to

$$42.3\,{\rm Hz} < |f_{\rm mode}| < 48.4\,{\rm Hz} \tag{33}$$

For J0537-6910:

$$86.0\,{\rm Hz} < |f_{\rm mode}| < 98.4\,{\rm Hz} \tag{34}$$

In addition to the mode frequency, Lockitch et al revisit the question of the gravitational wave signal produced by the various inertial modes. They find that the perturbed metric component h_2 is very small outside of the star for all but the r-modes, again indicating that the r-modes are the most efficient gravitational wave emitters. They also find that the $l = m = 2$ r-mode has a shorter CFS growth time than any other inertial mode. Both results show that the $l = m = 2$ r-mode remains an interesting mode to look for in gravitational wave searches, even when relativistic effects are taken into account.

5.2 The effects of the crust

The presence of the crust is very important for mode damping, as was shown by Bildsten & Ushomirsky (2000) and Levin & Ushomirsky (2001). The crust can have an affect on the mode frequency too. Basically, adding a crust changes things as now in addition to Coriolis restoring forces acting throughout the star, there are also elastic restoring forces in the crust. A useful dimensionless number to parameterise the important of the crust for r-modes is the ratio of the elastic sounds speed to the rotational equatorial speed:

$$\frac{v_{\rm elastic}}{v_{\rm rotation}} \sim \frac{1}{\Omega R}\sqrt{\frac{\mu}{\rho_{\rm crust}}} \tag{35}$$

where μ denotes the shear modulus of the crust. Parameterising

$$\frac{v_{\rm elastic}}{v_{\rm rotation}} \sim 0.159 \left(\frac{100\,{\rm Hz}}{f_{\rm star}}\right) \left(\frac{\mu}{10^{30}\,{\rm erg\,cm^{-3}}}\right)^{1/2} \left(\frac{10^{14}\,{\rm g\,cm^{-3}}}{\rho_{\rm crust}}\right)^{1/2} \tag{36}$$

This shows that the crust is something to worry about.

Information on how this influences the r-mode frequency can be extracted from Figure 1 of Levin & Ushomirsky (2001). The basic picture is as follows. For sufficiently slow rotation rates, the mode frequency is close to the standard $\kappa = 2/3$ result, with the fluid core but not the solid crust participating in the motion. For sufficiently high rotation rates the mode frequency is again close to $\kappa = 2/3$, but now the whole star, crust plus core, participates in the motion. For intermediate spin rates, where the dimensionless ratio given above is of order unity, there is an avoided crossing, which means that whatever passes as the 'r-mode' is really a hybrid inertial–elastic mode. From Figure 1 of Levin & Ushomirsky (2001) is seems that the departure from the $\kappa = 2/3$ result is significant (i.e. more than a few percent and can be discerned by eye) over the spin frequency interval

$$0.05 \lesssim \frac{\Omega}{\Omega_{\rm K}} \lesssim 0.1 \tag{37}$$

The Keplerian spin rate is not well constrained. Assuming $f_{\rm K} = \Omega_{\rm K}/(2\pi) =$ 1000 Hz, this corresponds to the spin interval

$$50\,{\rm Hz} \lesssim f_{\rm star} \lesssim 100\,{\rm Hz} \tag{38}$$

while for $f_{\rm K} = 2000$ Hz the interval becomes

$$100\,{\rm Hz} \lesssim f_{\rm star} \lesssim 200\,{\rm Hz} \tag{39}$$

These are clearly spin frequencies of potential interest, so we cannot rule out the possibility of the mode frequency being affected by the avoided crossing. Looking at the right hand panel of Figure 1 of Levin & Ushomirsky (2001), we see that departures from $\kappa = 2/3$ of $\sim \pm 20\%$ are possible:

$$1 - 0.2 \lesssim \kappa \lesssim 1 + 0.2 \tag{40}$$

Converting to the mode frequency using equation (5) gives a 10% uncertainty:

$$\frac{4}{3}(1 - 0.1) \lesssim \left| \frac{f_{\rm mode}}{f_{\rm star}} \right| \lesssim \frac{4}{3}(1 + 0.1) \tag{41}$$

This is smaller than the likely GR shift of section 5.1, but is double-sided, i.e. the mode frequency might be shifted up or down.

5.3 The effects of rapid rotation

A key reference is Lindblom et al. (1999). The relevant dimensionless number that parameterises the importance of rotation is

$$\frac{\Omega^2}{\pi G \bar{\rho}_0} = 2.82 \times 10^{-7} \left(\frac{f_{\rm star}}{\rm Hz} \right)^2 \left(\frac{R}{10^6\,{\rm cm}} \right)^3 \left(\frac{1.4 M_\odot}{M} \right), \tag{42}$$

where $\bar{\rho}_0$ is the average density of the non-rotating member of the stellar sequence. The numerical prefactor is equal to 2.6×10^{-4} when the frequency is set equal to the Crab rotation rate of 30.2 Hz.

The authors find that the leading order correction to the dimensionless mode frequency is quadratic in Ω, and so they write the dimensionless rotating frame mode frequency $\kappa = \sigma_{\rm R}/\Omega$ as

$$\kappa = \kappa_0 + \kappa_2 \frac{\Omega^2}{\pi G \bar{\rho}_0}, \tag{43}$$

where $\kappa_0 = 2/(m+1)$ and κ_2 parameterises the equation-of-state-dependent shift in frequency caused by the rotational corrections. Converting to the mode frequency as measured in the inertial frame:

$$|f_{\rm mode}| = \frac{4}{3} f_{\rm star} \left[1 - \frac{3\kappa_2}{4} \left(\frac{\Omega^2}{\pi G \bar{\rho}} \right) \right]. \tag{44}$$

They find positive values for κ_2, i.e. the effects of rapid rotation serve to *decrease* the inertial frame mode frequency; their results are given in their Table 1 for polytropes, and in their Figure 1 for realistic equations of state. For a $1.4 M_\odot$ star, their Figure 1 shows that κ_2 is only weakly equation of state dependent, varying over the interval 0.26 to 0.32 for the equations of state they examine, with $\kappa_2 = 0.29$ being a typical value. However, according to equation (43), the magnitude of the frequency shift depends not only upon κ_2, but also on the average density $\bar{\rho}_0$, which, for a given stellar mass, does vary significantly between different equations of state. We will therefore slightly inconsistently set $\kappa_2 = 0.29$, and make use of the upper and lower bounds on $\bar{\rho}_0$ of our equation (27) to give:

$$\frac{4}{3}\left[1 - 1.82\times 10^{-7}\left(\frac{f_{\rm star}}{1\,{\rm Hz}}\right)^2\right] \lesssim \left|\frac{f_{\rm mode}}{f_{\rm star}}\right| \lesssim \frac{4}{3}\left[1 - 6.51\times 10^{-9}\left(\frac{f_{\rm star}}{1\,{\rm Hz}}\right)^2\right] \tag{45}$$

For the Crab this becomes

$$\frac{4}{3}\left[1 - 1.64\times 10^{-4}\right] \lesssim \left|\frac{f_{\rm mode}}{f_{\rm star}}\right| \lesssim \frac{4}{3}\left[1 - 5.94\times 10^{-6}\right] \tag{46}$$

i.e. the effects of rotation on the mode frequency are negligible.

For the fastest observed pulsar, PSR J1748-2446ad:

$$\frac{4}{3}\left[1 - 0.093\right] \lesssim \left|\frac{f_{\rm mode}}{f_{\rm star}}\right| \lesssim \frac{4}{3}\left[1 - 3.3\times 10^{-3}\right] \tag{47}$$

which translates into

$$866\,{\rm Hz} \lesssim |f_{\rm mode}| \lesssim 952\,{\rm Hz} \tag{48}$$

Comparing with equation (32), we see that the effects of rotation on the mode frequency are likely to be smaller than those of General Relativity for young pulsars, but could be comparable for millisecond pulsars.

5.4 The effects of stratification

A key reference is Yoshida & Lee (2000), who consider the effect of the perturbations obeying a slight different equation of state from the background star. This introduces buoyancy forces that modify the mode frequencies.

The relevant dimensionless number that parameterises the importance of stratification is γ, defined by

$$\gamma = \Gamma^{-1} - (1 + 1/n)^{-1}, \tag{49}$$

where Γ is the adiabatic exponent governing the perturbations and n the polytropic index of the background star.

They find that while the majority of the inertial modes are significantly affected by stratification, the nodeless $l = m$ r-modes are relatively unaffected; see Fig. 4 of their paper. The figure shows that for $|\gamma| < 0.1$ at least, the effects of stratification are much less important that those of General Relativity.

The affect of stratification on inertial modes was also investigated by Passamonti et al. (2009), who also found the the r-mode frequency was affected only very slightly. This was shown to be true even for very rapidly rotating stars; see Figure 12 of Passamonti et al. (2009).

5.5 The effects of the magnetic field

Magnetic fields will alter the r-mode frequency slightly (Morsink & Rezania, 2002). A useful number that parameterised this affect is the ratio of the star's magnetic energy to its rotational energy:

$$\frac{E_{\rm magnetic}}{E_{\rm rotation}} \sim \frac{B^2 R^3}{I\Omega^2} \sim 2.5 \times 10^{-5} \left(\frac{B}{10^{12}\,{\rm G}}\right)^2 \left(\frac{1\,{\rm Hz}}{f_{\rm star}}\right)^2 \tag{50}$$

Clearly, this is very small for all stars aside from magnetars, and magnetars are of no interest as r-mode sources. So, magnetic corrections are unimportant. Physically this corresponds to the magnetic restoring forces being small compared the Coriolis restoring forces.

This is consistent with more detailed calculations, e.g. the numerical simulations of Lander et al. (2010). From their Figure 6 it can be seen that the avoided crossing between the $l = m = 2$ Alfvén mode and the $l = m = 2$ r-mode occurs for $\Omega/\Omega_{\rm K} \sim 0.1$, for a magnetic field $B \sim 3 \times 10^{16}$ G. Scaling down to $B \sim 3 \times 10^{12}$ G, the avoided crossing will occur for $\Omega/\Omega_{\rm K} \sim 10^{-5}$, which implies a stellar spin frequency of $f_{\rm star} \sim 0.01$ Hz. This is consistent with magnetic restoring forces being unimportant for r-modes in all stars of GW interest.

5.6 The effects of a possible superfluid component

Need to think about this.

6 Putting it all together: search parameters

The $l = m = 2$ r-mode seems to be a good mode to look for in a gravitational wave search: it has a shorter CFS instability growth time than other inertial modes, and is an efficient gravitational wave emitter. Another nice feature is that for a Newtonian single fluid slowly-rotating star, the mode frequency is simply related to the spin frequency: $|f_{\rm mode}| = 4f_{\rm star}/3$.

However, we have found that more realistic models, incorporating more physics alters this frequency relationship. The single largest effect came from including General Relativity, as described in section 5.1. Writing the fractional correction as $\epsilon_{\rm GR}$:

$$|f_{\rm mode}| = \frac{4}{3}f_{\rm star}(1 + \epsilon_{\rm GR}) \quad (51)$$

where the uncertainty in stellar compactness translated into an uncertainty in $\epsilon_{\rm GR}$:

$$0.04 \lesssim \epsilon_{\rm GR} \lesssim 0.19 \quad (52)$$

Differentiating with respect to time, we obtain the corresponding variation in $|\dot{f}_{\rm mode}|$, the mode frequency derivative. The parameter $\epsilon_{\rm GR}$ will be (nearly) time-independent, so we have

$$|\dot{f}_{\rm mode}| = \frac{4}{3}|\dot{f}_{\rm star}|(1 + \epsilon_{\rm GR}) \quad (53)$$

i.e. the fractional shift in mode frequency derivative is the same as the fractional shift in mode frequency. If GR corrections were the only ones we had to worry about, we would therefore search over a whole range of $\epsilon_{\rm GR}$ values, but for each mode frequency there would be a single frequency derivative.

The other potentially important contributions come from the crust and from rapid rotation. The effects of the crust can be large, comparable to those of GR, but only over a certain range of spin frequencies. Let us therefore next consider rotation. In section (5.3) we found

$$|f_{\rm mode}| = \frac{4}{3}f_{\rm star}(1 + \epsilon_{\rm rot}) \quad (54)$$

where the rotational correction is given by

$$\epsilon_{\rm rot} = -\frac{3\pi\kappa_2 f_{\rm star}^2}{G\bar{\rho}_0} \tag{55}$$

Inserting $\kappa = 0.29$ and (to maximise this effect) the lower bound on density of equation (27) gives

$$\epsilon_{\rm rot} = -1.82 \times 10^{-7} \left(\frac{\kappa_2}{0.29}\right) \left(\frac{f_{\rm star}}{1\,{\rm Hz}}\right)^2 \left(\frac{2.25 \times 10^{14}\,{\rm g\,cm^{-3}}}{\bar{\rho}_0}\right) \tag{56}$$

Clearly, this is a small effect in all but the most rapidly spinning stars. Differentiating to obtain the mode frequency derivative, we need to be careful, as $\epsilon_{\rm rot} \sim f_{\rm star}^2$. Taking this into account we obtain

$$|\dot{f}_{\rm mode}| = \frac{4}{3}|\dot{f}_{\rm star}|(1 + 3\epsilon_{\rm rot}) \tag{57}$$

i.e. the fractional shift in frequency derivative is three times the fractional shift in mode frequency. As for the GR case above, for each mode frequency shift there is a unique frequency derivative shift.

Now consider combining the effects of GR and rotation. Assuming we can simply multiply the correction factors [**DIJ: Is this the right thing to do?**] we have

$$|f_{\rm mode}| = \frac{4}{3}f_{\rm star}(1 + \epsilon_{\rm GR})(1 + \epsilon_{\rm rot}) \tag{58}$$

We therefore have

$$|f_{\rm mode}| = \frac{4}{3}f_{\rm star}(1 + \epsilon_{\rm GR} + \epsilon_{\rm rot} + \mathcal{O}(\epsilon_{\rm GR}\epsilon_{\rm rot})) \tag{59}$$

Differentiating gives the corresponding $|\dot{f}_{\rm mode}|$. Assembling these two results in a convenient form we have:

$$|f_{\rm mode}| = \frac{4}{3}f_{\rm star}(1 + \epsilon_{\rm GR} + \epsilon_{\rm rot}) \tag{60}$$

$$|\dot{f}_{\rm mode}| = \frac{4}{3}|\dot{f}_{\rm star}|[(1 + \epsilon_{\rm GR} + \epsilon_{\rm rot}) + 2\epsilon_{\rm rot}] \tag{61}$$

where fractional corrections of order $\mathcal{O}(\epsilon_{\rm GR}\epsilon_{\rm rot})$ have been suppressed. In this case of combined GR plus rotation, the fractional change in mode frequency differs form that in mode frequency derivative, by an amount of order $\epsilon_{\rm rot}$ that isn't known a priori. Given that for all but the fastest stars $\epsilon_{\rm GR} \gg \epsilon_{\rm rot}$ we can set $\epsilon_{\rm GR} + \epsilon_{\rm rot} \approx \epsilon_{\rm GR}$. The following strategy then suggests itself. A range in $f_{\rm mode}$ is searched over, with fractional variation given by the $\epsilon_{\rm GR}$ of equation (52). For each mode frequency, the corresponding frequency derivative would then be offset by an additional amount $2\epsilon_{\rm rot}$. As is made explicit in equation (56), the value of $2\epsilon_{\rm rot}$ is proportional to $\bar{\rho}_0^{-1}$. This can take a range of values, generating a range of values in $|\dot{f}_{\rm mode}|$ to be searched over for each $|f_{\rm mode}|$.

We can then make some simple estimates of the ranges in $f_{\rm mode}$ and $\dot{f}_{\rm mode}$ to be searched over. Given equation (52) we will consider a range in mode frequencies of

$$|\Delta f_{\rm mode}| = \frac{4}{3} f_{\rm star} \times 0.2 \qquad (62)$$

Given the difference between equations (60) and (61) we will, for each mode frequency $f_{\rm mode}$, consider a range of frequency derivatives of width

$$|\Delta \dot{f}_{\rm mode}| = \frac{4}{3} f_{\rm star} \times (2\epsilon_{\rm rot}) \qquad (63)$$

and will evaluate $\epsilon_{\rm rot}$ for the lower bound in density identified in equation (27). These can then be used to estimate the corresponding number of templates to be searched over.

For the Crab:

$$\Delta f_{\rm mode} = 8.05\,{\rm Hz} \quad \Rightarrow \quad \frac{\Delta f_{\rm mode}}{\delta f} = 2.6 \times 10^8 \left(\frac{T_{\rm obs}}{1\,{\rm yr}}\right) \qquad (64)$$

$$\Delta \dot{f}_{\rm mode} = 1.7 \times 10^{-13}\,{\rm Hz\,s^{-1}} \quad \Rightarrow \quad \frac{\Delta \dot{f}_{\rm mode}}{\delta f_1} = 85 \left(\frac{T_{\rm obs}}{1\,{\rm yr}}\right)^2 \qquad (65)$$

For J0537-6910:

$$\Delta f_{\rm mode} = 16.5\,{\rm Hz} \quad \Rightarrow \quad \frac{\Delta f_{\rm mode}}{\delta f} = 5.2 \times 10^8 \left(\frac{T_{\rm obs}}{1\,{\rm yr}}\right) \qquad (66)$$

$$\Delta \dot{f}_{\rm mode} = 3.7 \times 10^{-13}\,{\rm Hz\,s^{-1}} \quad \Rightarrow \quad \frac{\Delta \dot{f}_{\rm mode}}{\delta f_1} = 190 \left(\frac{T_{\rm obs}}{1\,{\rm yr}}\right)^2 \qquad (67)$$

For J1748-2446ad:

$$\Delta f_{\rm mode} = 190\,{\rm Hz} \quad \Rightarrow \quad \frac{\Delta f_{\rm mode}}{\delta f} = 6.0 \times 10^9 \left(\frac{T_{\rm obs}}{1\,{\rm yr}}\right) \qquad (68)$$

$$\Delta \dot{f}_{\rm mode} = 7.5 \times 10^{-14}\,{\rm Hz\,s^{-1}} \quad \Rightarrow \quad \frac{\Delta \dot{f}_{\rm mode}}{\delta f_1} = 37 \left(\frac{T_{\rm obs}}{1\,{\rm yr}}\right)^2 \qquad (69)$$

Adding in the crust, again assuming we can simply multiply the relevant factors, we would have

$$|f_{\mathrm{mode}}| = \frac{4}{3} f_{\mathrm{star}} (1 + \epsilon_{\mathrm{GR}})(1 + \epsilon_{\mathrm{rot}})(1 + \epsilon_{\mathrm{crust}}) \tag{70}$$

where, according to equation (41) of section 5.2,

$$-0.1 \lesssim \epsilon_{\mathrm{crust}} \lesssim 0.1 \tag{71}$$

Again neglecting high order terms, we have

$$|f_{\mathrm{mode}}| = \frac{4}{3} f_{\mathrm{star}} (1 + \epsilon_{\mathrm{GR}} + \epsilon_{\mathrm{crust}} + \epsilon_{\mathrm{rot}}) \tag{72}$$

Combining the range in ϵ_{GR} of equation (52) and the range in $\epsilon_{\mathrm{crust}}$ of equation (71), we would want to search over a frequency range given by

$$-0.06 \lesssim \epsilon_{\mathrm{GR}} + \epsilon_{\mathrm{crust}} \lesssim 0.29 \tag{73}$$

only a factor of order unity wider than the range to be searched over away from resonance (compare equation (52)).

However, the influence of the crust on the mode frequency derivative may be much greater. Rather than differentiating equation (73), we can note the following: according to Figure 1 of Levin & Ushomirsky (2001), close to the resonance frequencies, the mode frequency verses spin frequency curve is close to flat. If it were exactly flat, then there would be no evolution in mode frequency, no matter how large the stellar spin-down rate $\dot{f}_{\mathrm{star}}$. It follows that, to look at a 'worst case' scenario (i.e. largest band to search over) we would put

$$\Delta \dot{f}_{\mathrm{mode}} = |\dot{f}_{\mathrm{mode}}| \tag{74}$$

which increases the number of $\dot{f}_{\mathrm{mode}}$ templates to be searched over:

$$\text{Crab} \qquad \frac{\Delta \dot{f}_{\mathrm{mode}}}{\delta f_1} = 1.9 \times 10^5 \left(\frac{T_{\mathrm{obs}}}{1\,\mathrm{yr}} \right)^2 \tag{75}$$

$$\text{J0537} - 6910 \qquad \frac{\Delta \dot{f}_{\mathrm{mode}}}{\delta f_1} = 1.0 \times 10^5 \left(\frac{T_{\mathrm{obs}}}{1\,\mathrm{yr}} \right)^2 \tag{76}$$

$$\text{J1748} - 2446\text{ad} \qquad \frac{\Delta \dot{f}_{\mathrm{mode}}}{\delta f_1} < 150 \left(\frac{T_{\mathrm{obs}}}{1\,\mathrm{yr}} \right)^2 \tag{77}$$

To sum up:

- If GR and rotational effects are included, template numbers are:

$$\text{Young pulsars}: \qquad \sim 10^8 \times 10^2 \sim 10^{10} \left(\frac{T_{\rm obs}}{1\,\rm yr}\right)^3 \qquad (78)$$

$$\text{MSPs}: \qquad \sim 10^9 \times 10^1 \sim 10^{10} \left(\frac{T_{\rm obs}}{1\,\rm yr}\right)^3 \qquad (79)$$

- If GR, rotation and crustal effects are included template numbers are:

$$\text{Young pulsars}: \qquad \sim 10^8 \times 10^5 \sim 10^{13} \left(\frac{T_{\rm obs}}{1\,\rm yr}\right)^3 \qquad (80)$$

$$\text{MSPs}: \qquad \sim 10^9 \times 10^2 \sim 10^{11} \left(\frac{T_{\rm obs}}{1\,\rm yr}\right)^3 \qquad (81)$$

Either way, there see enormous numbers to me ...

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