An Overview of Control Theory and Digital Signal Processing

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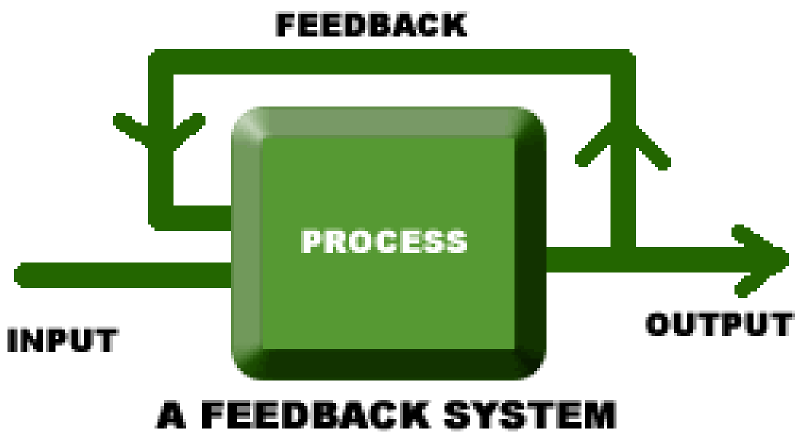
Syllabus (tentative)

Day	Topic	Textbooks
1	Control theory: Physical systems, models, linear systems, block diagrams, differential equations, feedback loops, cruise control example, MATLAB implementation.	Chau, Pao C. *Process Control: A First Course with MATLAB®.* Cambridge University Press, 2002. ISBN 0-521-00255-9. Ingle, Vinay K. and John G. Proakis. *Digital Signal Processing using Matlab®.* Brooks/Cole 2000. ISBN 0-534-37174-4. Smith, Steven W. *The Scientist and Engineer's Guide to Digital Signal Processing.* California Technical Publishing 1999. http://www.dspguide.com/
2	Control theory: Laplace transform and its inverse , transfer functions, partial fraction expansion, first-order and second-order systems, dynamic response, bode plots, stability criteria, MATLAB implementation.	
3	Control theory: robustness, typical compensators, noise suppression, one arm cavity lock example, MATLAB implementation and time-domain simulations with SIMULINK.	
4	DSP: Discrete-time signals and systems, impulse response, system stability, convolution and correlation, differential to difference equations, the Z transform, the Discrete-time Fourier Transform (DTFT), the Discrete Fourier Transform (DFT), MATLAB implementation.	
5	DSP: The Fast-Fourier Transform (FFT), power spectral density, sampling theorem, aliasing, analog-to-digital transformations, digital filtering, FIR filters, IIR filters, moving average filter, filter design, ADC and DACs, MATLAB implementation.	

Objective

Control System

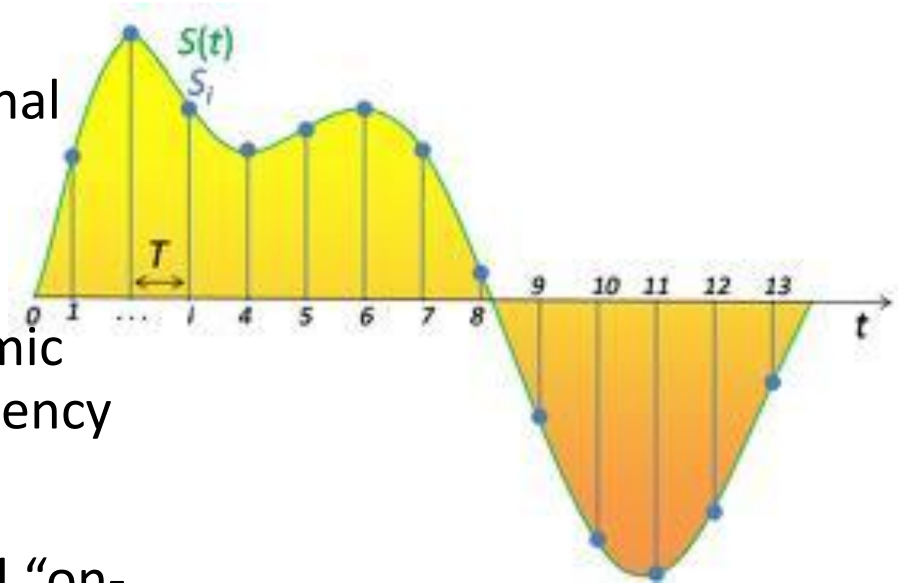
- Manages and regulates a set of variables in a system
  - SISO – single-input-single-output
  - MIMO – multiple-input-multiple-output
- A quantity is measured then controlled
- Requirements
  - Bandwidth
  - Rise time
  - Overshoot
  - Steady state error
  - …



Objective

Digital Signal Processing

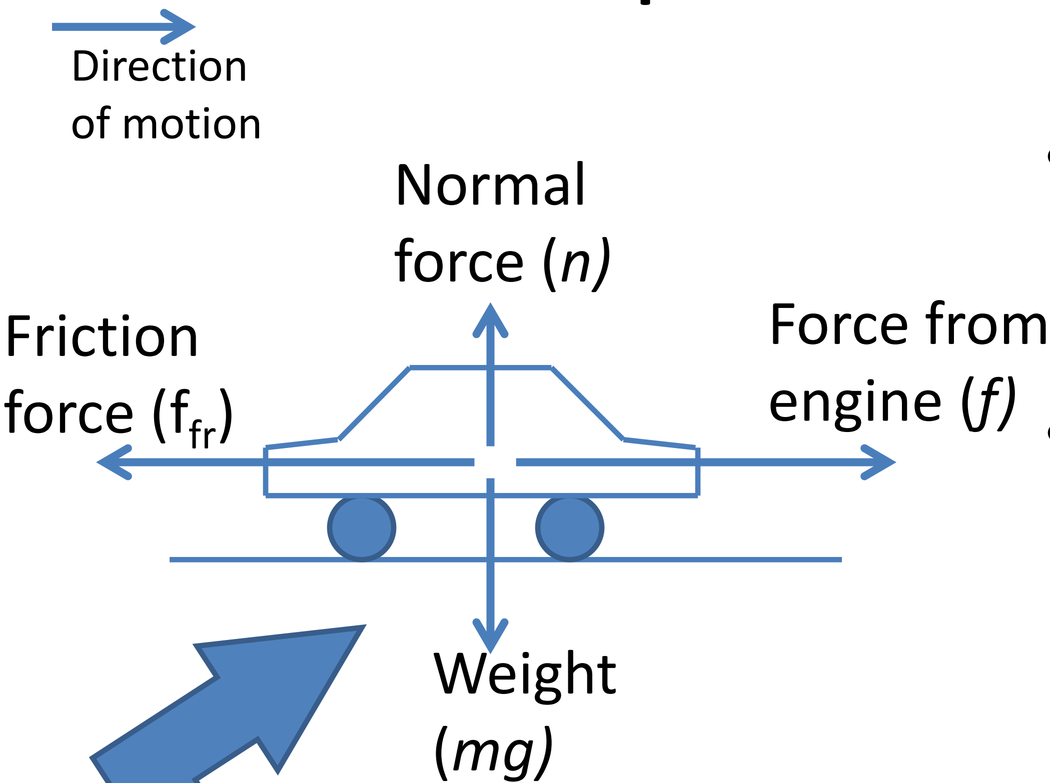
- Measure and filter an analog signal
- Digital signal
  - Created by sampling an analog signal
  - Can be stored
- Analog filters
  - Cheap, fast and have a large dynamic range in both amplitude and frequency
- Digital filters
  - Can be designed and implemented “on-the-fly”
  - Superior level of performance.
    - Example: a low pass digital filter can have a gain of 1 ± 0.0002, a frequency cutoff at 1000 Hz, and a gain of less than 0.0002 for frequencies above 1001 Hz. A transition of 1 Hz!



Control Theory 1

- Given a physical system
  - Objective: sense and control a variable in the system
- Examples
  - As basic as
    - a car's cruise control (SISO) or
  - Not so basic as
    - Locking the full LIGO interferometer (MIMO)

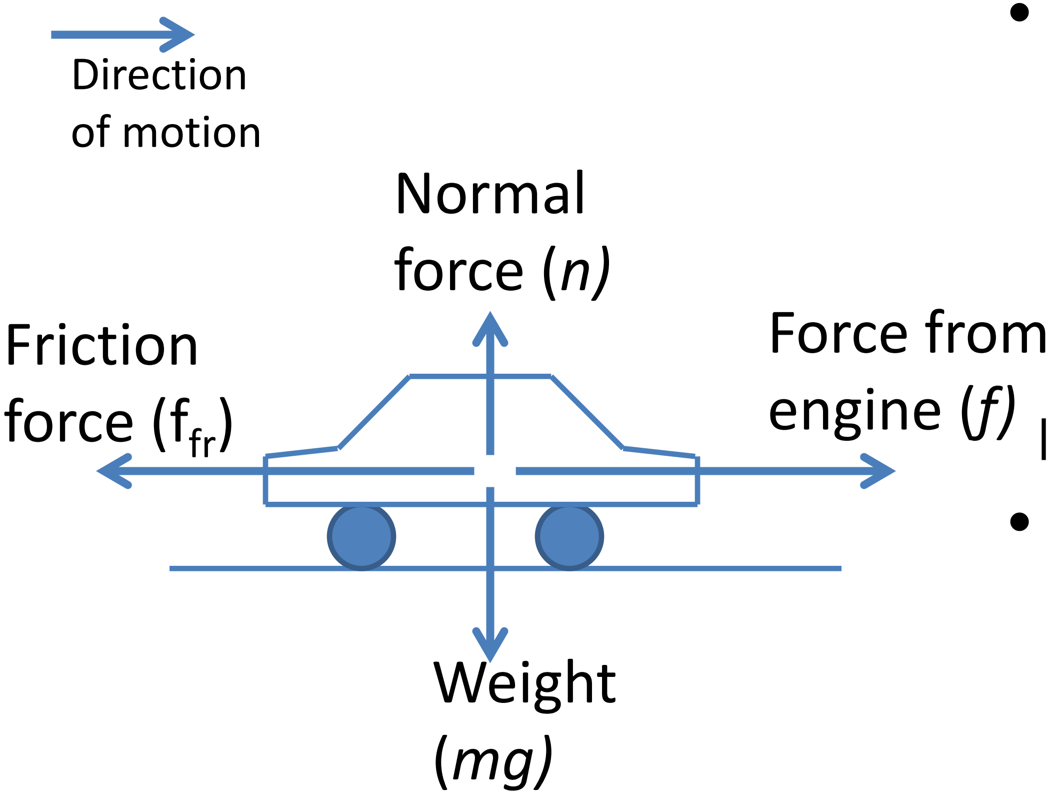
Example: Cruise Control



- Objective
  - Car needs to maintain a given speed
- Physical system includes
  - car's inertia
  - friction

Force or free body diagram: allows to analyze the forces at play

Physical model



- Physical system described by the following equation of motion

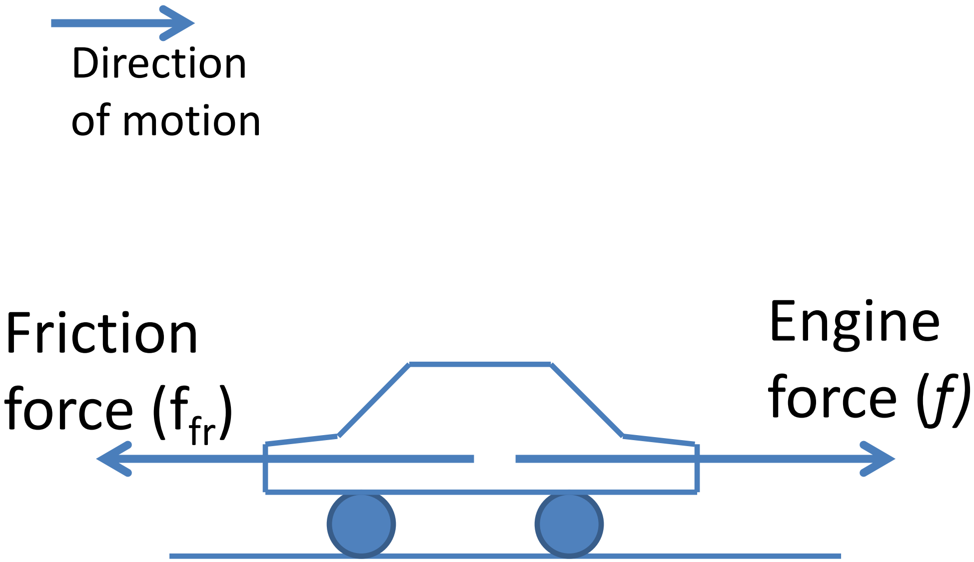
$$\sum_{\parallel \, to\, road} f = f - f_{fr} = ma$$

- Simplifying and assuming friction force f_{fr} is proportional to speed

$$f_{fr} = bv$$

$$\boxed{ma = m\frac{dv}{dt} = f - bv}$$

First-order differential equation



- Solving for first-order differential equation (assuming f is a constant)

$$m\frac{dv}{dt} = f - bv$$

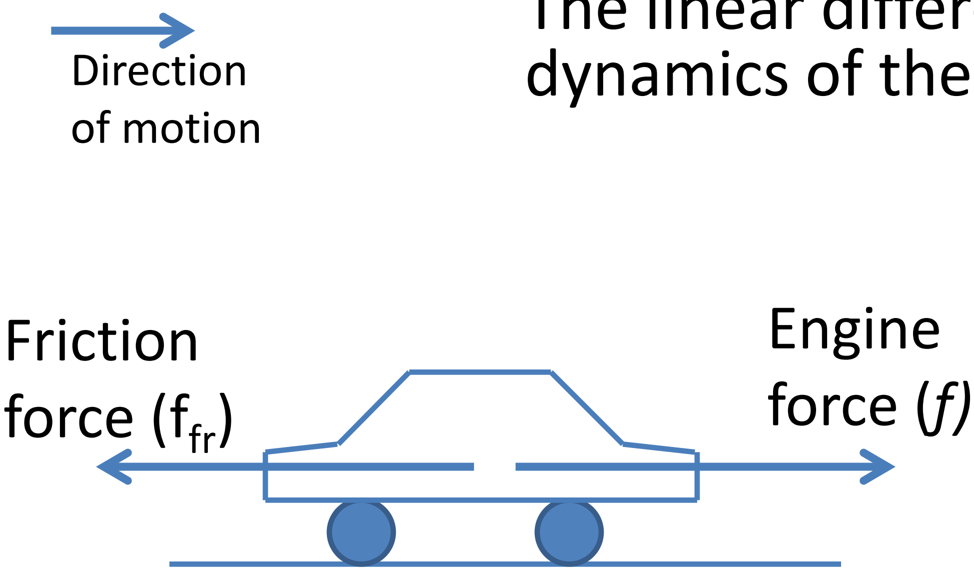
yields the solution

$$v = v_r\left(1 - e^{-t/\tau}\right)$$

Speed at regime $v_r = f/b$

Time constant $\tau = m/b$

MATLAB implementation



The linear differential equation describing the dynamics of the system

$$m\frac{dv}{dt} = f - bv$$

Using MATLAB's Symbolic Math Toolbox

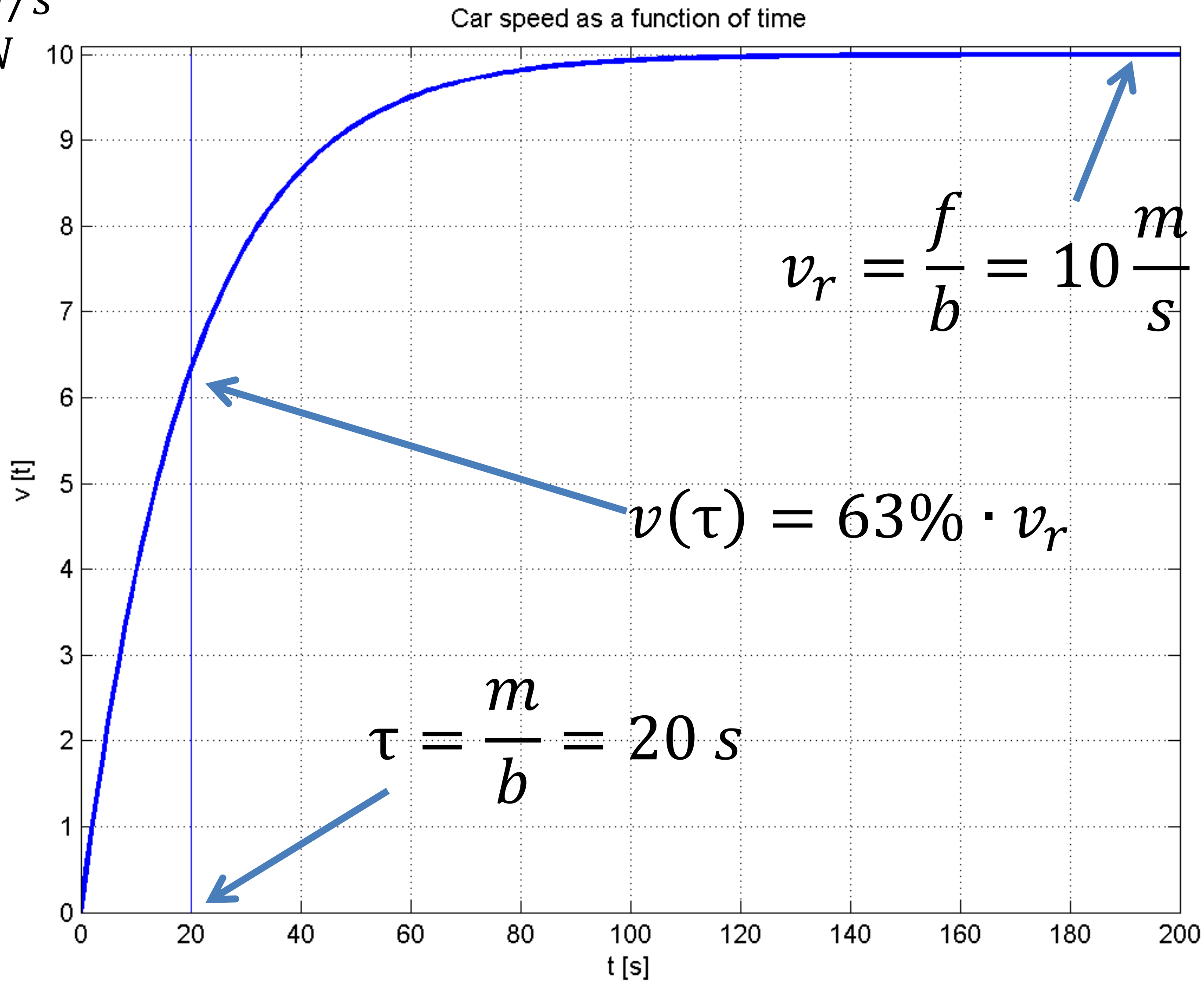
```
>> dsolve('m*Dy=f-b*y','y(0)=0')
ans =
(f - f/exp((b*t)/m))/b
```

Results: $v = v_r \cdot \left(1 - e^{-t/\tau}\right)$

$m = 1000\ kg$

$b = 50\ kg/s$

$f = 500\ N$



$$v_r = \frac{f}{b} = 10\,\frac{m}{s}$$

$$v(\tau) = 63\% \cdot v_r$$

$$\tau = \frac{m}{b} = 20\ s$$

cruise_timedomain.m

Block diagram: *representing the physical system*

- To illustrate a cause-and-effect relationship
- A single block represents a physical system
- Blocks are connected by lines
  - Lines represent how signals flow in the system
- In general, a physical system *G* has signal *x(t)* as input and signal *y(t)* as output
- *G* is the transfer function of the system

Car's body

Transfer function *G* represents the car's body

– *G* converts the force from the engine f (input signal, N) to the car's actual speed v (output signal, m/s)

$$v = G \cdot f$$

with $\mathrm{G} = \frac{1}{b}\left(1 - e^{-\frac{b}{m}t}\right)$

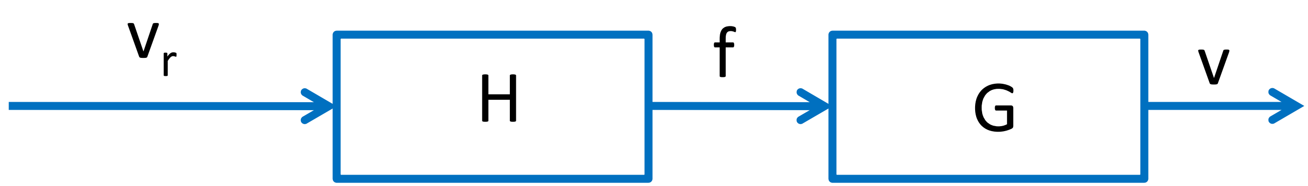
– Units: s/kg

Setting the desired speed

- Second transfer function *H* (the controller)
  - Converts the desired speed (or reference) v_r to a required force f
  - Sets the throttle
  - For simplicity, *H* is set to a constant

$$\left.\begin{array}{r} f = H \cdot v_r \\ v = G \cdot f \end{array}\right\} \rightarrow v = G \cdot H \cdot v_r$$

  - $G \cdot H$ must be dimensionless

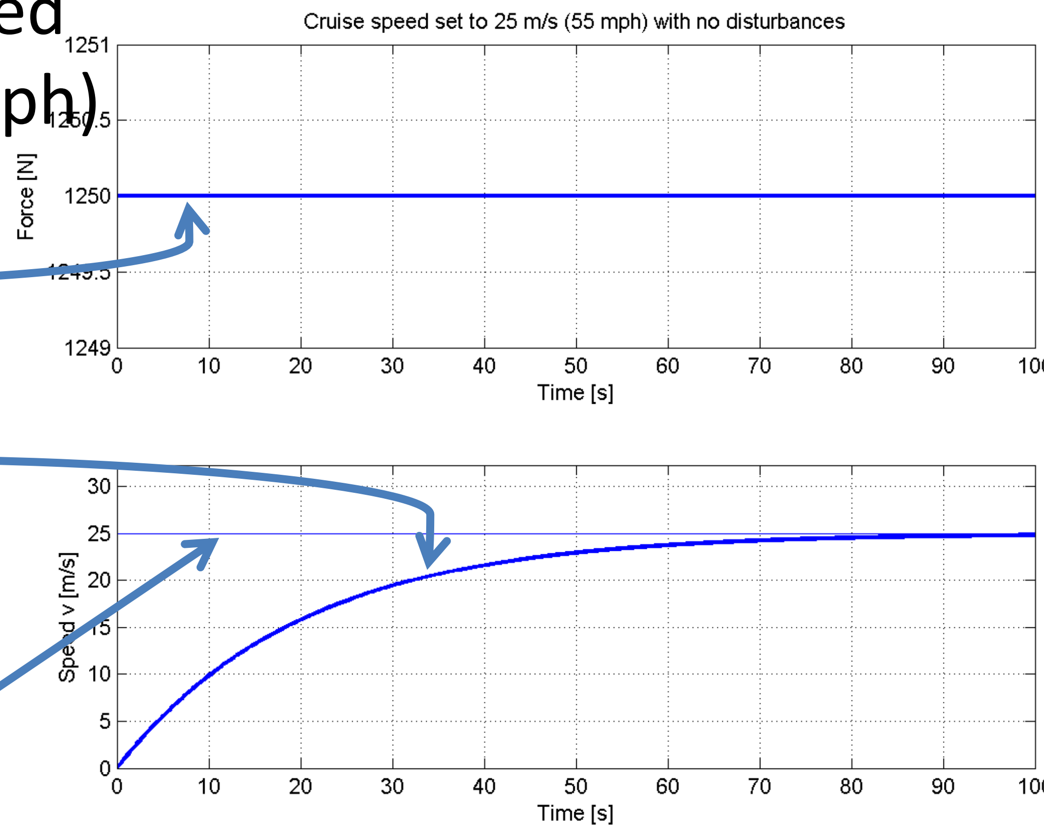


Plotting results

- With $H = b$ the actual speed is the reference: $v = v_r$
- Simulate: setting desired speed to 25 m/s (55 mph)

Generated force by controller H

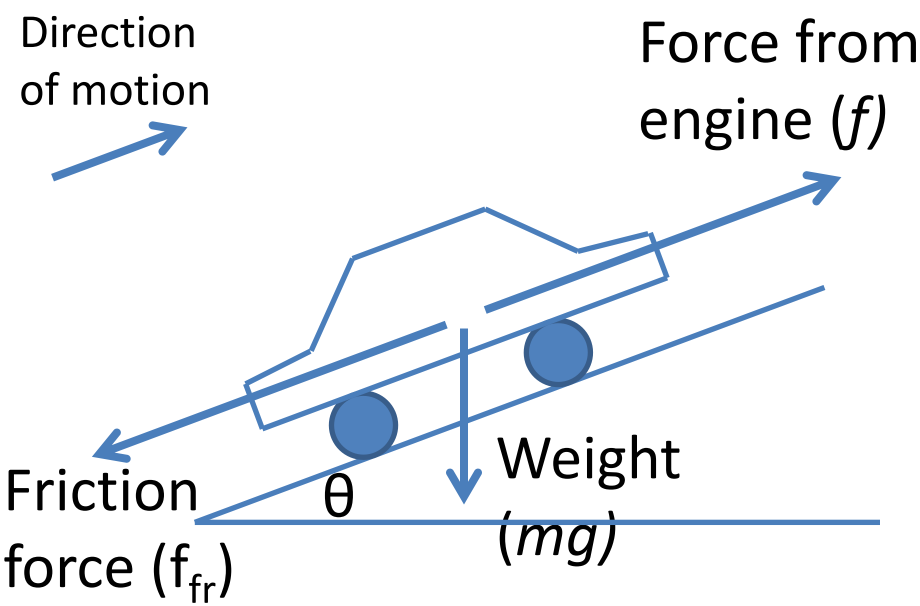
Resulting speed v



Desired speed v_r

cruisefeedback_timedomain.m

Introducing a disturbance – a hill



- In the presence of a hill the equation of motion needs to be re-visited
- Assuming a small angle θ

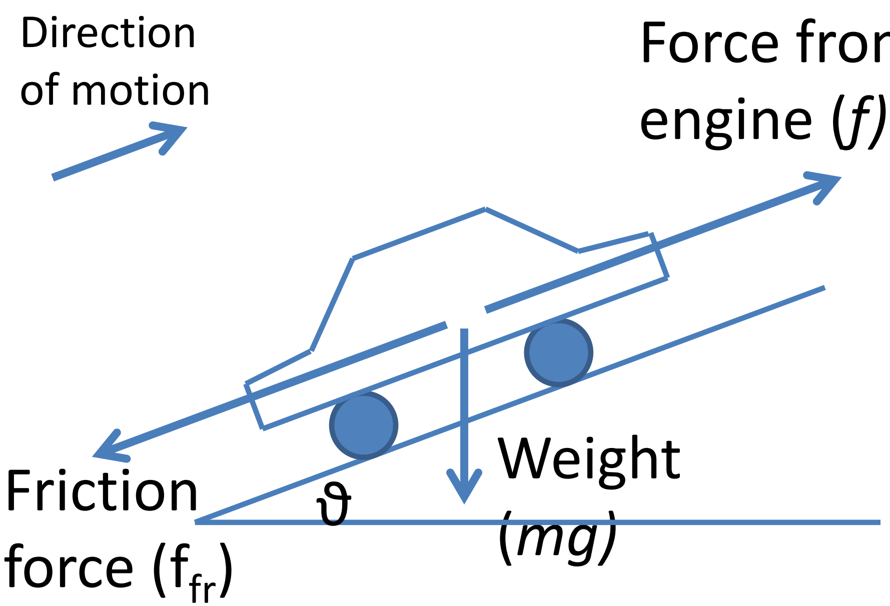
$$m\frac{dv}{dt} = f - bv - mg \cdot \theta$$

Added term

Introducing a disturbance – a hill

Assuming f and θ are constants

$$m\frac{dv}{dt} = f - bv - mg \cdot \theta$$



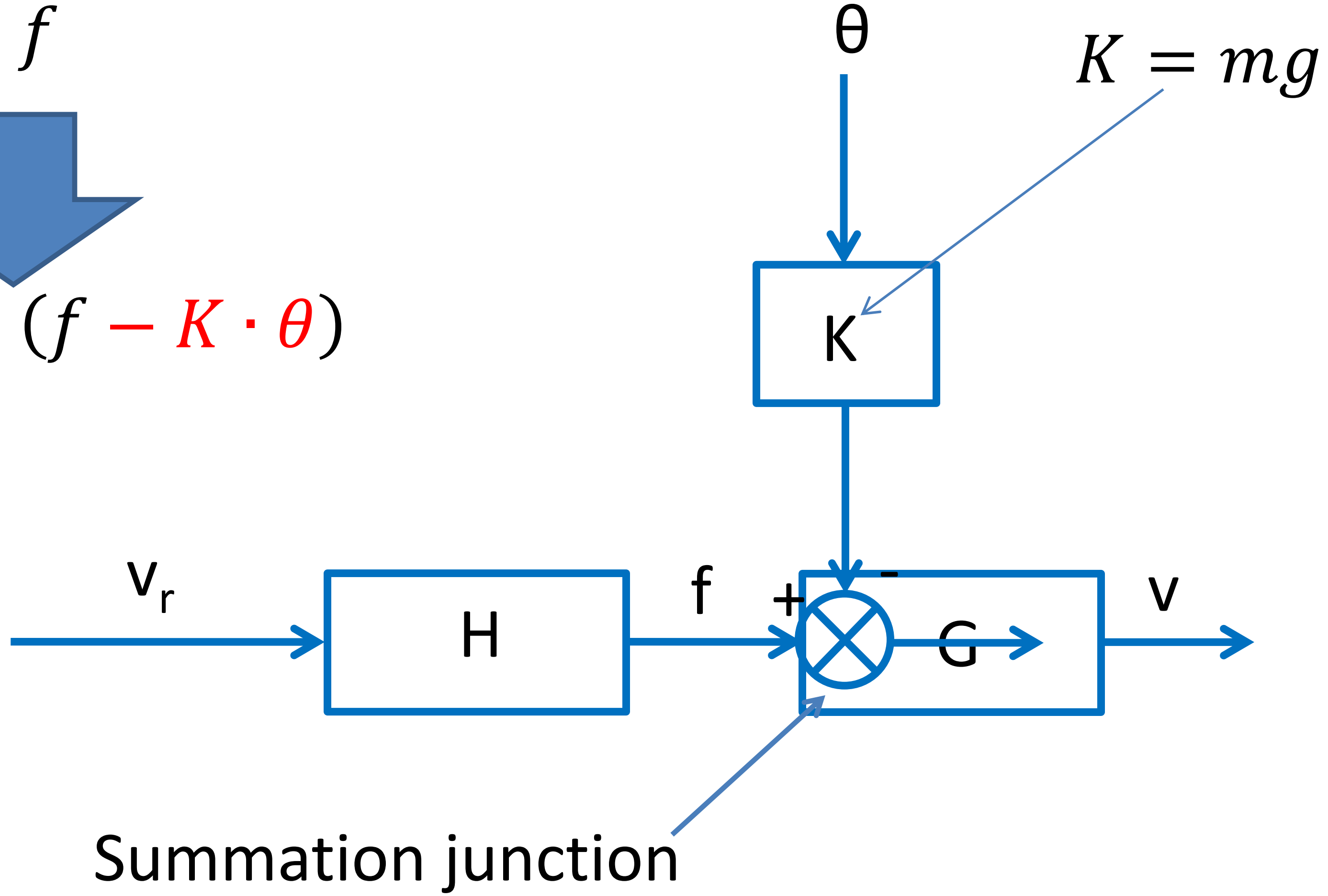
$$v = G \cdot (f - mg \cdot \theta)$$

Added term

Modifying the block diagram

$$v = G \cdot f$$

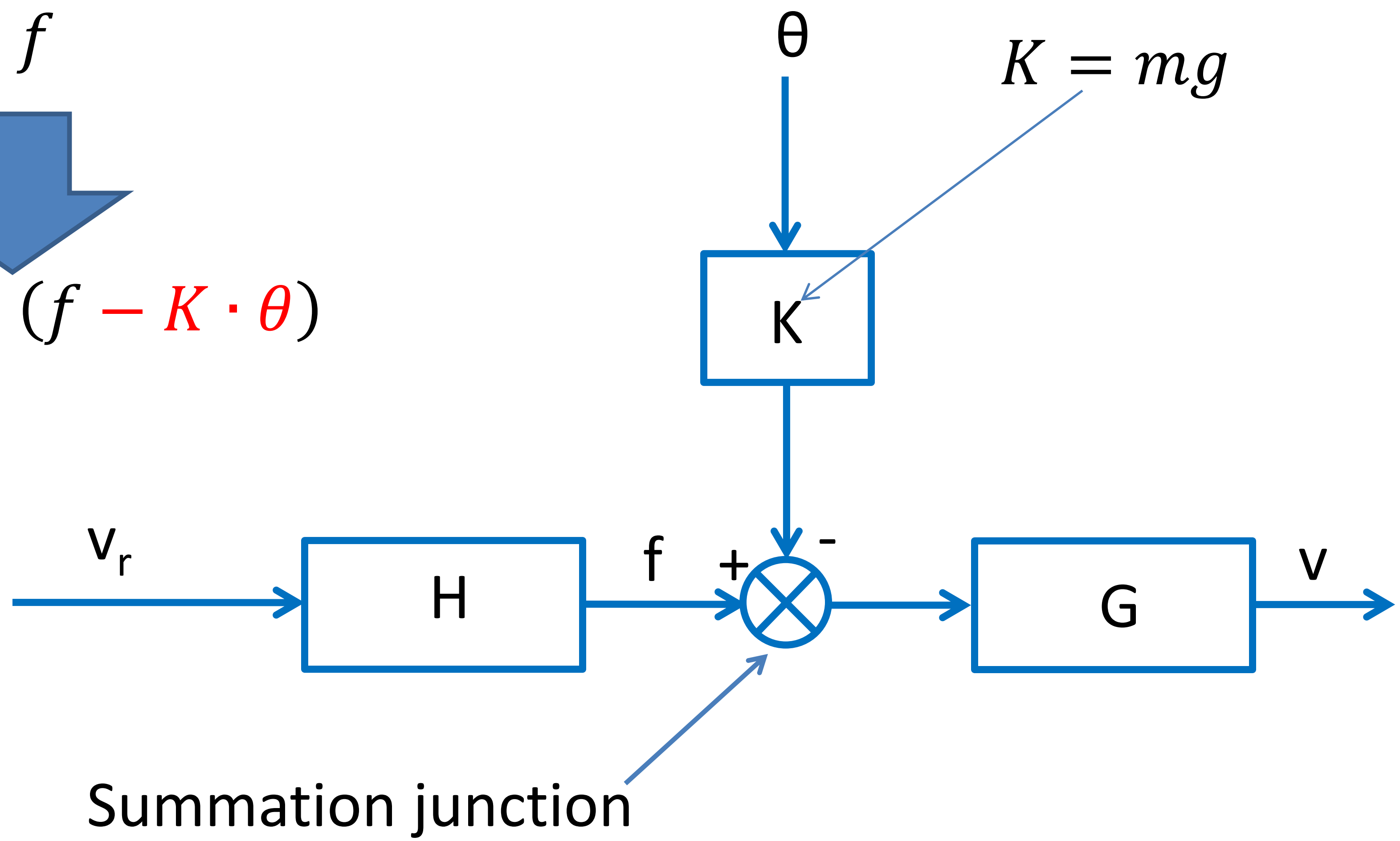
$$v = G \cdot (f - K \cdot \theta)$$



Modifying the block diagram

$$v = G \cdot f$$

$$v = G \cdot (f - K \cdot \theta)$$



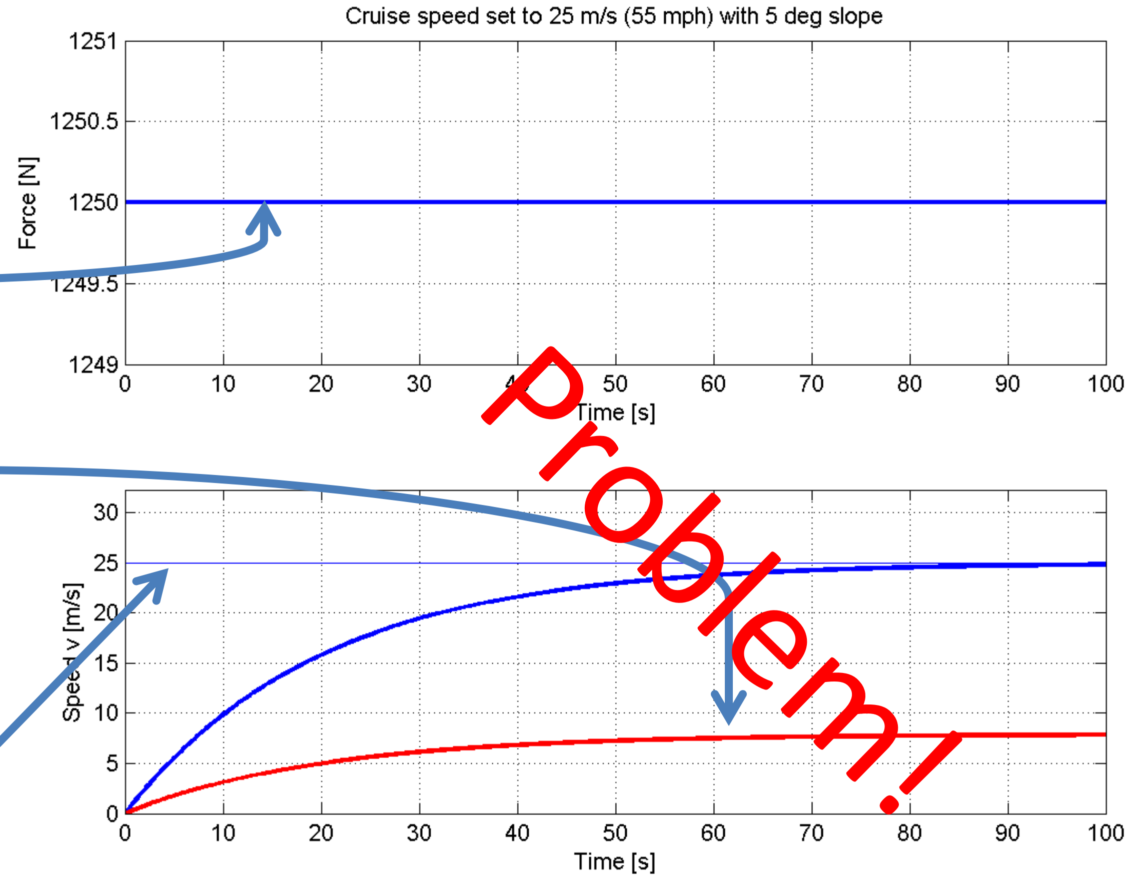
Plotting results

Setting desired speed to 25 m/s and slope of $\theta = 5°$

Generated force by controller H

Resulting speed v

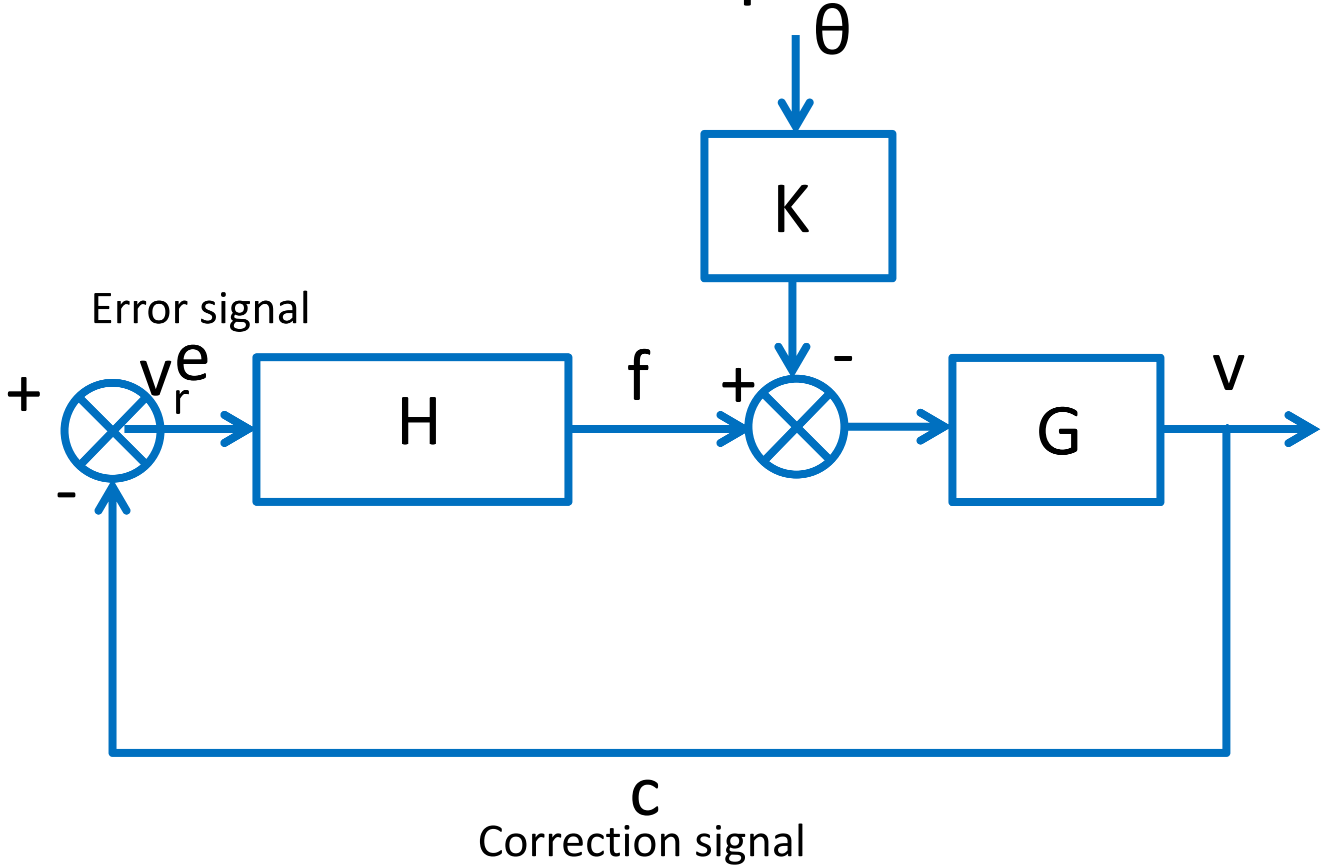
Desired speed



cruisefeedback_timedomain.m

Negative Feedback

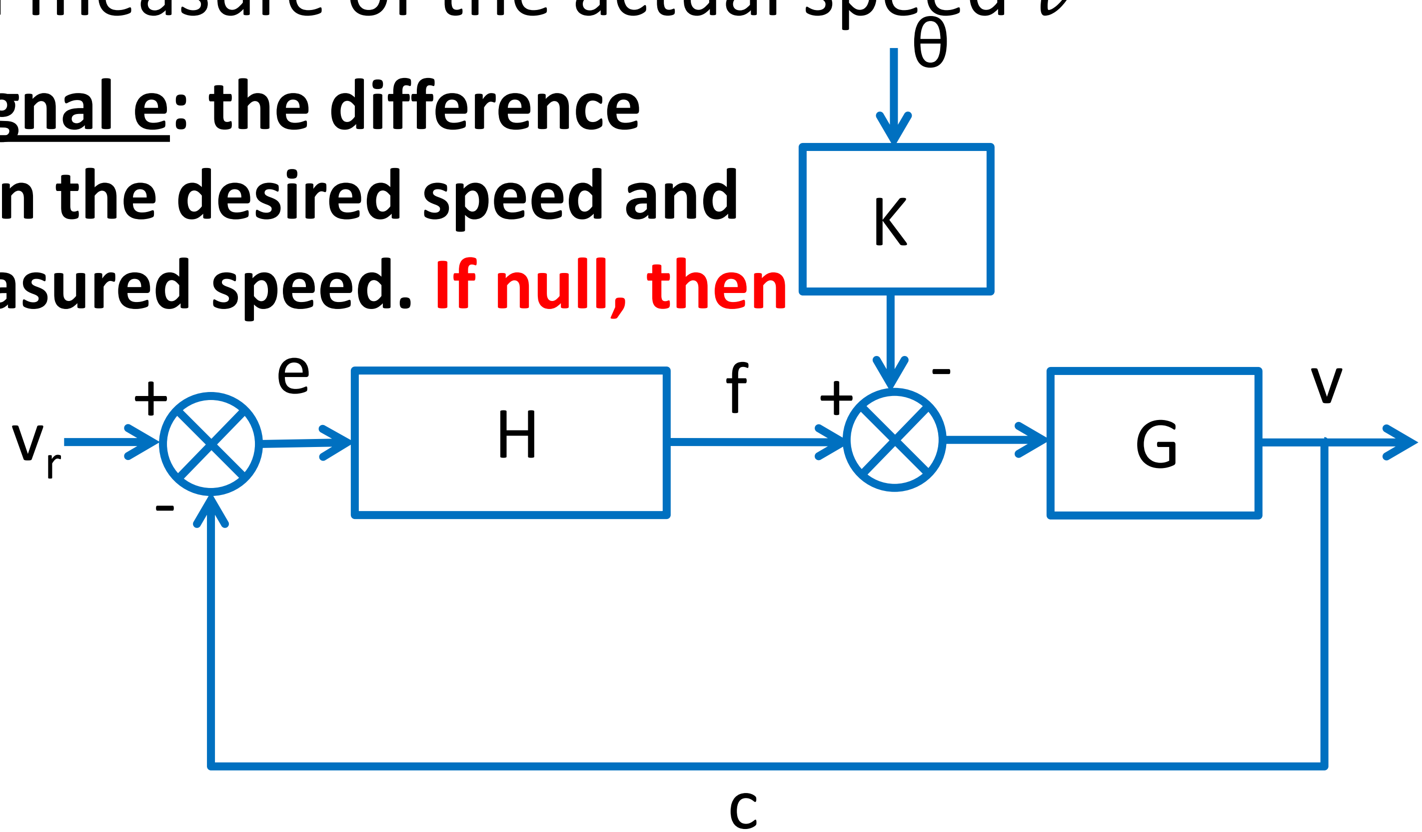
1. Let's measure the car's speed and
2. Correct for it by feeding back into the system a measure of the actual speed v



Negative Feedback

1. Let's measure the car's speed and
2. Correct for it by feeding back into the system a measure of the actual speed v

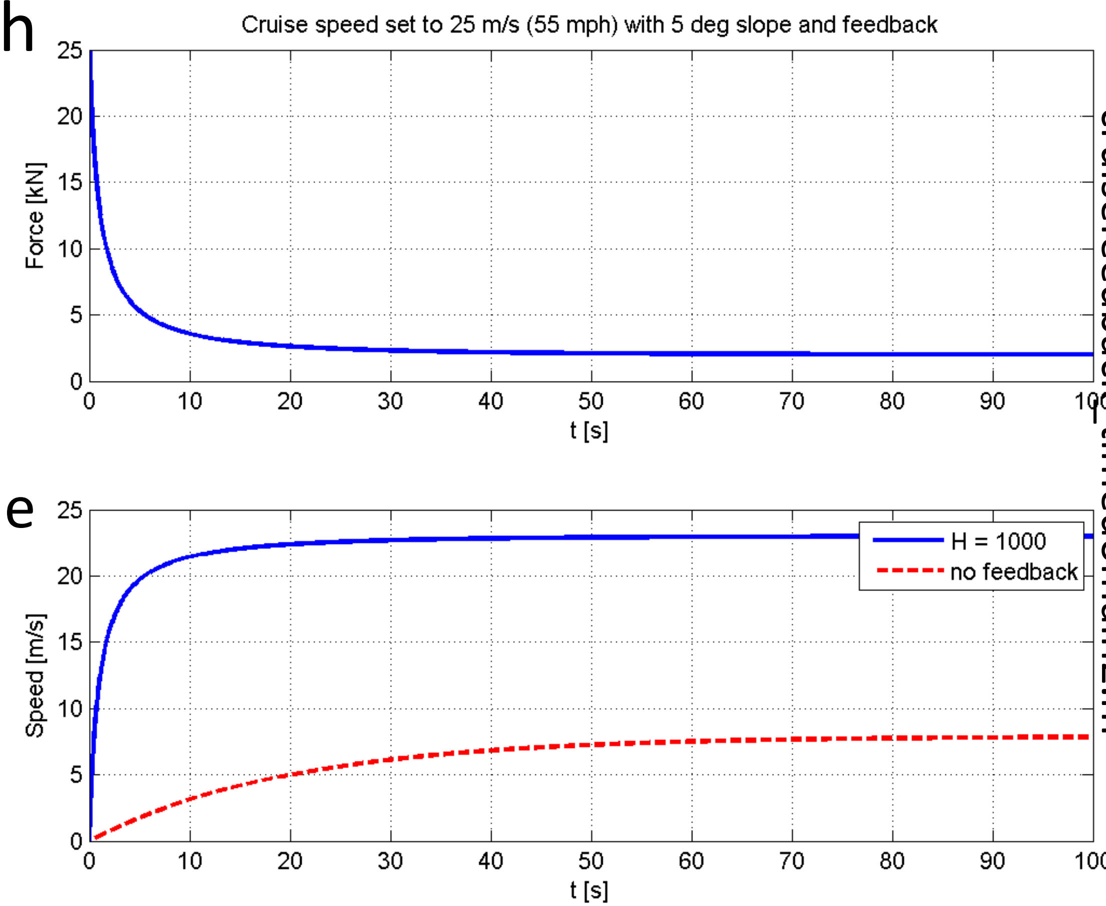
<u>Error signal e</u>: the difference between the desired speed and the measured speed. **If null, then** $\boldsymbol{v = v_r}$



<u>Correction signal c</u>: in this case it is just a measure of the actual speed

Negative feedback

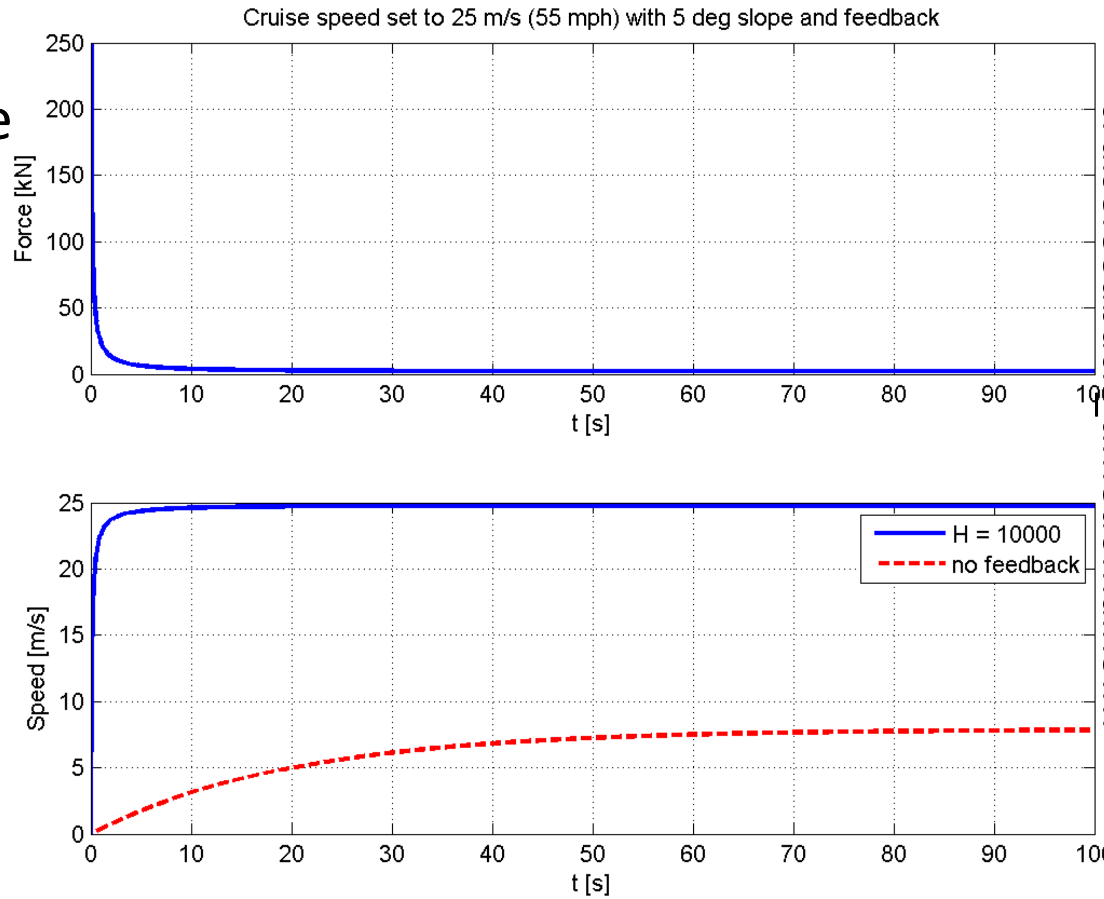
- Plot of *force* vs. *time* and *speed* vs. *time* with negative feedback
- Setting $H = 10^3 \; kg/s$
- Result:
  - Faster response with feedback (compare blue against red curves)
  - Speed at regime: $23 \; m/s$ (error of $\sim 10\%$)



cruisefeedback_timedomain2.m

Negative feedback

- Increasing the controller's gain (H)
  - decreases the rise time
  - while decreasing the steady state error
- Setting $H = 10^4 \; kg/s$
- Result:
  - Even faster response
  - Speed at regime: $24.8 \; m/s$ (error of $\sim 1\%$)



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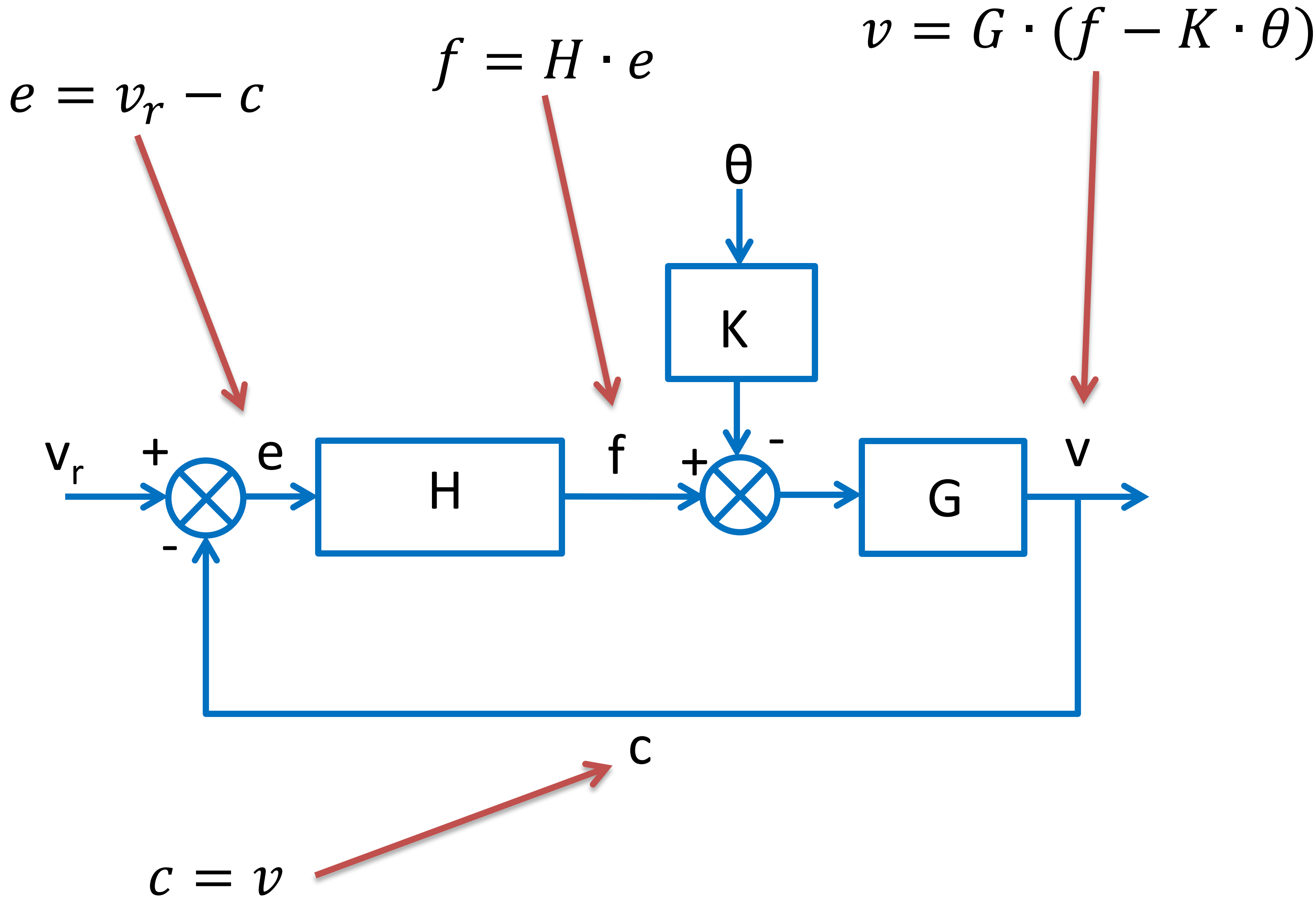
Yes but… how does it work?

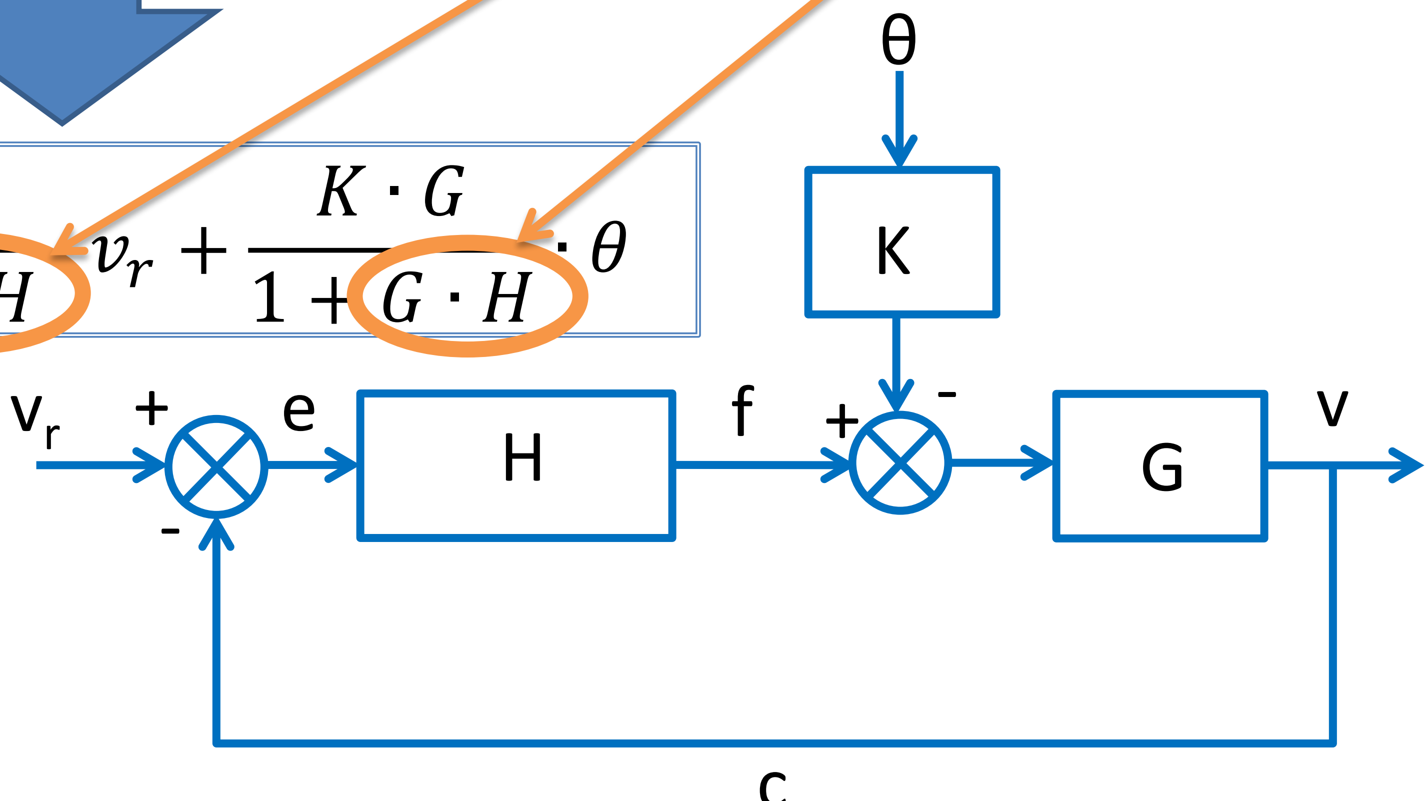
Signal flow and block diagrams



$$e = v_r - c$$
$$= v_r - v$$
$$= v_r - G \cdot (f - K \cdot \theta)$$
$$= v_r - G \cdot (H \cdot e - K \cdot \theta)$$

$$e = \frac{1}{1 + G \cdot H} v_r + \frac{K \cdot G}{1 + G \cdot H} \cdot \theta$$

System’s open loop gain (dimensionless)



$$e = \frac{1}{1 + G \cdot H} \cdot v_r + \frac{K \cdot G}{1 + G \cdot H} \cdot \theta$$

IF $G \cdot H \gg 1$ (high gain, closed loop, with feedback)

$$e \approx 0 \cdot v_r + \frac{K \cdot G}{G \cdot H} \cdot \theta \approx \frac{K}{H} \theta$$

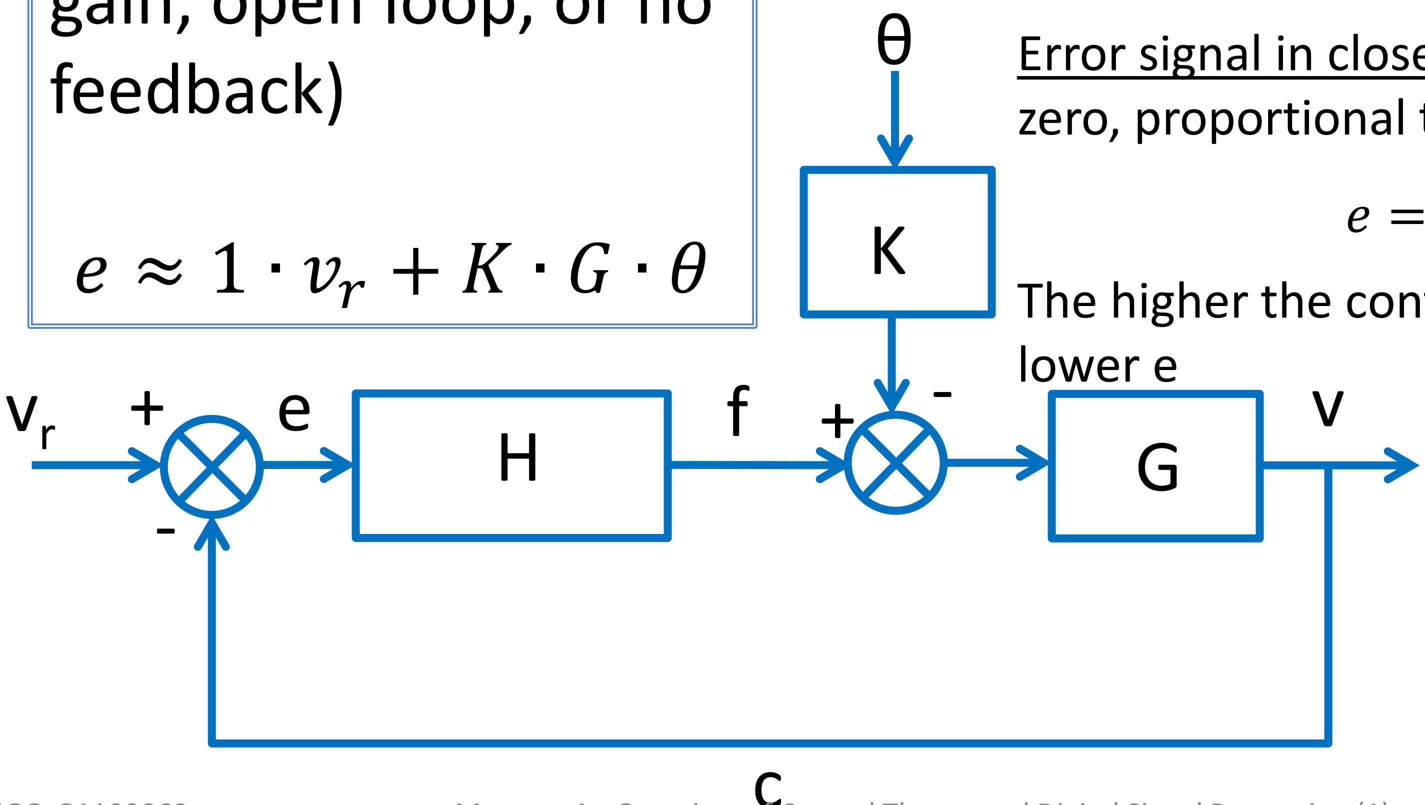
IF $G \cdot H \ll 1$(low gain, open loop, or no feedback)

$$e \approx 1 \cdot v_r + K \cdot G \cdot \theta$$

Error signal in closed loop: close to zero, proportional to angle θ

$$e = \frac{K}{H} \theta$$

The higher the controller's gain, the lower e

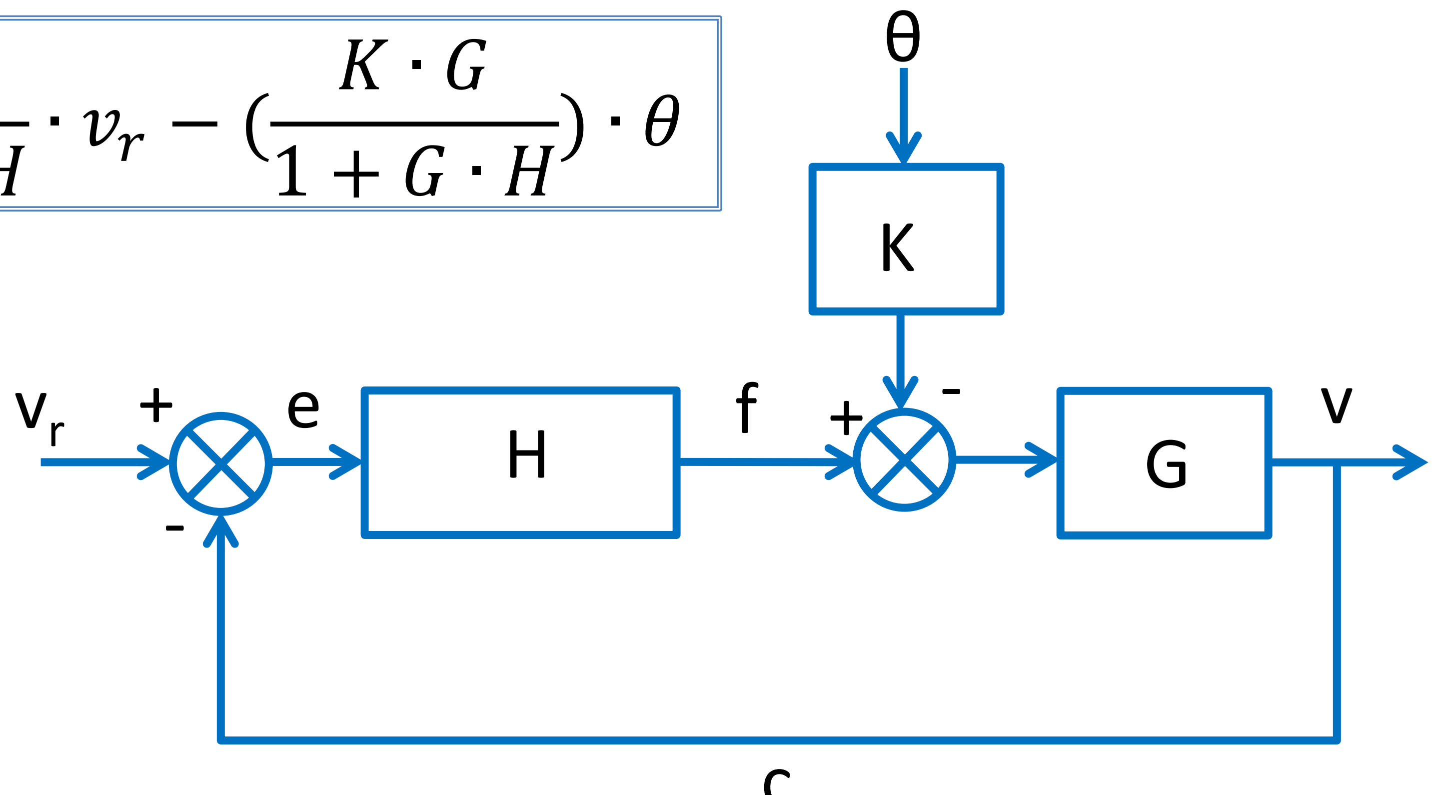


$$v = G \cdot H \cdot e - K \cdot G \cdot \theta$$
$$= G \cdot H \cdot (v_r - v) - K \cdot G \cdot \theta$$
$$= G \cdot H \cdot v_r - G \cdot H \cdot v - K \cdot G \cdot \theta$$

$$v = \frac{G \cdot H}{1 + G \cdot H} \cdot v_r - \left(\frac{K \cdot G}{1 + G \cdot H}\right) \cdot \theta$$

<u>With no feedback</u>

$$v = G \cdot H \cdot v_r - K \cdot G \cdot \theta$$



$$v = \frac{G \cdot H}{1 + G \cdot H} v_r - \frac{K \cdot G}{1 + G \cdot H} \theta$$

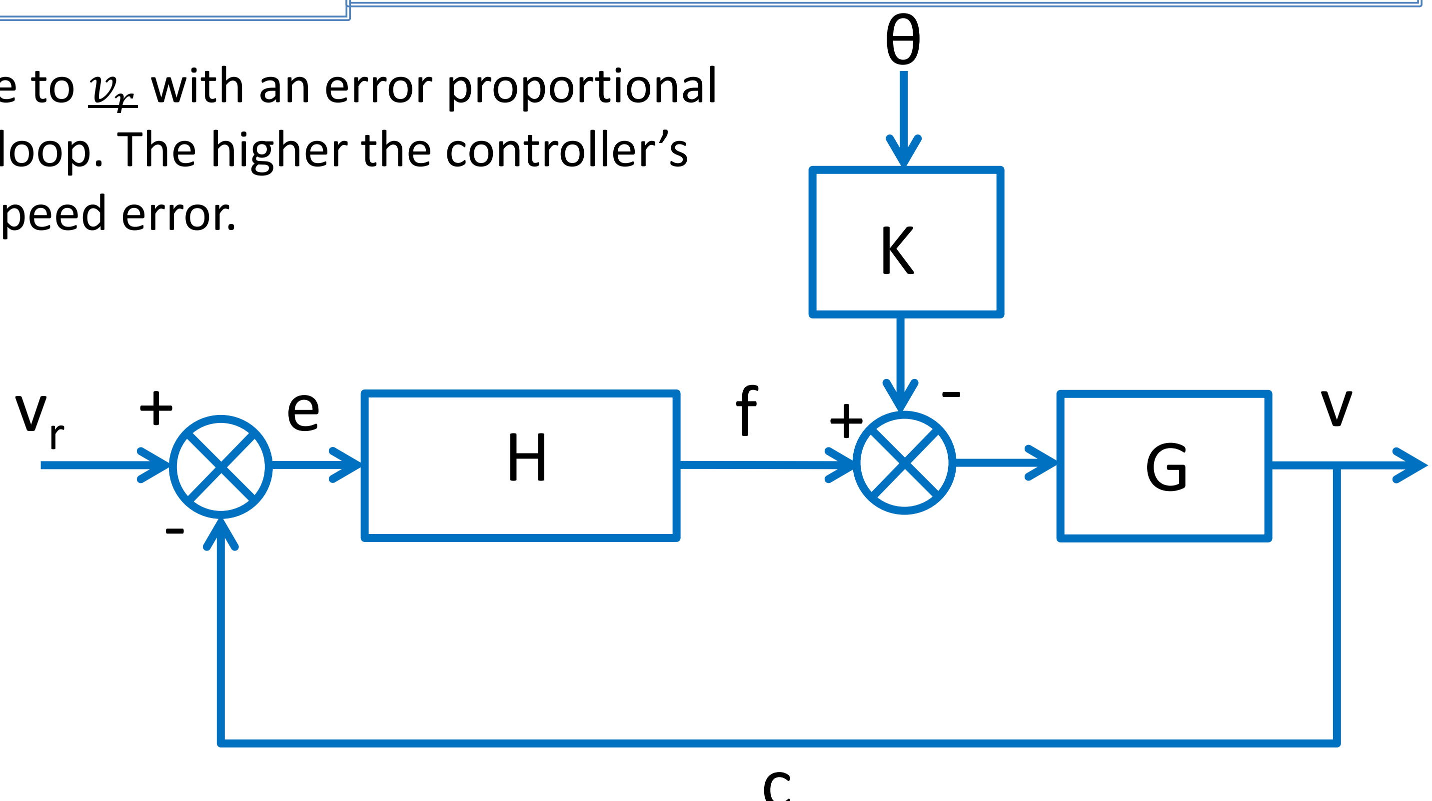
IF $G \cdot H \ll 1$ (low gain, open loop or no feedback)

$$v \approx G \cdot H \cdot v_r - K \cdot G \cdot \theta$$

IF $G \cdot H \gg 1$ (high gain, closed loop, with feedback)

$$v \approx 1 \cdot v_r - \frac{K \cdot G}{G \cdot H} \cdot \theta \approx v_r - \frac{K}{H} \theta$$

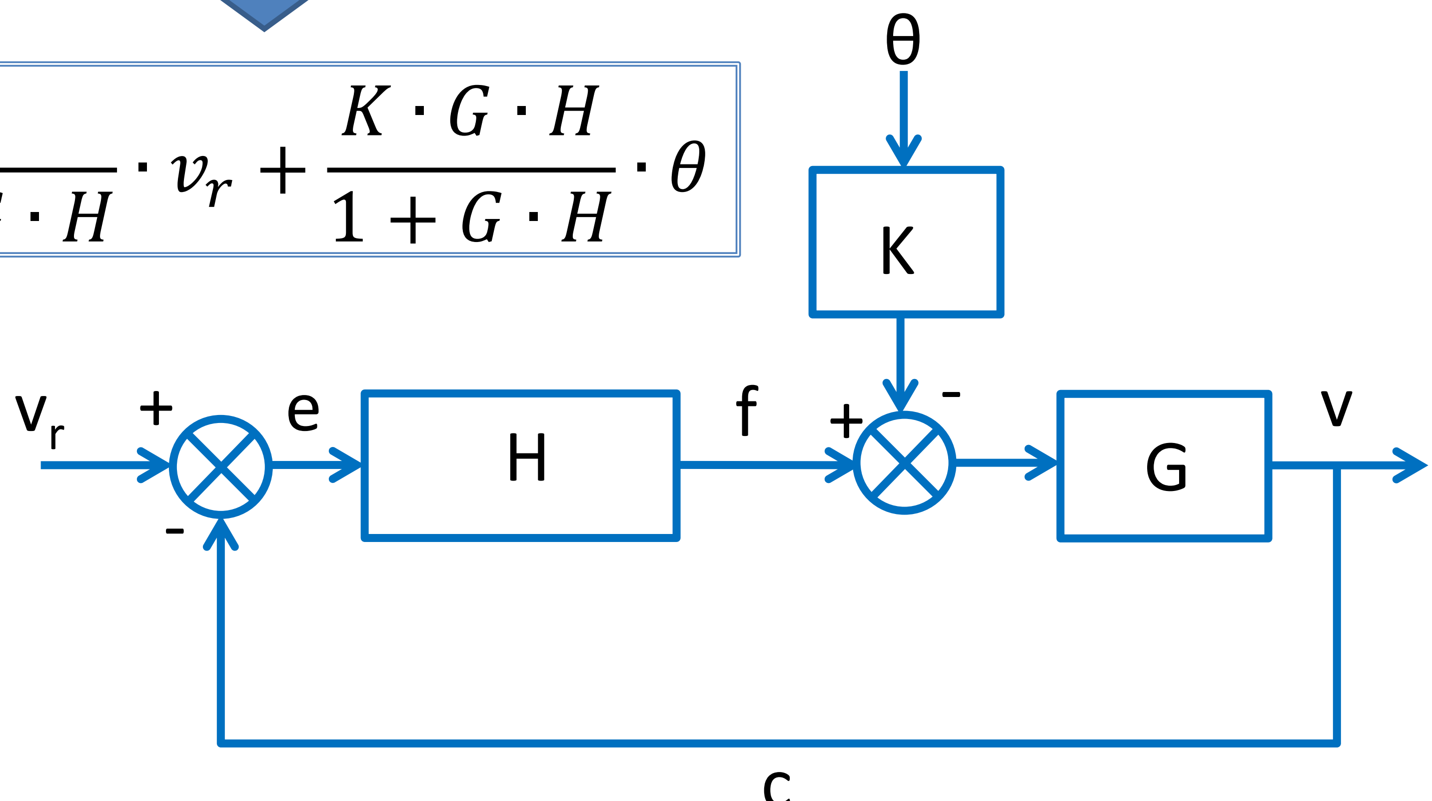
<u>Actual speed v</u>: close to $\underline{v_r}$ with an error proportional to θ when in closed loop. The higher the controller's gain, the lower the speed error.



$$f = H \cdot e$$

$$= H \cdot \left(\frac{1}{1 + G \cdot H} v_r + \frac{K \cdot G}{1 + G \cdot H} \theta\right)$$

$$f = \frac{H}{1 + G \cdot H} \cdot v_r + \frac{K \cdot G \cdot H}{1 + G \cdot H} \cdot \theta$$



$$f = \frac{H}{1 + G \cdot H} \cdot v_r + \frac{K \cdot G \cdot H}{1 + G \cdot H} \cdot \theta$$

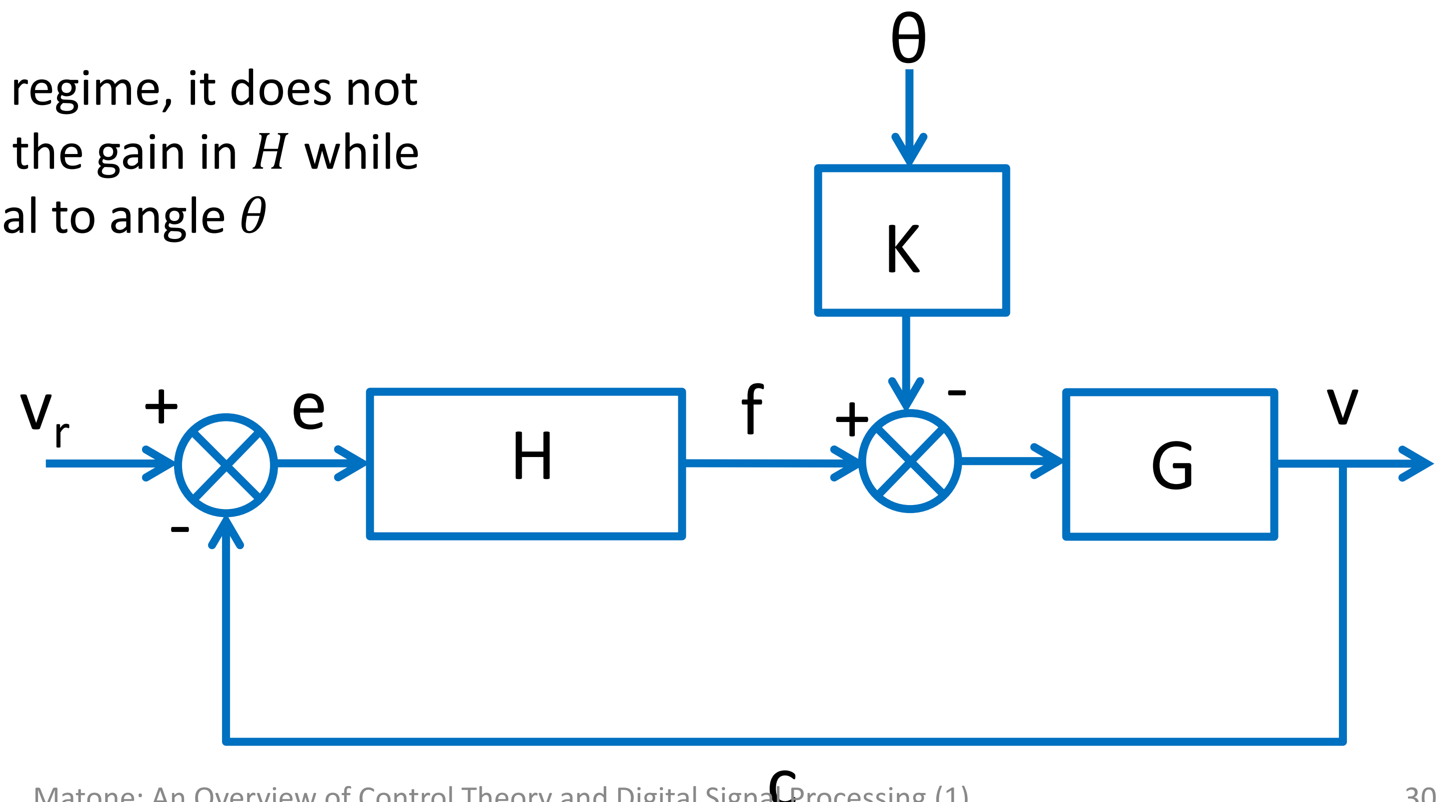
IF $G \cdot H \ll 1$ (low gain, open loop, or no feedback)

$$f \approx H \cdot v_r + K \cdot G \cdot H \cdot \theta$$

IF $G \cdot H \gg 1$ (high gain, closed loop, with feedback)

$$f \approx \frac{1}{G} v_r + K \cdot \theta$$

<ins>Force f</ins>: at regime, it does not depend on the gain in H while proportional to angle θ



Plotting GH and $GH/(1+GH)$

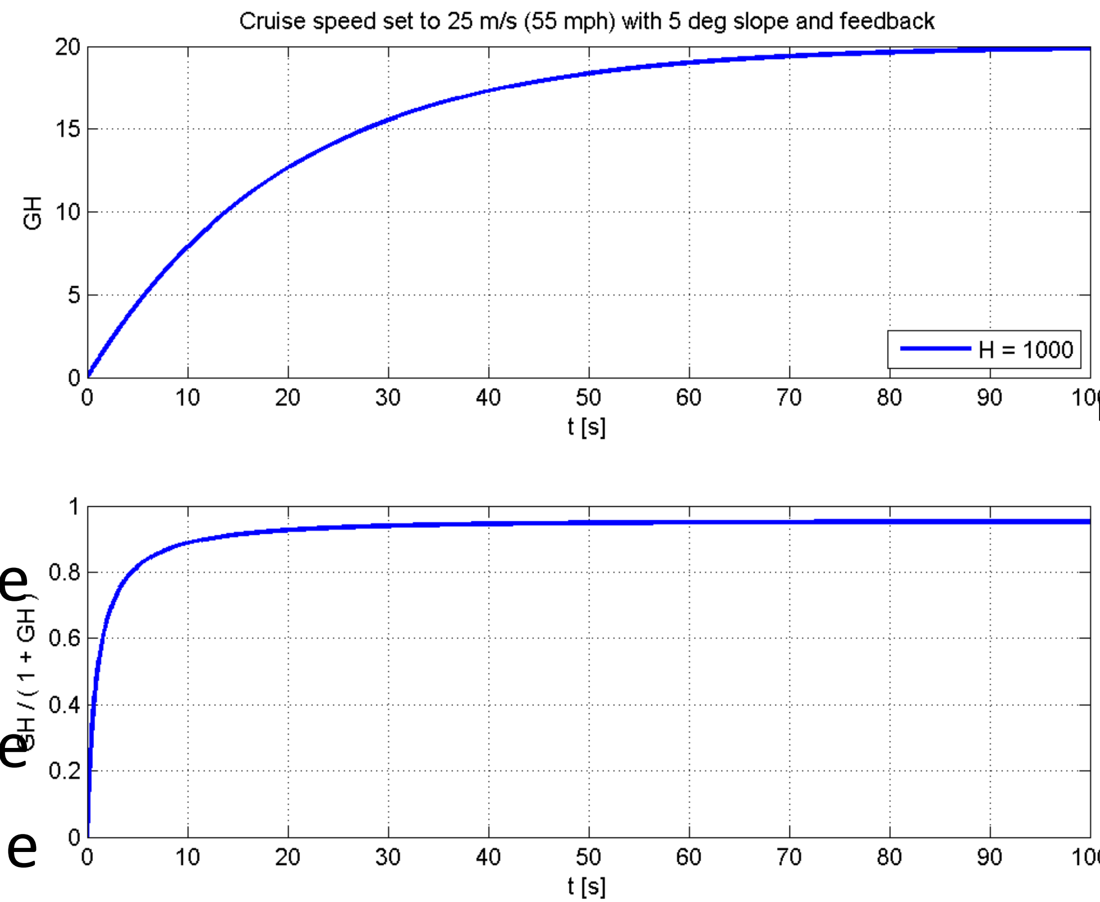
- Open loop

$$v = G \cdot H \cdot v_r$$

- Closed loop

$$v = \frac{G \cdot H}{1 + G \cdot H} \cdot v_r$$

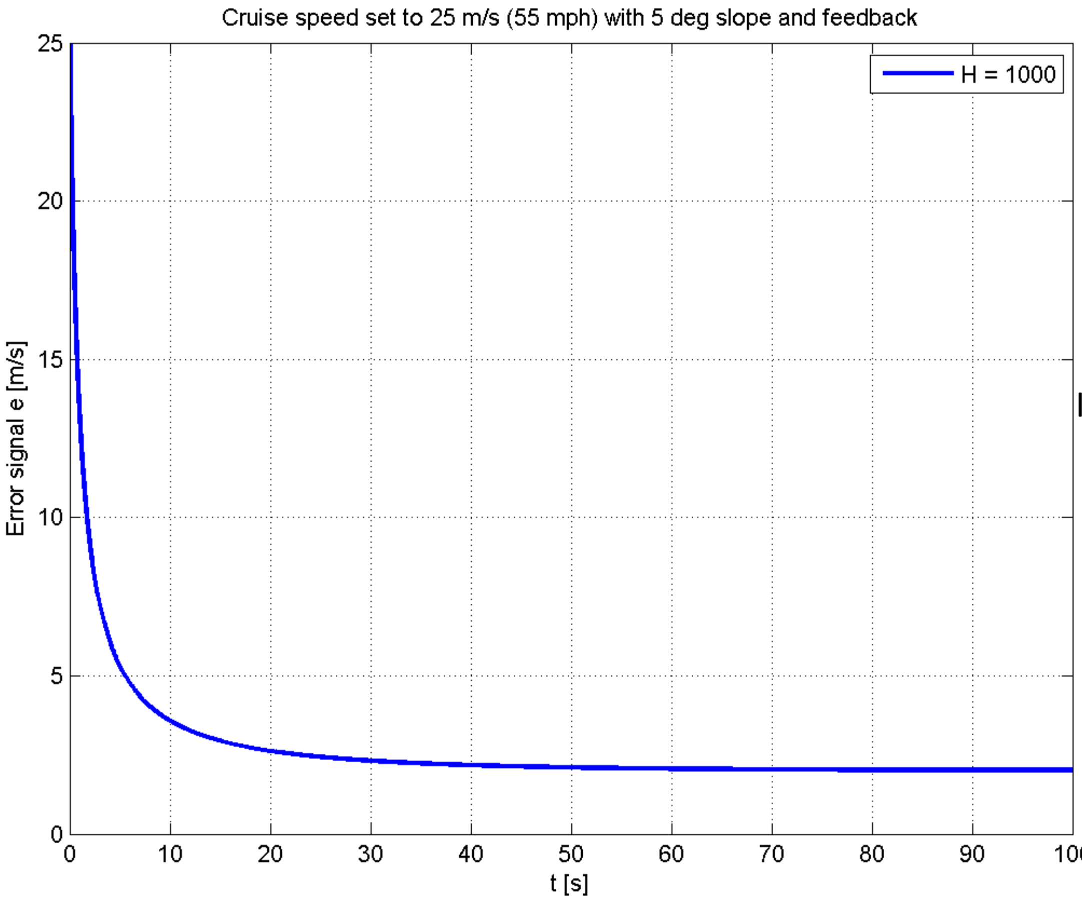
- Setting $H = 10^3\ kg/s$
- Plotting the open loop transfer function vs. time and the closed loop transfer function vs. time
- Notice the rapid rise time for the closed loop case



cruisefeedback_timedomain2.m

The error signal *e*

- Plot of error signal *e* vs *time*
- Error signal decreases to 3 m/s.
- Notice a ~10% *steady state error*



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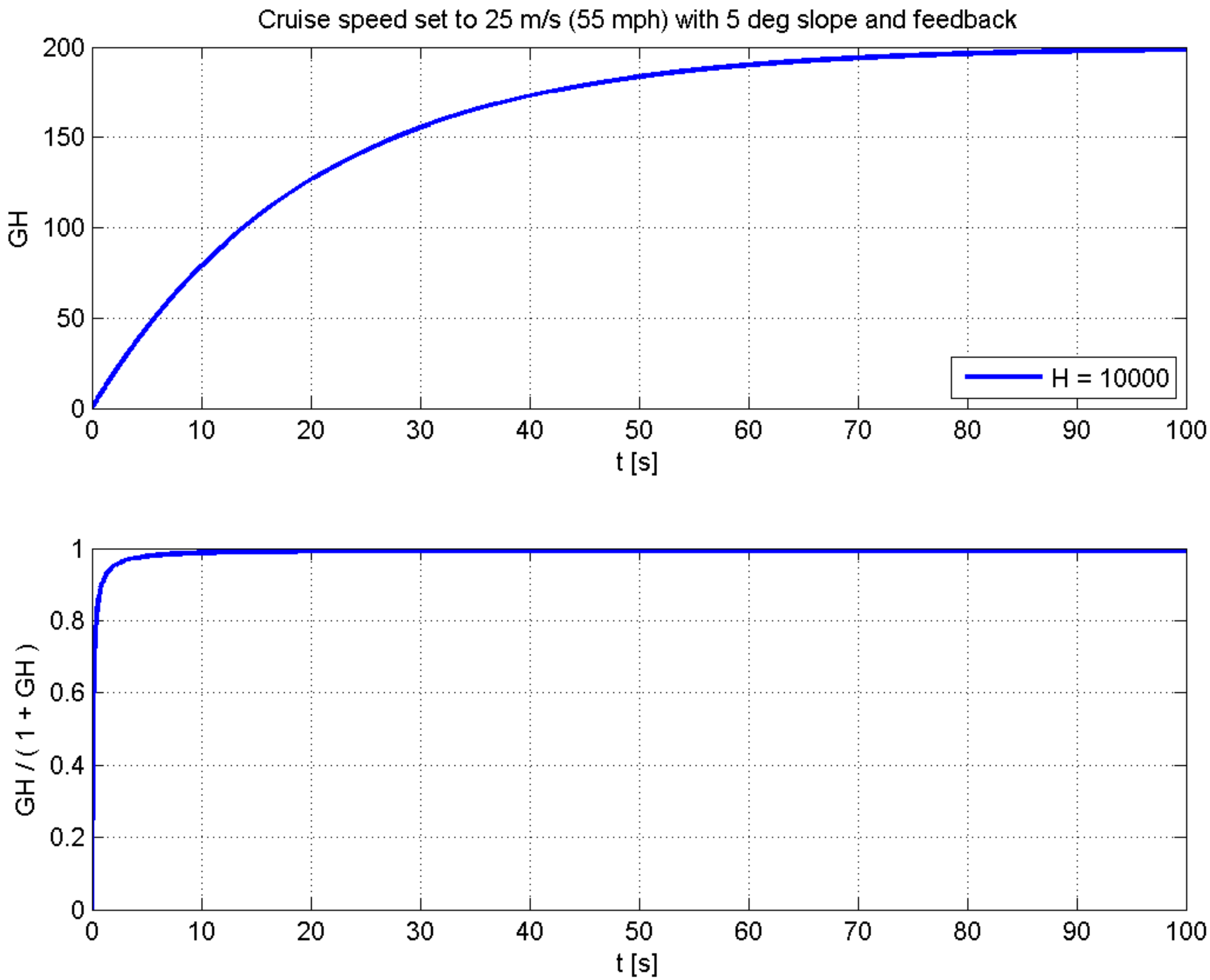
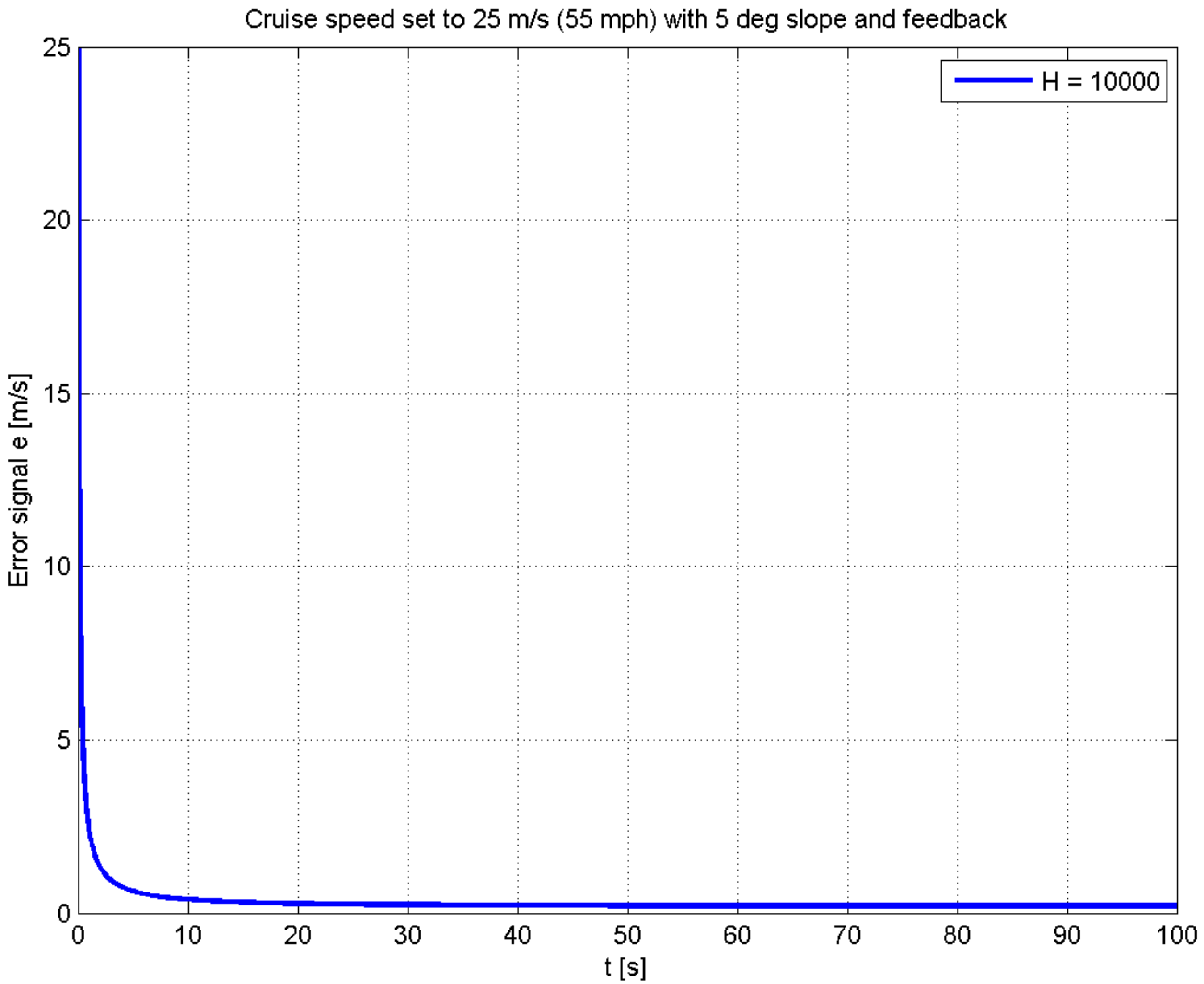
Open and closed loop TF with $H = 10^4\ kg/s$



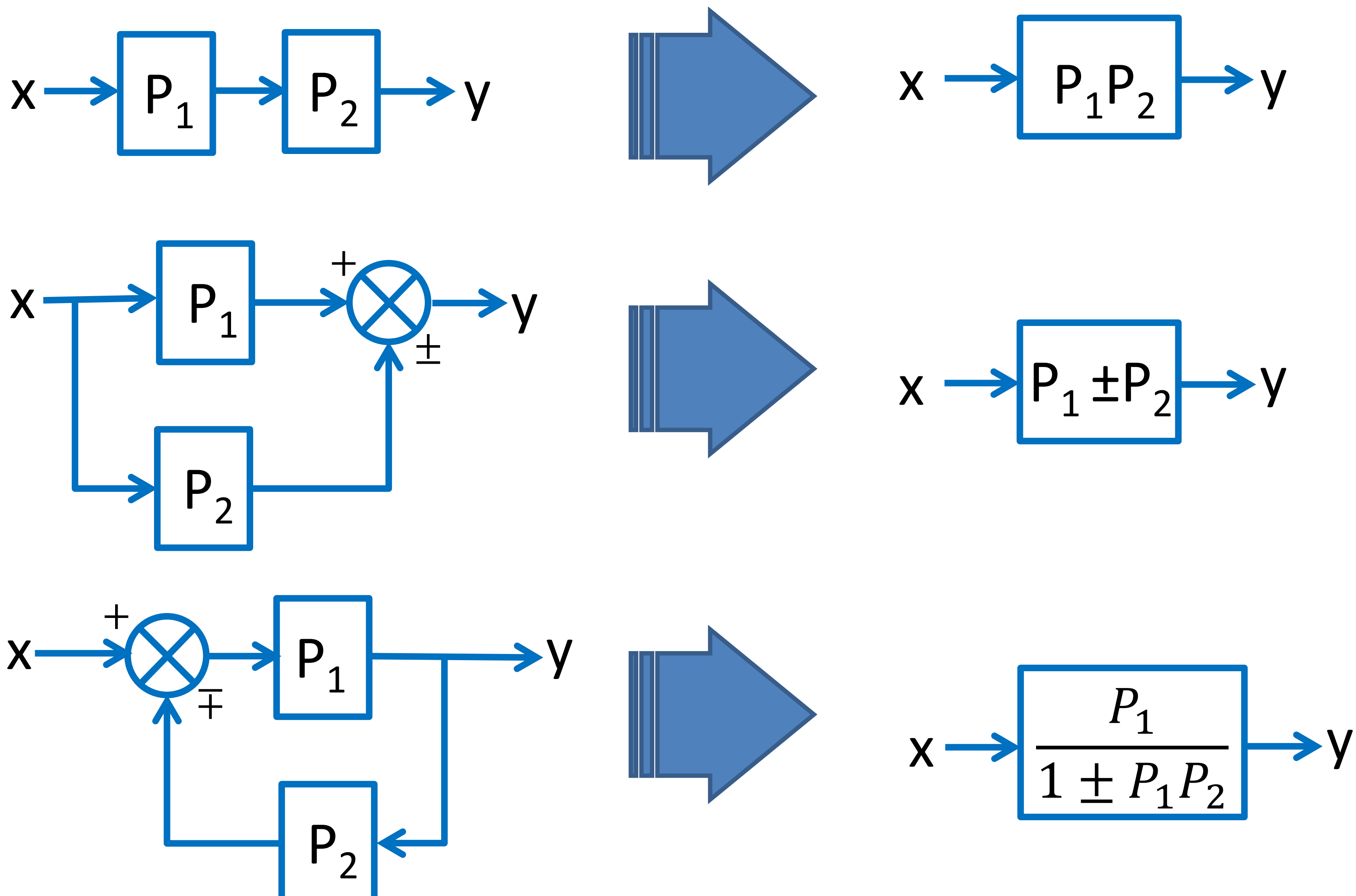
Error signal e with $H = 10^4\ kg/s$



Cruise control example

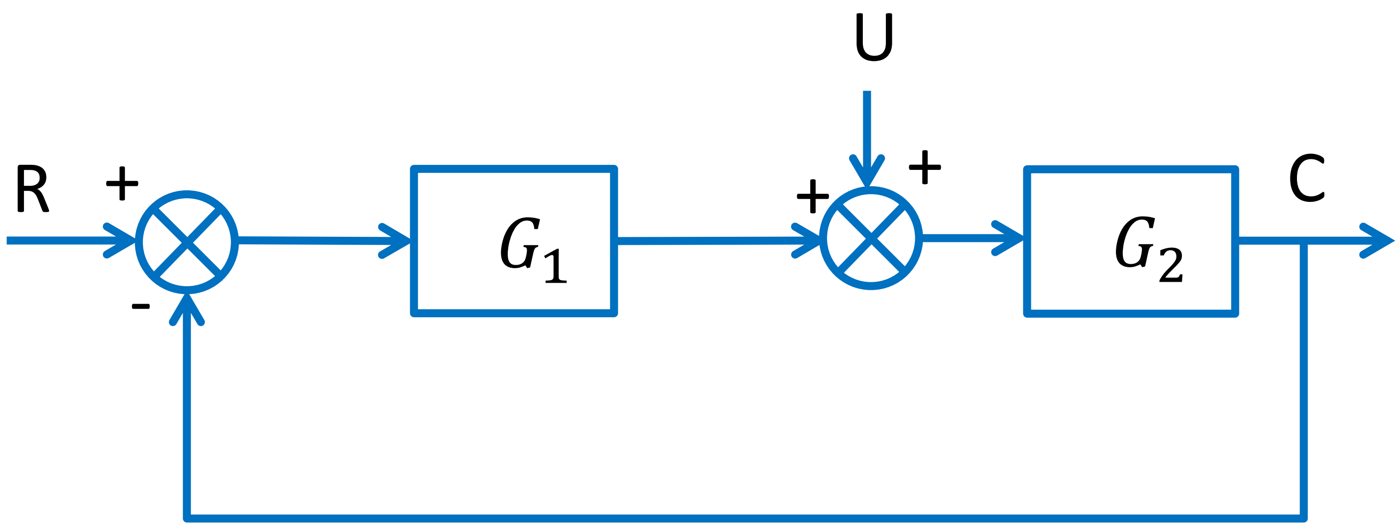
- First-order differential equation
- Simplest controller: simply a gain with no time constants involved
- How to handle more complicated problems?

Block diagram reduction



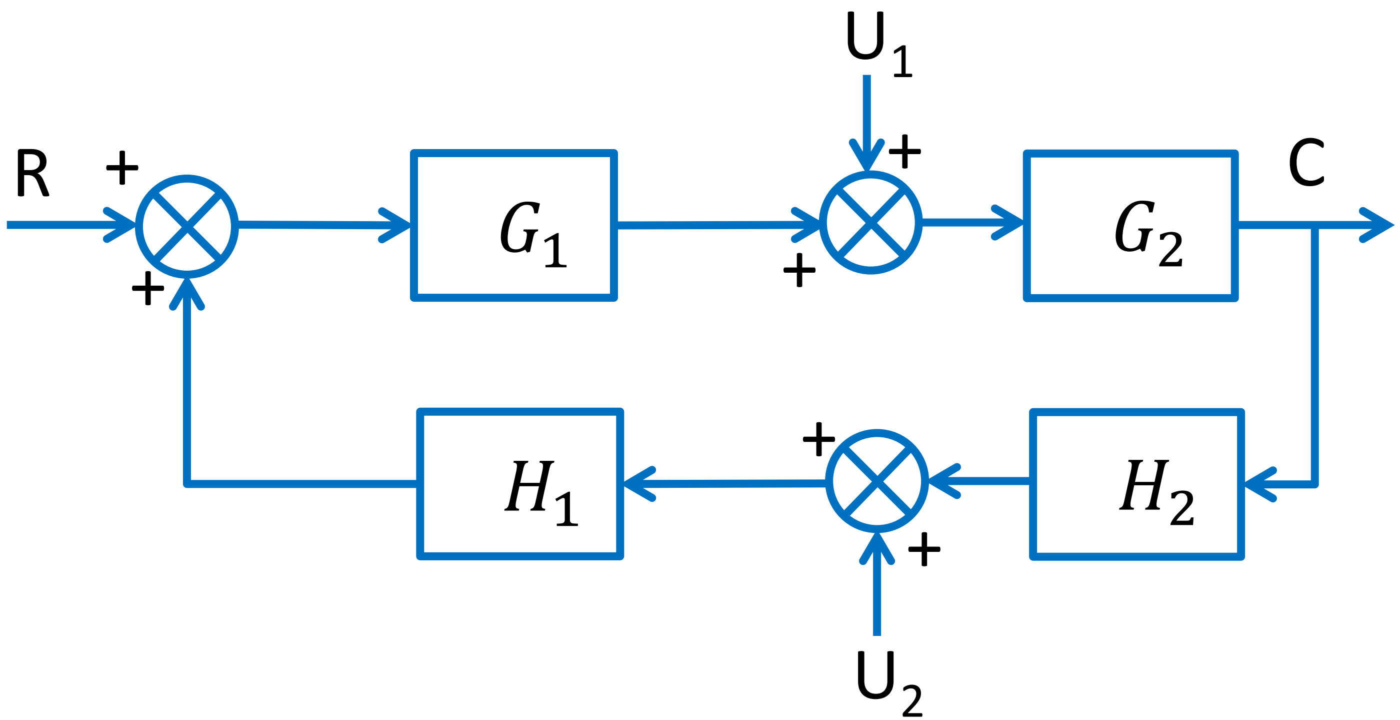
Practice

Determine the output C in terms of inputs U and R.



Practice

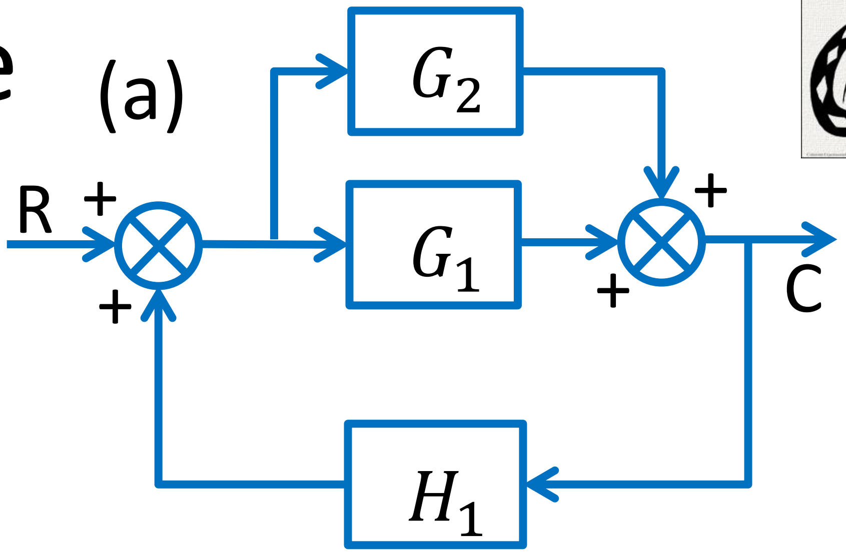
Determine the output C in terms of inputs U_1, U_2 and R.



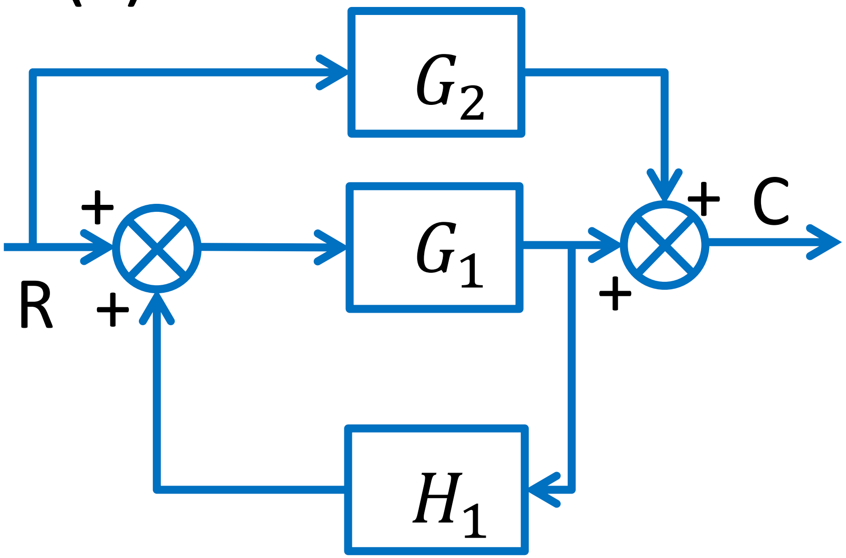
More practice

Determine C/R for the following systems.

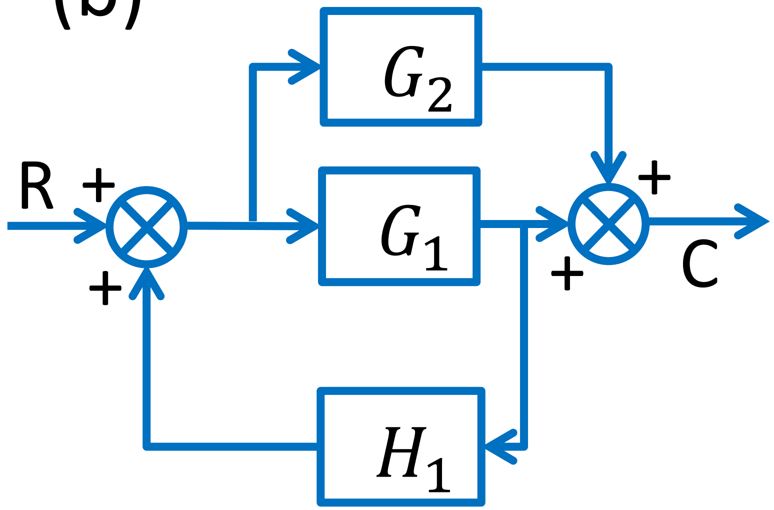
(a)



(c)

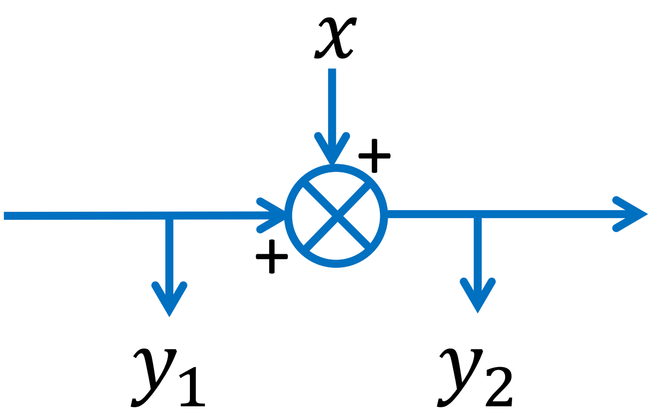


(b)



How do we MEASURE the OL TF of a system when the loop is closed?

1. Add an injection point in a closed loop system
2. Inject signal x and read signal y_1 (just before the injection) and y_2 (right after the injection)
3. Solve for the ratio $\frac{y_1}{y_2}$



So far…

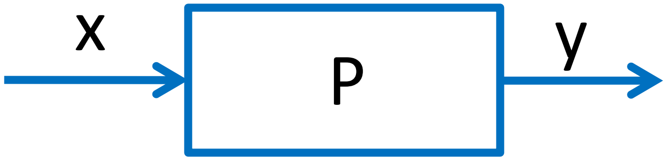
- Control theory builds on differential equations
- Block diagrams help visualize the signal flow in a physical system
- The cause-and-effect relationship between variables is referred to as a transfer function (TF)
- The system's open-loop TF is the product of transfer functions
  - cruise control example: $G \cdot H$
  - Two cases: $G \cdot H \ll 1$ and $G \cdot H \gg 1$
- MATLAB implementation
  - Functions used: dsolve

Laplace Transforms

- The technique of Laplace transform (and its inverse) facilitates the solution of ordinary differential equations (ODE).
- Transformation from the time-domain to the frequency-domain.
- Functions are complex, often described in terms of magnitude and phase

Linear systems

- To map a model to frequency space
  - System must be linear
  - Output proportional to input
- Given system P
  - Input signals: x_1 and x_2
  - Output signals (response): y_1 and y_2



- System P is linear
  - If input signal: $a\, x_1 + b\, x_2$
  - Then output signal: $a\, y_1 + b\, y_2$
  - Superposition principle

Example

- Is $y = \frac{dx}{dt}$ a linear system?
  - Knowing that $y_1 = \frac{dx_1}{dt}$ and $y_2 = \frac{dx_2}{dt}$
  - If input is $c_1 x_1 + c_2 x_2$, output is
    $\frac{d}{dt}(c_1 x_1 + c_2 x_2) =$
    $c_1 \frac{d}{dt} x_1 + c_2 \frac{d}{dt} x_2 =$
    $c_1 \; y_1 + c_2 \; y_2$
  - System is linear

Example

- Is $y = x^2$ a linear system?
  - Knowing that $y_1 = (x_1)^2$ and $y_2 = (x_2)^2$
  - If input is $c_1 x_1 + c_2 x_2$, output is $(c_1 x_1 + c_2 x_2)^2 \neq c_1 \ y_1 + c_2 \ y_2$
  - System is not linear

In general

Output

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y$$

$$= b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_0 x$$

Input

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i x(t)$$

$$n \geq m$$

For a stable system

Laplace Transform $\mathcal{L}$

- Transforms a *linear differential equation* into an *algebraic equation*
- Tool in solving differential equations
- Laplace transform of function *f*

$$F(s) = \mathcal{L}[f(t)]$$

- Laplace inverse transform of function *F*

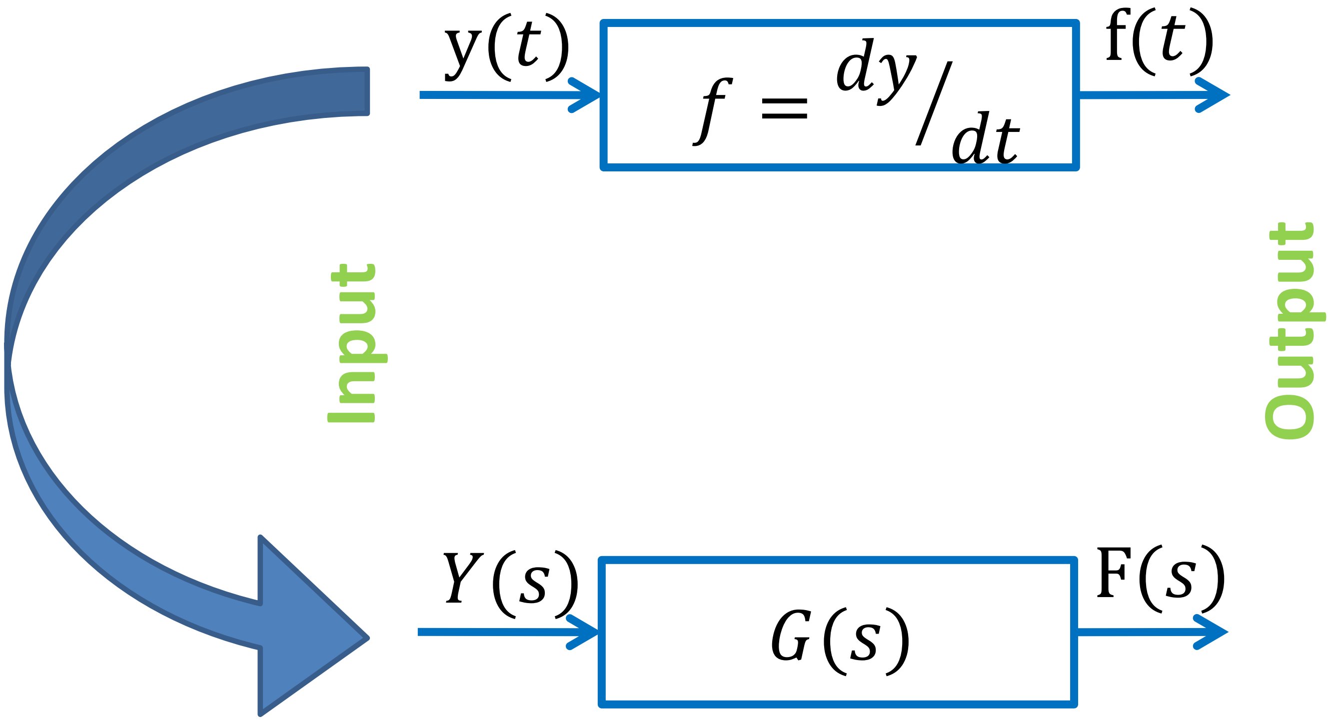
$$f(t) = \mathcal{L}^{-1}[F(s)]$$

where $s = j\omega$ is the transform variable

Imaginary unit (j)

$2\pi f$ (ω)

Time domain ↔ Laplace domain



Laplace Transform $\mathcal{L}$

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{-j\omega}^{+j\omega} F(s) e^{st} ds$$

(Some) Laplace transform pairs

	$f(t)$	$F(s)$
Unit step	$u(t)$	$1/s$
Unit ramp	t	$1/s^2$
Exponential	e^{at}	$1/(s-a)$
Sinusoid	$\sin(\omega_0 t)$	$\omega/(s^2+\omega_0{}^2)$
	$(1/a)(1-e^{-at})$	$1/s(s+a)$
SHO	$\frac{\omega_0}{\sqrt{1-\delta^2}} e^{-\delta\omega_0 t} \times$ $\times \sin(\sqrt{1-\delta^2}\,\omega_0 t)$	$\frac{\omega_0{}^2}{s^2+2\delta\omega_0\, s+\omega_0{}^2}$

Laplace transform properties

- Linearity

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

- Derivatives
  - First-order: $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s)$
  - Second-order: $\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s)$
- Integral

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

Solution to ODEs

1. Laplace transform the system's ODE
2. Solve the algebraic equation in s
3. Inverse transform back to the time domain

Transfer Function $G(s)$

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i x(t)$$

$$\sum_{i=0}^{n} a_i s^i Y(s) = \sum_{i=0}^{m} b_i s^i X(s)$$

Using derivative property

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

Algebraic equation in s, the ratio of 2 polynomials in s

Transfer function $G(s)$ relates input $X(s)$ to output $Y(s)$.

Transfer Function $G(s)$

The roots of the numerator are referred to as *zeros*.

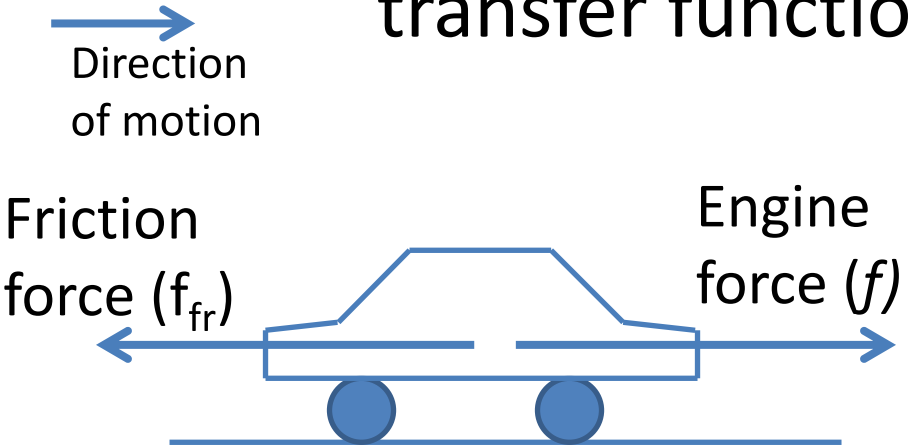
Transfer function $G(s)$ can be defined by

- The coefficients of *s* or
- Its poles and zeros

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

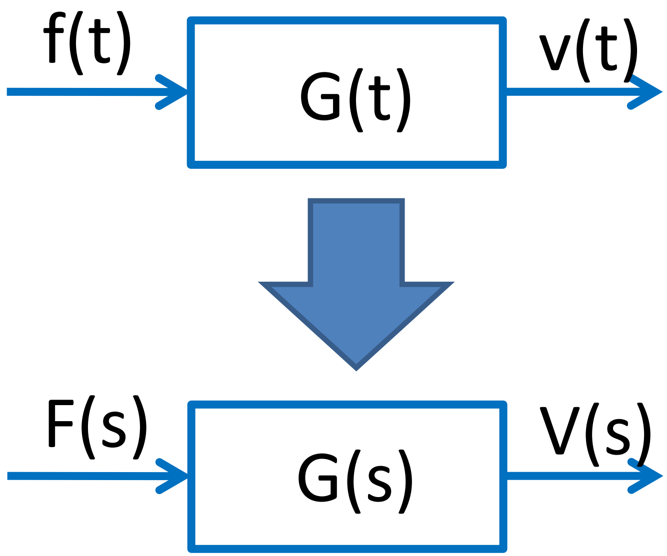
The roots of the denominator are referred to as *poles*.

Let's apply it to the cruise control example: transfer function G (the car's body)



$$m\frac{dv(t)}{dt} = f(t) - bv(t)$$

$$m\, s\, V(s) = F(s) - b\, V(s)$$



$$V(s) = G(s)\, F(s)$$

$$\text{where } G(s) = \frac{1/m}{s + b/m}$$

Pole at $-b/m$

Dynamic response: using lookup tables to inverse transform

Laplace inverse transform using lookup tables

$$\frac{1}{s\,(s+a)}$$

$$\Updownarrow$$

$$\frac{1}{a}\left(1-e^{-at}\right)$$

Input: step function, amplitude F_0

$$f(t) = F_0\,u(t)$$

$$F(s) = \mathcal{L}[f(t)] = {}^{F_0}/_{s}$$

The response (in frequency space) is

$$V(s) = G(s)\cdot F(s) = \frac{1/m}{s+b/m}\cdot\frac{F_0}{s}$$

The time-domain response is

$$v(t) = \mathcal{L}^{-1}[V(s)] =$$

$$= \mathcal{L}^{-1}\left[\frac{F_0}{m}\cdot\frac{1}{s\,(s+b/m)}\right]$$

Dynamic response: using lookup tables to inverse transform

Laplace inverse transform using lookup tables

$$\frac{1}{s\,(s+a)}$$

$$\Updownarrow$$

$$\frac{1}{a}(1-e^{-at})$$

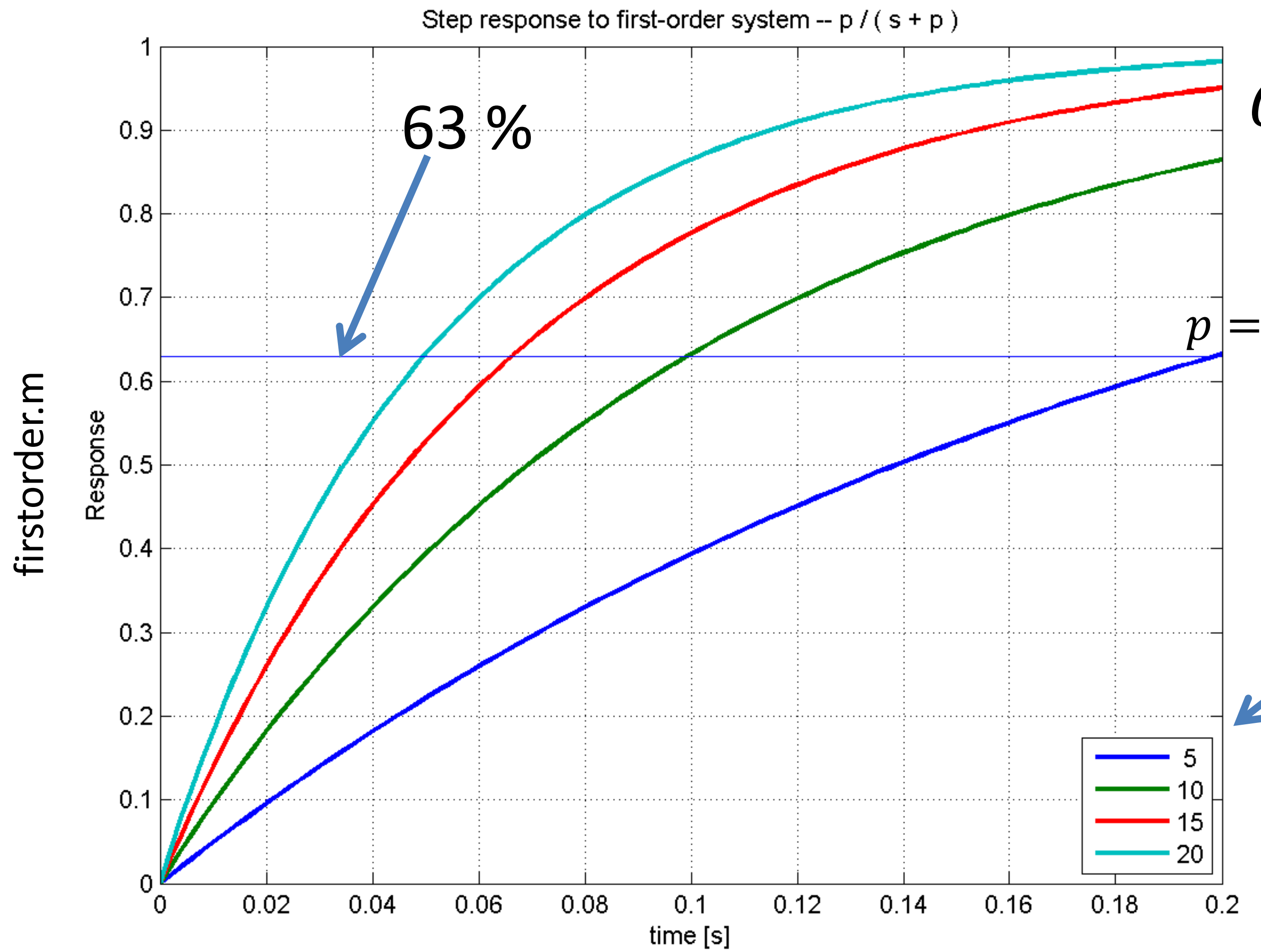
$$v(t) = \mathcal{L}^{-1}\left[\frac{F_0}{m}\cdot\frac{1}{s\,(s+b/m)}\right]$$

$$= \frac{F_0}{m}\cdot\mathcal{L}^{-1}\left[\frac{1}{s\,(s+b/m)}\right]$$

$$v(t) = \frac{F_0}{b}\left[1-e^{-\frac{b}{m}t}\right]$$

Pole $a = \frac{1}{\tau}$

First-order system step response



$$G(s) = \frac{p}{s+p}$$

$p = 5\ rad/s, \tau = 0.2\ s$

Pole p

MATLAB implementation

The step response of transfer function

$$G(s) = \frac{5}{s+5}$$

```
>> G=tf(5, [1 5]);
>> step(G);
```

Partial fraction expansion

1. Reduce a complex function to a collection of simpler ones
2. Then use lookup table

Order m

$$F(s) = \frac{Q(s)}{P(s)} = \sum_i \frac{\alpha_i}{s + a_i} \qquad m \leq n$$

Order n

$$f(t) = \mathcal{L}^{-1}\left[\frac{\alpha_1}{s + a_1}\right] + \cdots + \mathcal{L}^{-1}\left[\frac{\alpha_n}{s + a_n}\right]$$

$$= \alpha_1 \, e^{-a_1 t} + \cdots + \alpha_n \, e^{-a_n t}$$

$$f(t) = \sum_{i=0}^{n} \alpha_i \, e^{-a_i t}$$

Comments

$$F(s) = \sum_{i} \frac{\alpha_i}{s + a_i} \quad \Longleftrightarrow \quad f(t) = \sum_{i=0}^{n} \alpha_i \, e^{-a_i t}$$

1. Poles of *F(s)* determine the time evolution of *f(t)*
2. Zeros of *F(s)* affect coefficients
3. Poles closer to origin → larger time constants

Example

Find $f(t)$ of the Laplace transform

$$F(s) = \frac{(6s^2 - 12)}{s^3 + s^2 - 4s - 4}$$

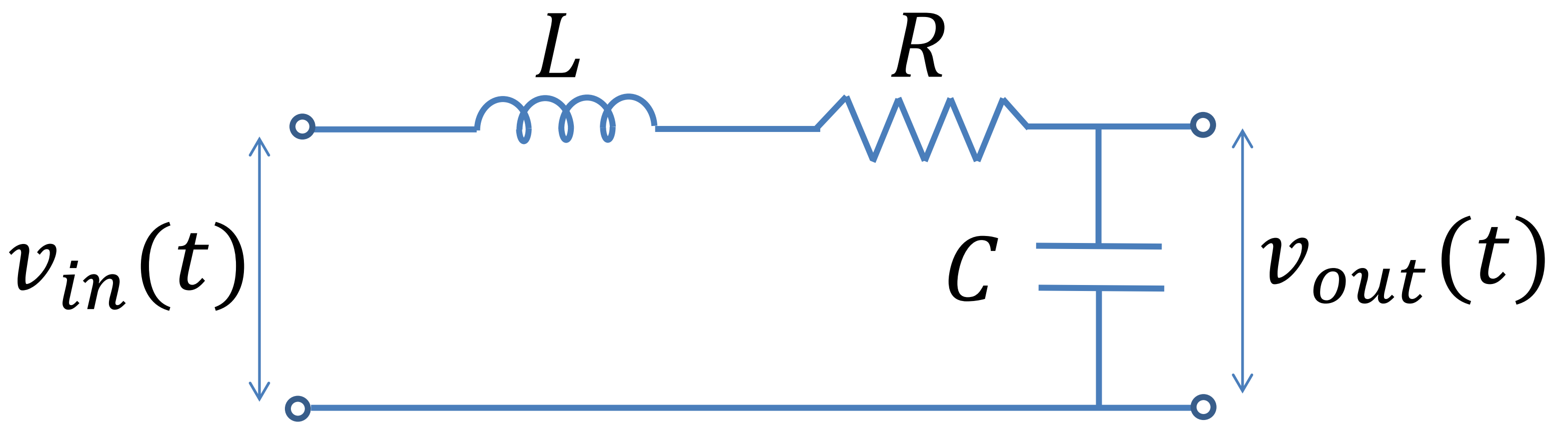
Sol: Using MATLAB

```
>> [R,P,K]=residue([6 0 -12],[1 1 -4 -4])
R =
      3.0000
      1.0000
      2.0000
P =
     -2.0000
      2.0000
     -1.0000
K =
        []
```

$$F(s) = \frac{3}{s+2} + \frac{1}{s-2} + \frac{2}{s+1}$$

$$f(t) = 3\,e^{-2t} + e^{2t} + 2\,e^{-t}$$

Example: LRC circuit

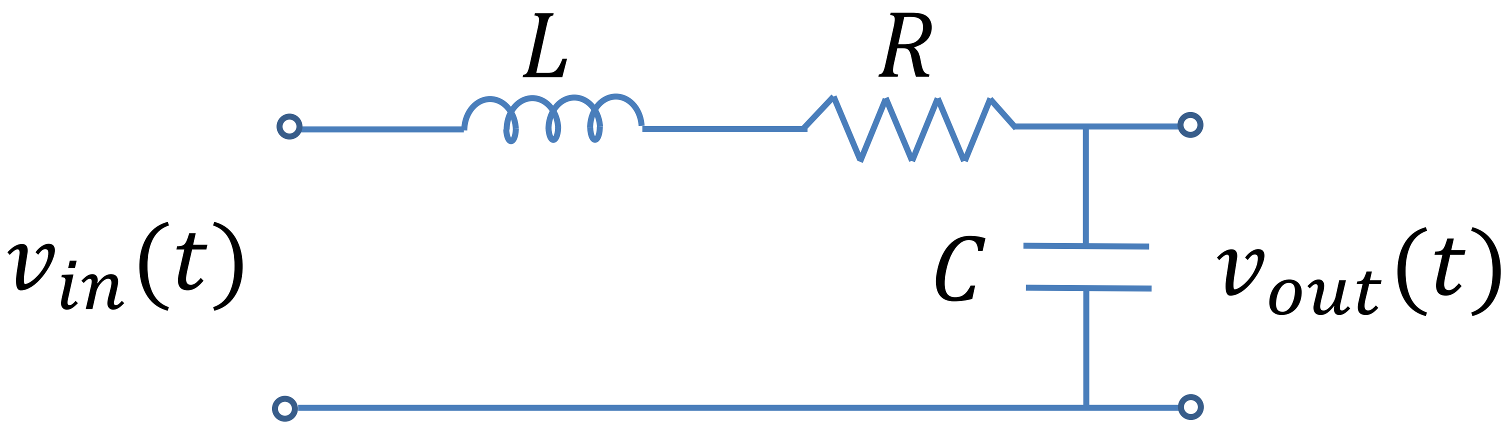


$$v_{in}(t) = v_L(t) + v_R(t) + v_C(t)$$

$$= L\frac{d}{dt}i(t) + R\,i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

$$= L\,D\,i(t) + R\,i(t) + \frac{1}{C}\frac{1}{D}i(t)$$

Example: LRC circuit



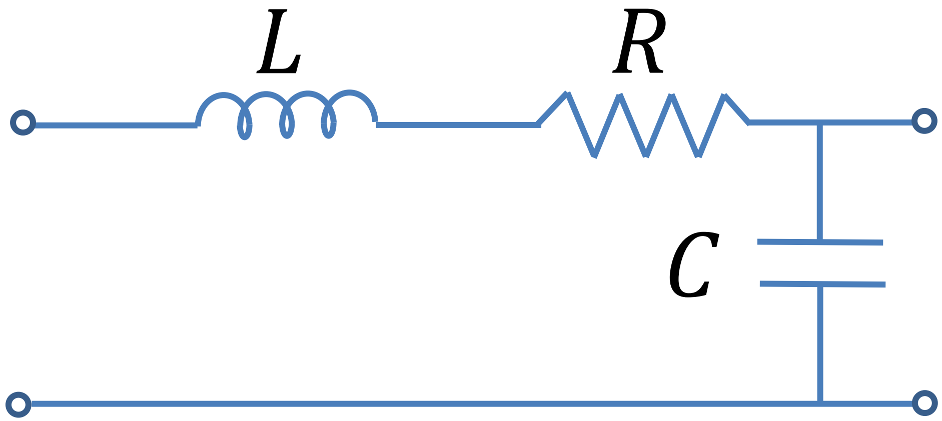
$$v_{in}(t) = L\,D\,i(t) + R\,i(t) + \frac{1}{C}\frac{1}{D}i(t)$$

$$i(t) = C\frac{d}{dt}v_{out}(t) = C\,D\,v_{out}(t)$$

$$v_{in}(t) = (L\,C\,D^2 + R\,C\,D + 1)\,v_{out}(t)$$

Example: LRC circuit

$v_{in}(t)$



$v_{out}(t)$

$$v_{in}(t) = (L\,C\,D^2 + R\,C\,D + 1)\,v_{out}(t)$$

$\mathcal{L}$

$$V_{in}(s) = (L\,C\,s^2 + R\,C\,s + 1)\,V_{out}(s)$$

LRC circuit: transfer function

$$V_{out}(s) = \frac{1}{L\,C\,s^2 + R\,C\,s + 1} \cdot V_{in}(s)$$

Setting L = 1 H, C = 1 F and R = 1 Ω

$$V_{out}(s) = \frac{1}{s^2 + \ s + 1} \cdot V_{in}(s)$$

LRC circuit: dynamic response to step

Setting the input to a step of amplitude 1 V

$$V_{in}(s) = \frac{1}{s}$$

The unit step response is

$$V_{out}(s) = \frac{1}{s^2 + s + 1} \cdot \frac{1}{s} = \frac{1}{s^3 + s^2 + s}$$

LRC circuit: dynamic response to step

$$V_{out}(s) = \frac{1}{s^3 + s^2 + s} = \sum_i \frac{\alpha_i}{s + a_i}$$

Using MATLAB for the solution

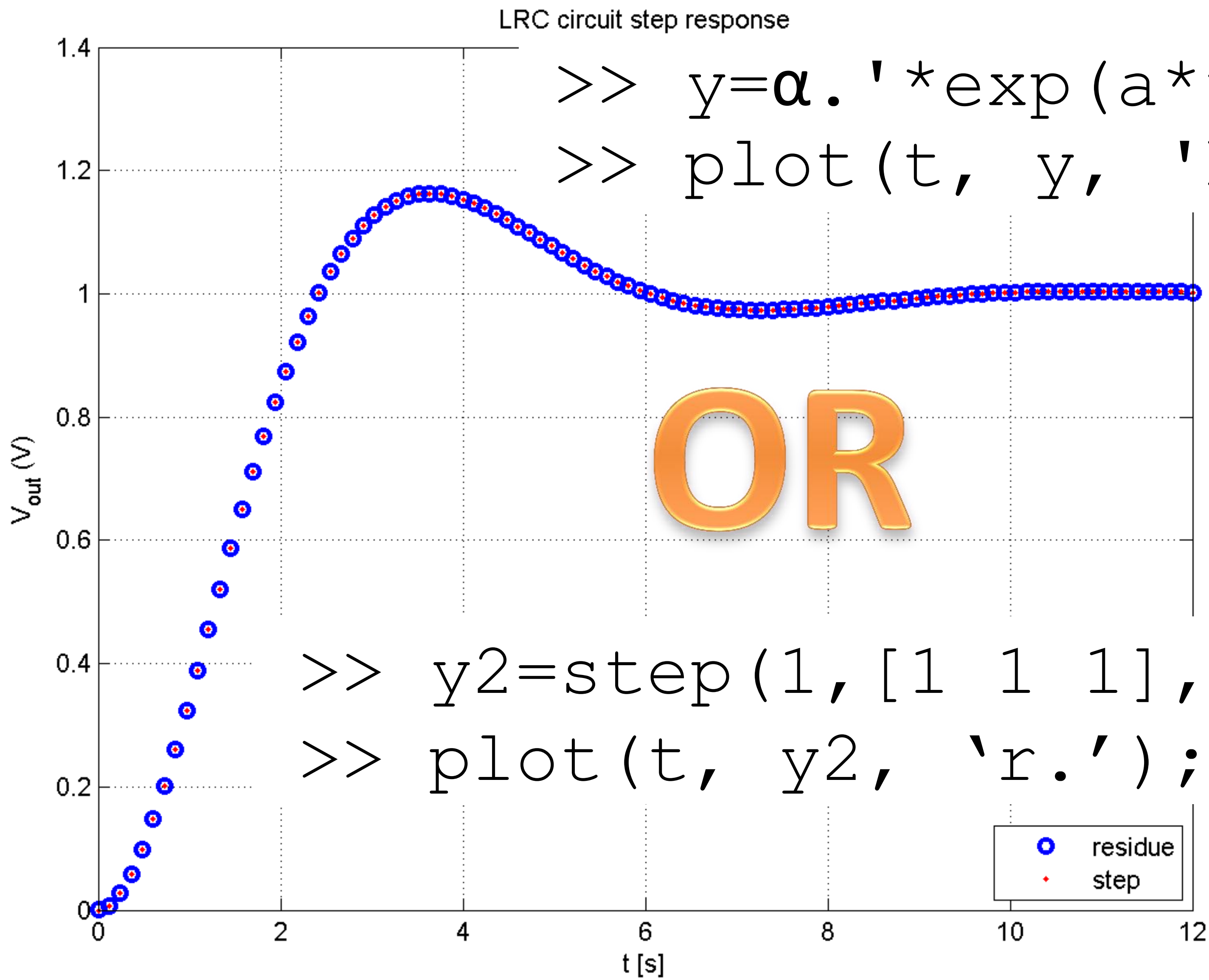
Coefficient vector for polynomial at numerator

Coefficient vector for polynomial at denominator

```
>> n = [1];
>> d = [1 1 1 0];
>> [α, a, k] = residue(n, d);
```

$$v_{out}(t) = \alpha_1\, e^{a_1 t} + \alpha_2\, e^{a_2 t} + \alpha_3\, e^{a_3 t}$$

Plotting results of two methods



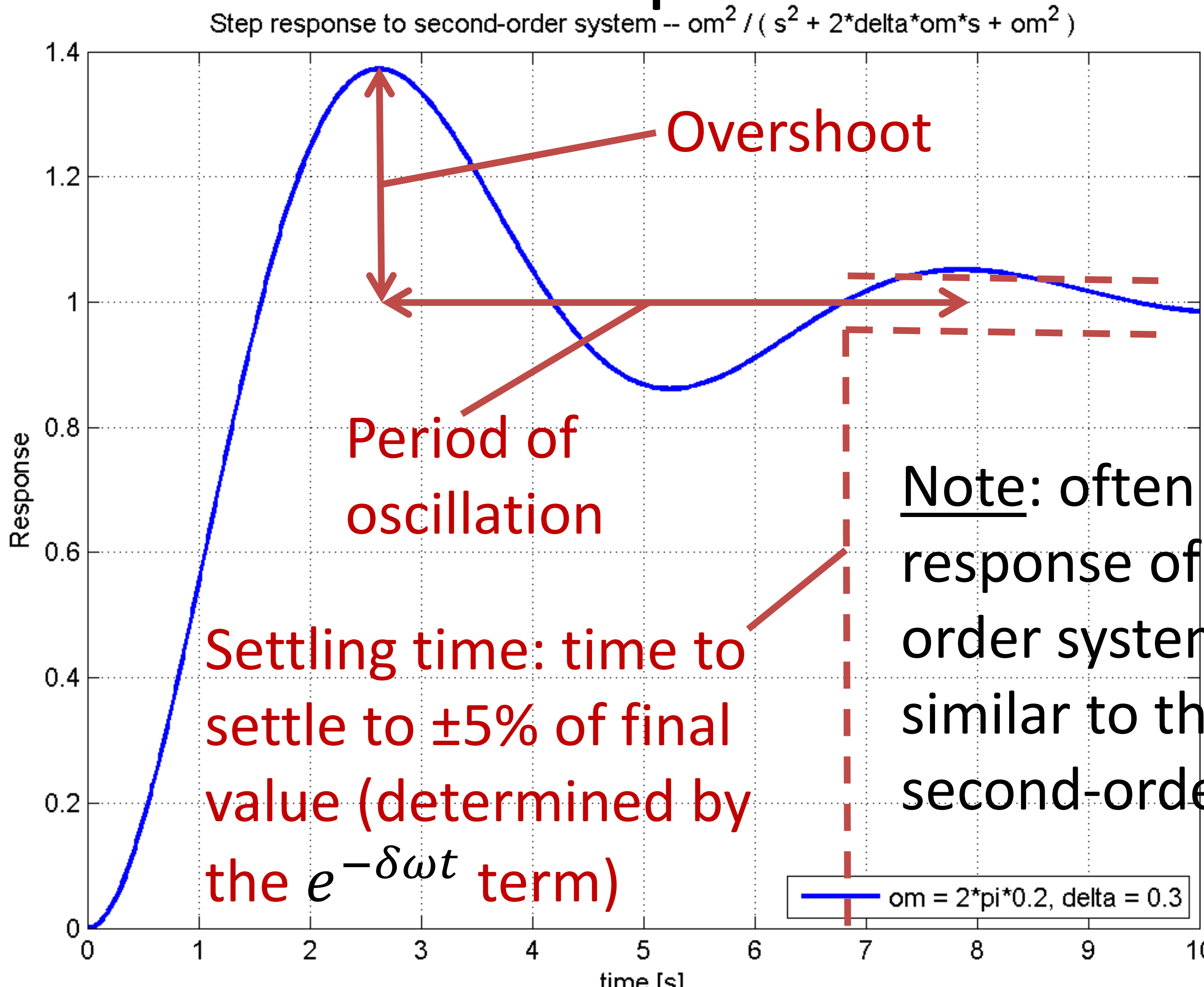

```
>> y=α.'*exp(a*t);
>> plot(t, y, 'bo');
```

OR

```
>> y2=step(1,[1 1 1],t);
>> plot(t, y2, 'r.');
```

Second-order system step response

secondorder.m



Note: often the response of a high-order system is similar to the second-order one

Verify the following

$$F(s) = \frac{6s^2 - 12}{s^3 + s^2 - 4s - 4}$$
$$f(t) = 2e^{-t} + 3e^{-2t} + e^{2t}$$

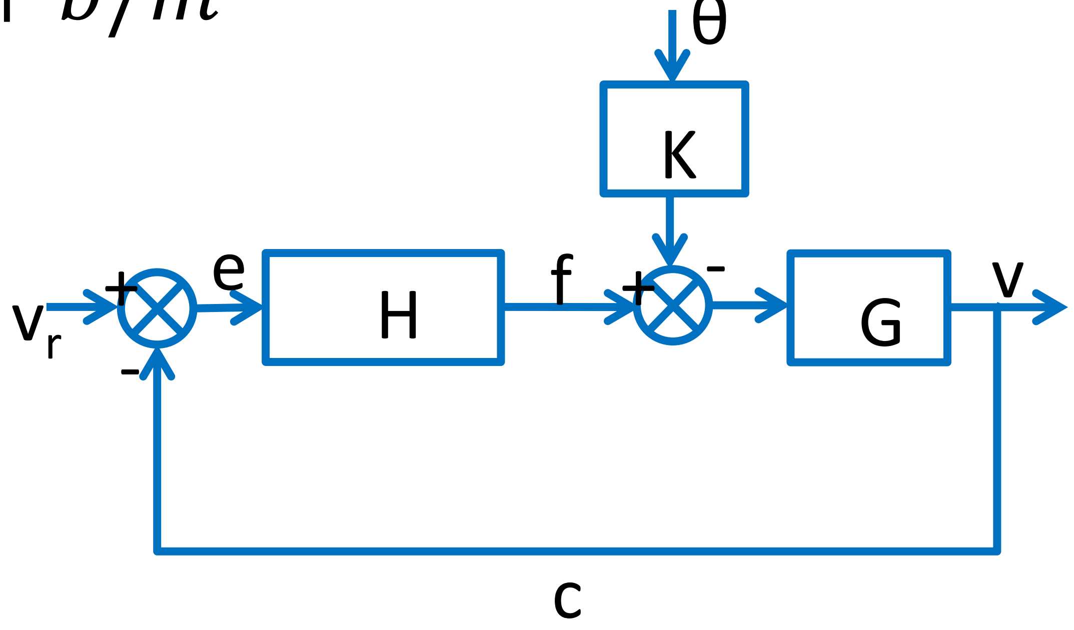
$$F(s) = \frac{6s}{s^3 + s^2 - 4s - 4}$$
$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}$$

$$F(s) = \frac{s + 5}{s^2 + 4s + 13}$$
$$f(t) = 2e^{-t} - 3e^{-2t} + e^{2t}$$

Back to cruise control: system's step response

$$V(s) = \frac{G \cdot H}{1 + G \cdot H} \cdot V_r(s)$$

where $G(s) = \frac{1/m}{s + b/m}$ and $H(s) = K_{gain}$

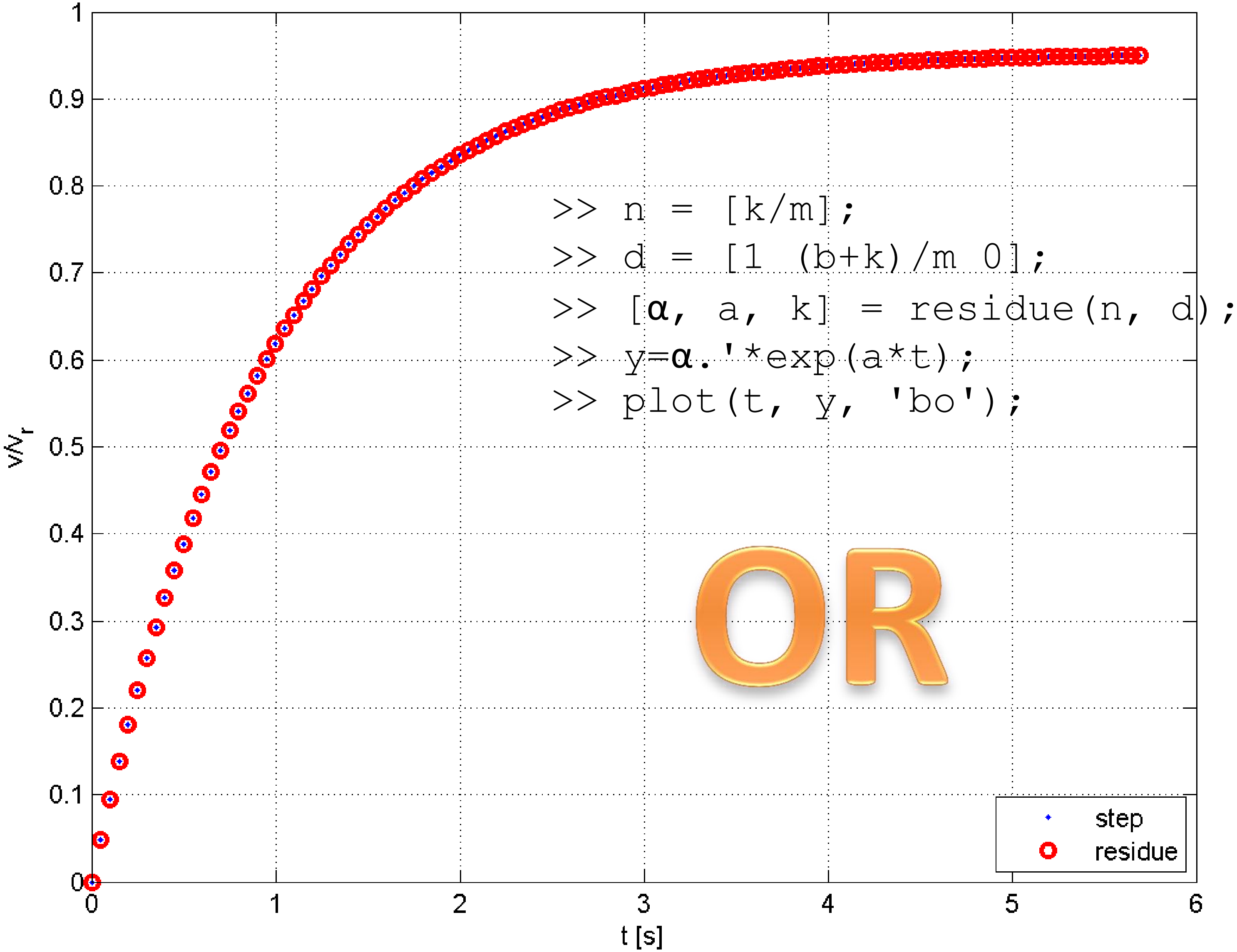


Step response

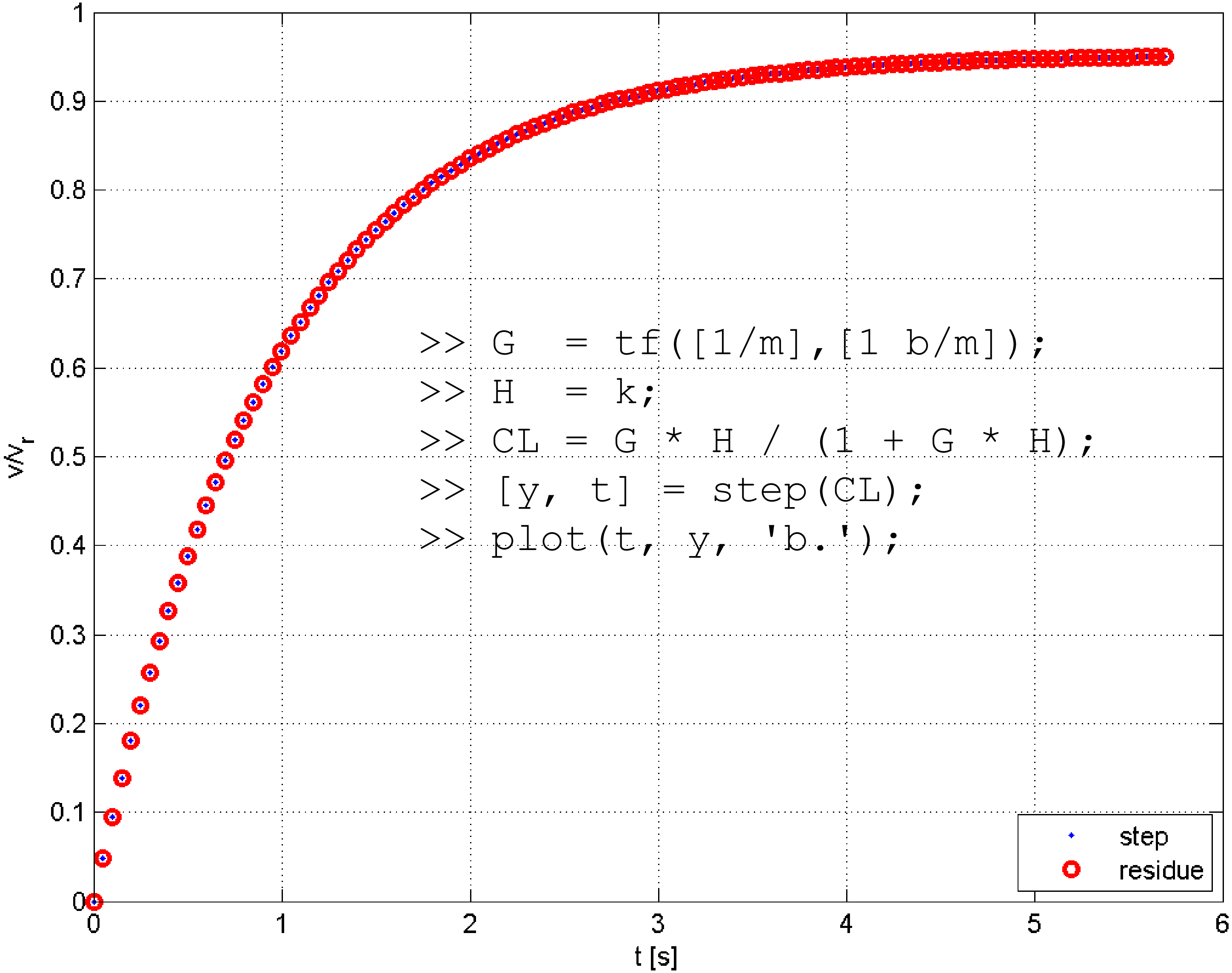
$$V(s) = \frac{G \cdot H}{1 + G \cdot H} \cdot \frac{1}{s} = \frac{K_{gain}/m}{s^2 + \frac{(b + K_{gain})}{m} s}$$

$$v(t) = \frac{K_{gain}}{K_{gain} + m}\left(1 - e^{-t/\tau}\right) \text{ with } \tau = \frac{m}{b + K_{gain}}$$

Back to cruise control



Back to cruise control



Transfer function, poles and zeros

- It is convenient to express a transfer function G(s) in terms of its poles and zeros:

$$G(s) = \frac{Q(s)}{P(s)}$$

$$= k \cdot \frac{(s - z_1) \cdot (s - z_2) \dots (s - z_m)}{(s - p_1) \cdot (s - p_2) \dots (s - p_n)}$$

- *k* is the gain of the transfer function

Summary of pole characteristics

- ***Real distinct poles (often negative)***

$$\frac{c_i}{s-p_i} \quad \leftrightarrow \quad c_i\, e^{p_i t}$$

- ***Real poles, repeated m times (often negative)***

$$\left[\frac{c_{i,1}}{s-p_{i,1}} + \frac{c_{i,2}}{\left(s-p_{i,2}\right)^2} + \cdots + \frac{c_{i,3}}{\left(s-p_{i,3}\right)^3} + \frac{c_{i,m}}{\left(s-p_{i,m}\right)^m}\right]$$

$$\updownarrow$$

$$\left[c_{i,1} + c_{i,2}t + \frac{1}{2!}c_{i,3}t^2 + \cdots + \frac{c_{i,m}}{(m-1)!}t^{m-1}\right] \cdot e^{p_i t}$$

Summary of pole characteristics

- ***Complex-conjugate poles***

$$\frac{c_i}{s-p_i}+\frac{(c_i)^*}{s-(p_i)^*} \quad \leftrightarrow \quad c_i\, e^{p_i t}+(c_i)^* e^{(p_i)^* t}$$

often re-written as a second-order term

$$\frac{\omega^2}{s^2+2\delta\omega\, s+\omega^2} \quad \leftrightarrow \quad \sim e^{\alpha t}\cdot \sin(\beta t+\varphi)$$

- ***Poles on imaginary axis***
  - Sinusoid
  - Pole at zero: step function
- ***Poles with a positive real part***
  - Unstable time-domain solution

Summary

- The Laplace transform is a tool to facilitate solving for ODEs.
- Systems need to be linear
- No need to do the transform (integral)
  - Use transform pairs, transform tables
  - Laplace transform properties: linearity, derivatives and integrals.
- Once in the Laplace domain, a TF is simply the ratio of two polynomials in s. Carry out algebra to solve the problem.
- No need to do the inverse transform
  - Use transform pairs, transform tables
  - For high-order TFs, use the partial-fraction expansion to reduce the problem to simpler parts
- IMPORTANT:
  - Poles of a transfer function determine the time evolution of the system
  - Poles with a real positive part correspond to unstable and unphysical systems
  - The system TF needs to have poles with a negative real part
- MATLAB implementation
  - Functions used: step, residue, tf

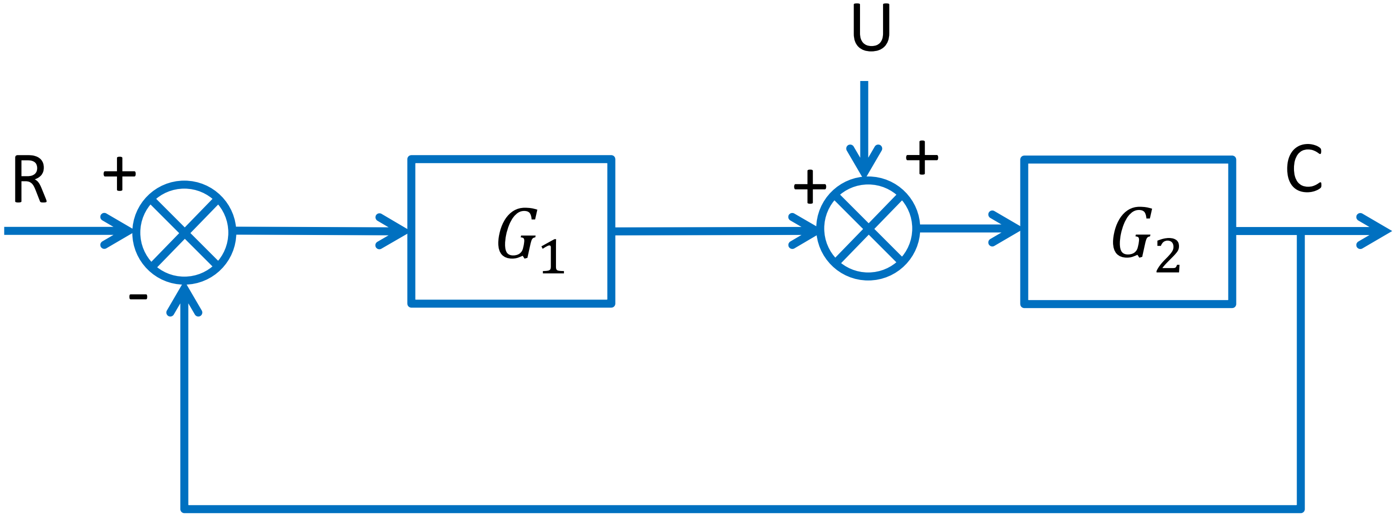
Solutions

Practice

Determine the output C in terms of inputs U and R.

Sol:

$$C = \left(\frac{G_2}{1 + G_1 G_2}\right) \cdot (G_1\, R + U)$$

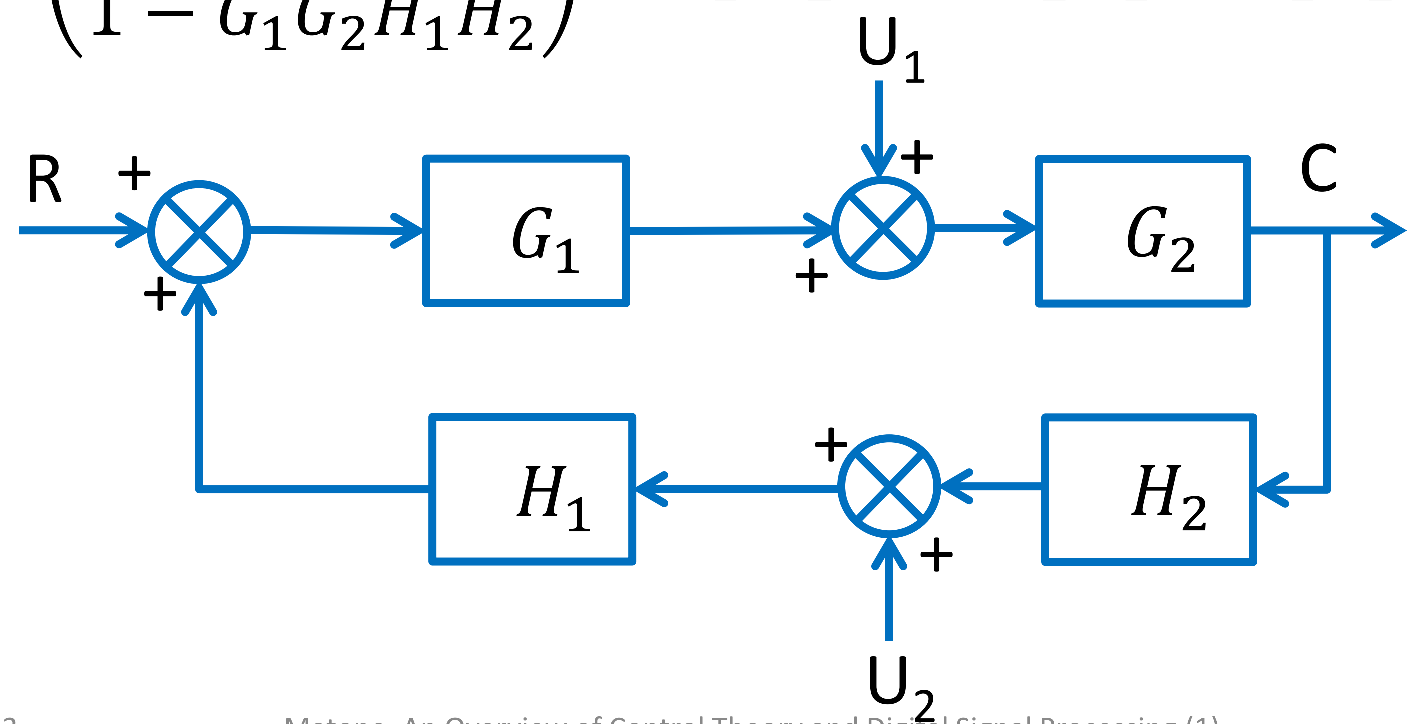


Practice

Determine the output C in terms of inputs U_1, U_2 and R.

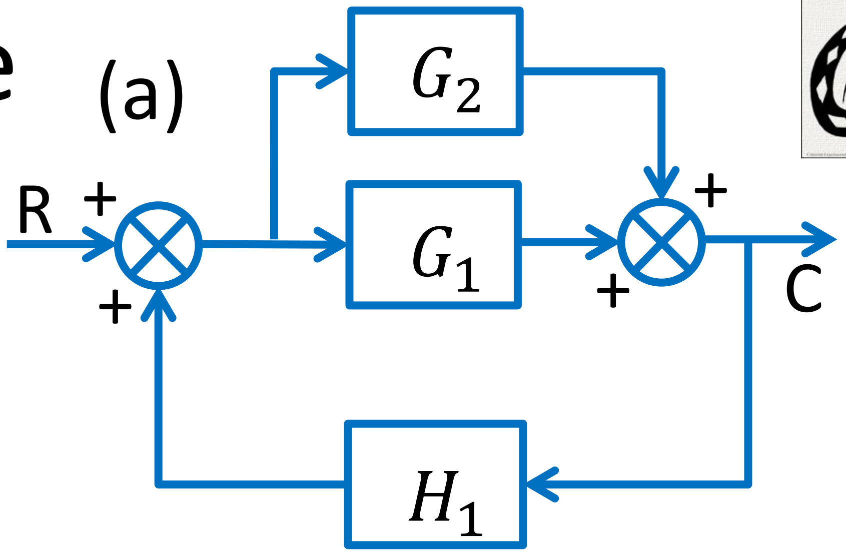
Sol:

$$C = \left(\frac{1}{1 - G_1 G_2 H_1 H_2}\right) \cdot (G_1\, G_2\, R + G_2 U_1 + G_1 G_2 H_1 U_2)$$



More practice

(a)



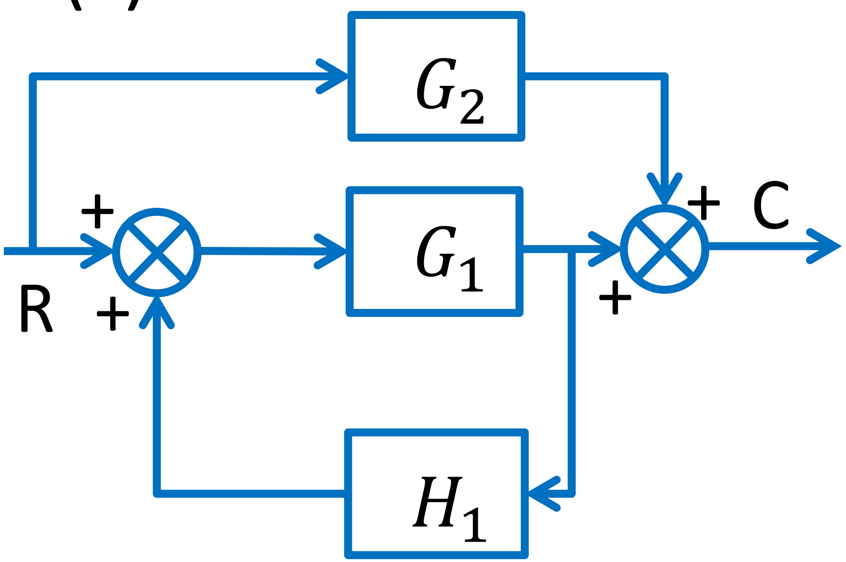
Determine C/R for the following systems. Sol:

a) $$\frac{C}{R} = \frac{G_1+G_2}{1-G_1H_1-G_2H_2}$$

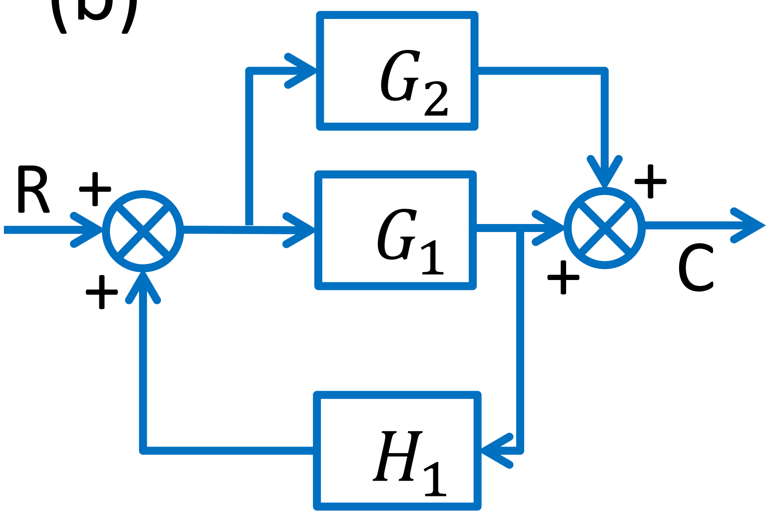
b) $$\frac{C}{R} = \frac{G_1+G_2}{1-G_1H_1}$$

c) $$\frac{C}{R} = \frac{G_1+G_2(1-G_1H_1)}{1-G_1H_1}$$

(c)



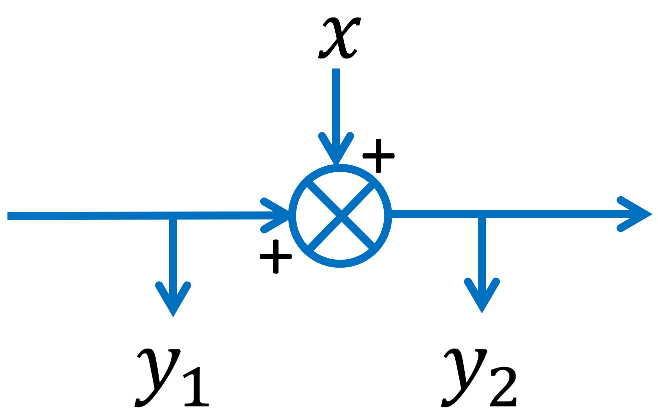
(b)



How do we MEASURE the OL TF of a system when the loop is closed?

1. Add an injection point in a closed loop system
2. Inject signal x and read signal y_1 (just before the injection) and y_2 (right after the injection)
3. Solve for the ratio $\frac{y_1}{y_2}$

Sol: $\frac{y_1}{y_2} = -G_{OL}$



Partial-fraction examples

- Denominator: has distinct, real roots
  - Example 2.4, 2.5, 2.6
- Denominator: complex roots
  - Example 2.7, 2.8
- Denominator: repeated roots
  - Example 2.9

Practice: verify the following

$$F(s) = \frac{6}{(s+1)(s+2)(s+3)}$$
$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}$$

$$F(s) = \frac{s+5}{s^2+4s+13}$$
$$f(t) = \sqrt{2}e^{-2t}\sin(3t + \frac{\pi}{4})$$

$$F(s) = \frac{2}{(s+1)^3(s+2)}$$
$$f(t) = 2\left[\left(1 - t + \frac{t^2}{2}\right)e^{-t} - e^{-2t}\right]$$