To recap

- Control theory builds on differential equations
- The Laplace transform is a tool to facilitate solving for ODEs.
- No need to do actually do the transform
  - Lookup tables
- System *G(s)* is stable if
  - Its response is bounded and finite
  - poles must have *negative real parts*
- Still, how does the cruise control example work?

Getting there…

Recall

What is the transfer function of a system whose input u and output y are related by the following differential equation?

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u + \frac{du}{dt}$$

Recall

Given the system's transfer function

$$P(s) = \frac{2\,s+1}{s^2+s+1}$$

determine the system's differential equation to input $u(t)$.

Recall

Determine which of the following transfer functions represent stable systems and which represent unstable systems. Use MATLAB's `step` to verify your answer.

$$a)\ P(s) = \frac{s-1}{(s+2)(s^2+4)}$$

$$b)\ P(s) = \frac{s-1}{(s+2)(s+4)}$$

$$c)\ P(s) = \frac{(s+2)(s-2)}{(s+1)(s-1)(s+4)}$$

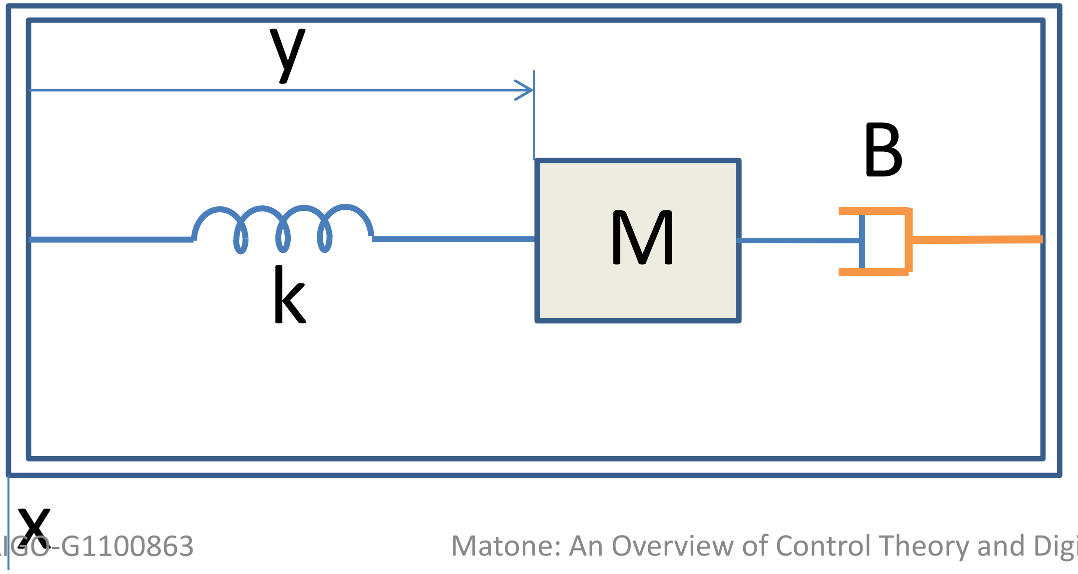
$$d)\ P(s) = \frac{6}{(s^2+s+1)(s+1)^2}$$

$$e)\ P(s) = \frac{5\,(s+10)}{(s^2-s+10)(s+5)}$$

Another example

A simple mechanical accelerometer is shown below. The position *y* is with respect of the case, the case's position is *x*. What is the transfer function between the input acceleration A ($a = d^2x/dt^2$) and the output *Y*?

$$-B\frac{dy}{dt} - ky = M\frac{d^2}{dt^2}(y - x)$$



Control Theory 2

The response of a stable system G(s) is characterized by its

Amplitude and Phase shift

to a sinusoidal excitation

Frequency response

It can be shown that if

$$x(t) = X\sin(\omega t)$$

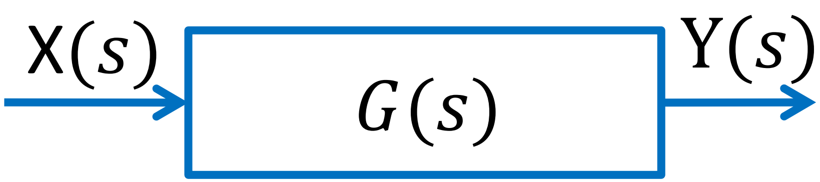
then

$$y(t) = |G(s)| \cdot X \cdot \sin\big(\omega t + \varphi(s)\big)$$

where

- $|G(s)|$ is the amplitude response and
- $\varphi(s)$ is the phase shift

This is how we measure transfer functions



Frequency response

1. The dynamic behavior of a physical system can be determined by measuring its transfer function with a sinusoidal excitation
2. Magnitude and phase response are a function of frequency ($s = j\omega$)
3. Frequency-response helps to understand the stability criteria

Graphical analysis tool: Bode Plot

- Common graphical representation of transfer function $G(s)$
- $G(s)$ is complex
  - plot of magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$
- Convention
  - Log-log scale for magnitude vs. frequency (Hz)
  - Semi-log scale for phase (deg) vs. frequency (Hz)
- Other conventions
  - Magnitude in dB ($X(dB) = 20\log_{10}X$) vs. angular frequency (rad/s)

Bode plot: $G(s) = {}^1/_s$

$$G(s) = \frac{1}{s} \rightarrow$$

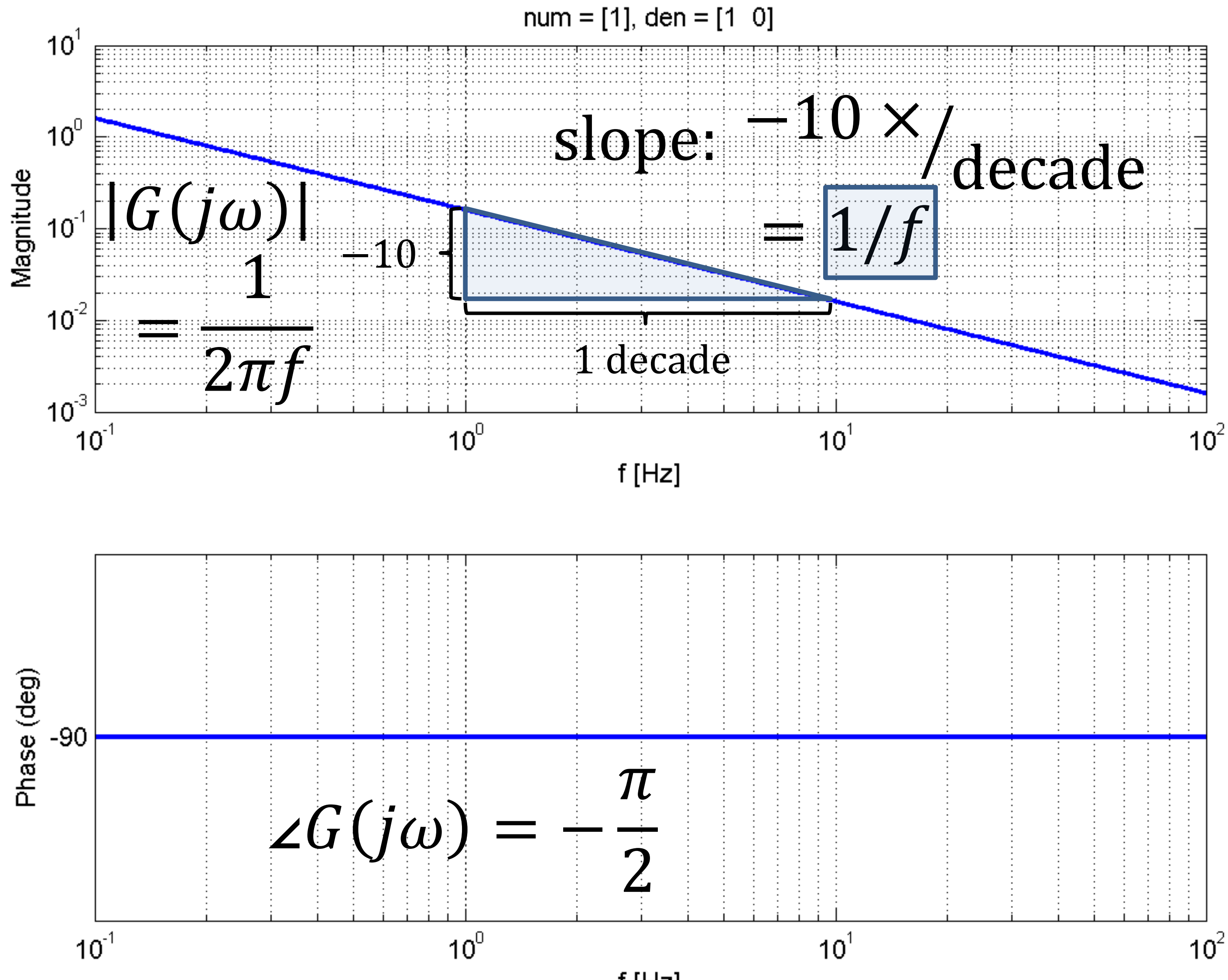
$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega}$$

$$|G(j\omega)| = GG^* = \frac{1}{\omega}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathcal{I}m(G)}{\mathcal{R}e(G)}\right) = -\frac{\pi}{2}$$

$$|G(j\omega)| = \frac{1}{\omega} \qquad \angle G(j\omega) = -\frac{\pi}{2}$$

Bode plot: $G(s) = {}^{1}/_{s}$



bodeexamples.m

Bode plot: $G(s) = {}^{1}/_{s^2}$

$$G(s) = \frac{1}{s^2} \rightarrow$$

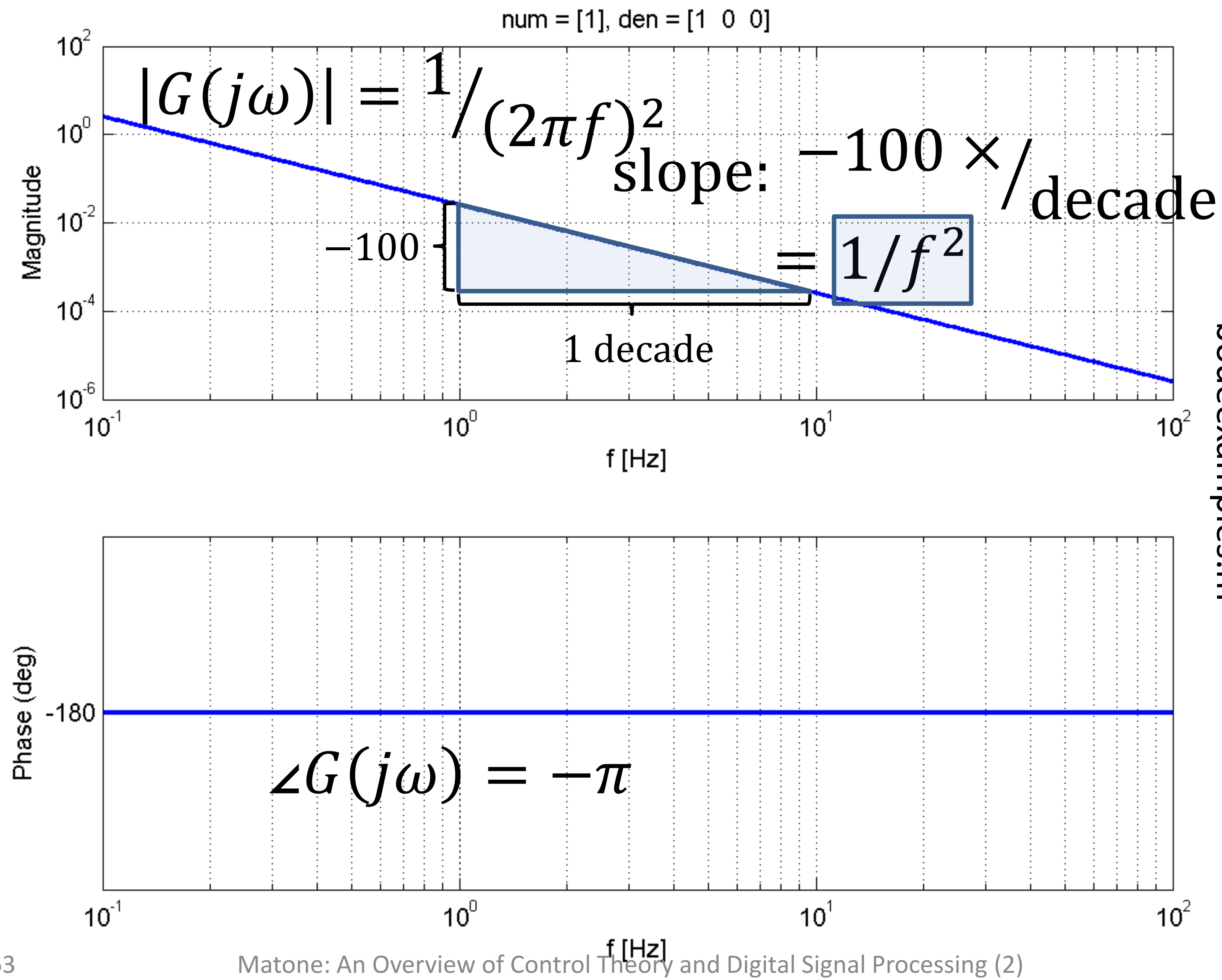
$$G(j\omega) = \frac{1}{(j\omega)^2} = -\frac{1}{\omega^2}$$

$$|G(j\omega)| = GG^* = \frac{1}{\omega^2}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{Im(G)}{Re(G)}\right) = -\pi$$

$$|G(j\omega)| = \frac{1}{\omega^2} \quad \angle G(j\omega) = -\pi$$

Bode plot: $G(s) = {}^{1}/_{s^2}$



bodeexamples.m

Bode plot: $G(s) = s$

$$G(s) = s \rightarrow$$

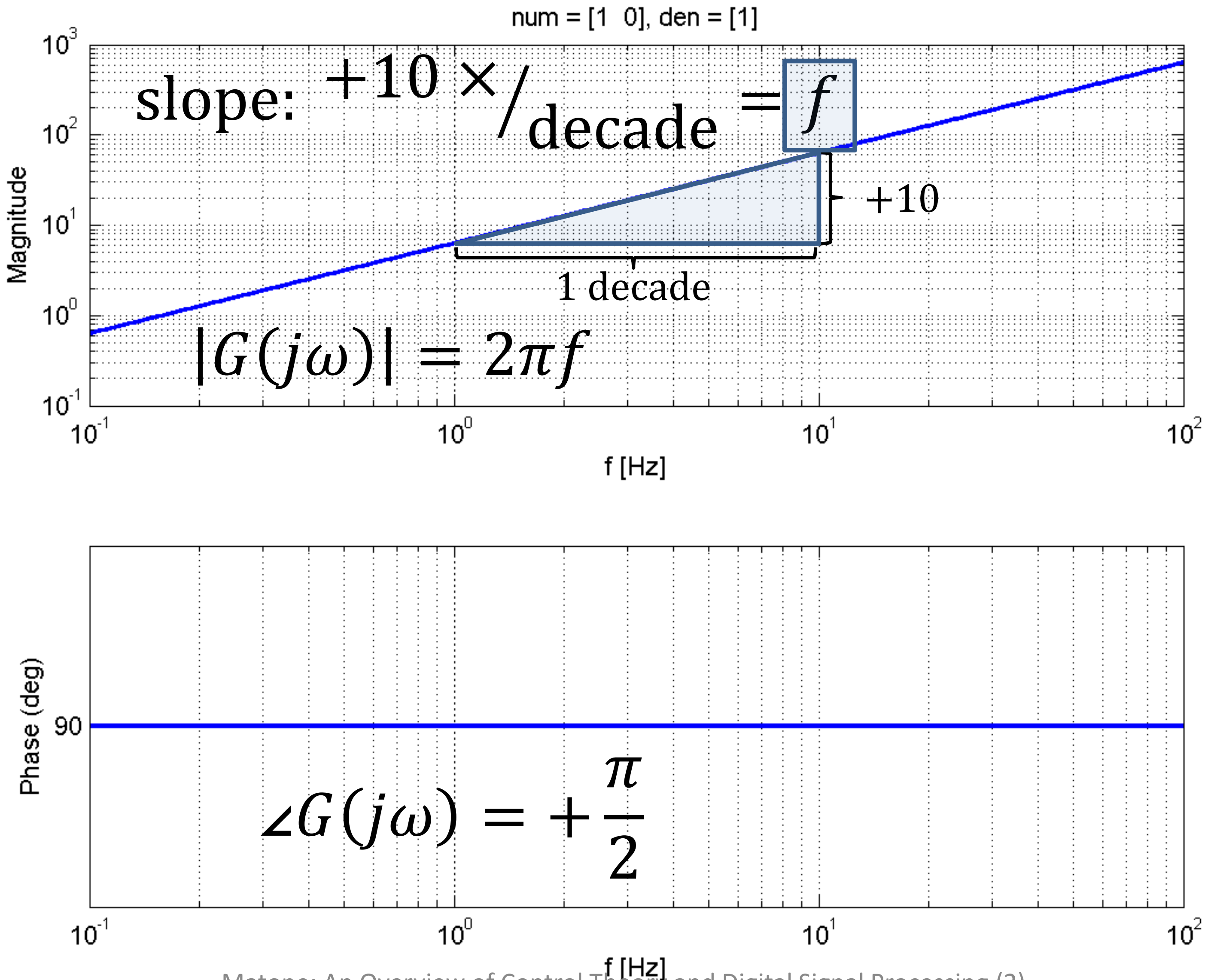
$$G(j\omega) = j\omega$$

$$|G(j\omega)| = GG^* = \omega$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathcal{Im}(G)}{\mathcal{Re}(G)}\right) = \frac{\pi}{2}$$

$$|G(j\omega)| = \omega \qquad \angle G(j\omega) = \frac{\pi}{2}$$

Bode plot: $G(s) = s$



bodeexamples.m

Bode plot: $G(s) = s^2$

$$G(s) = s^2 \rightarrow$$

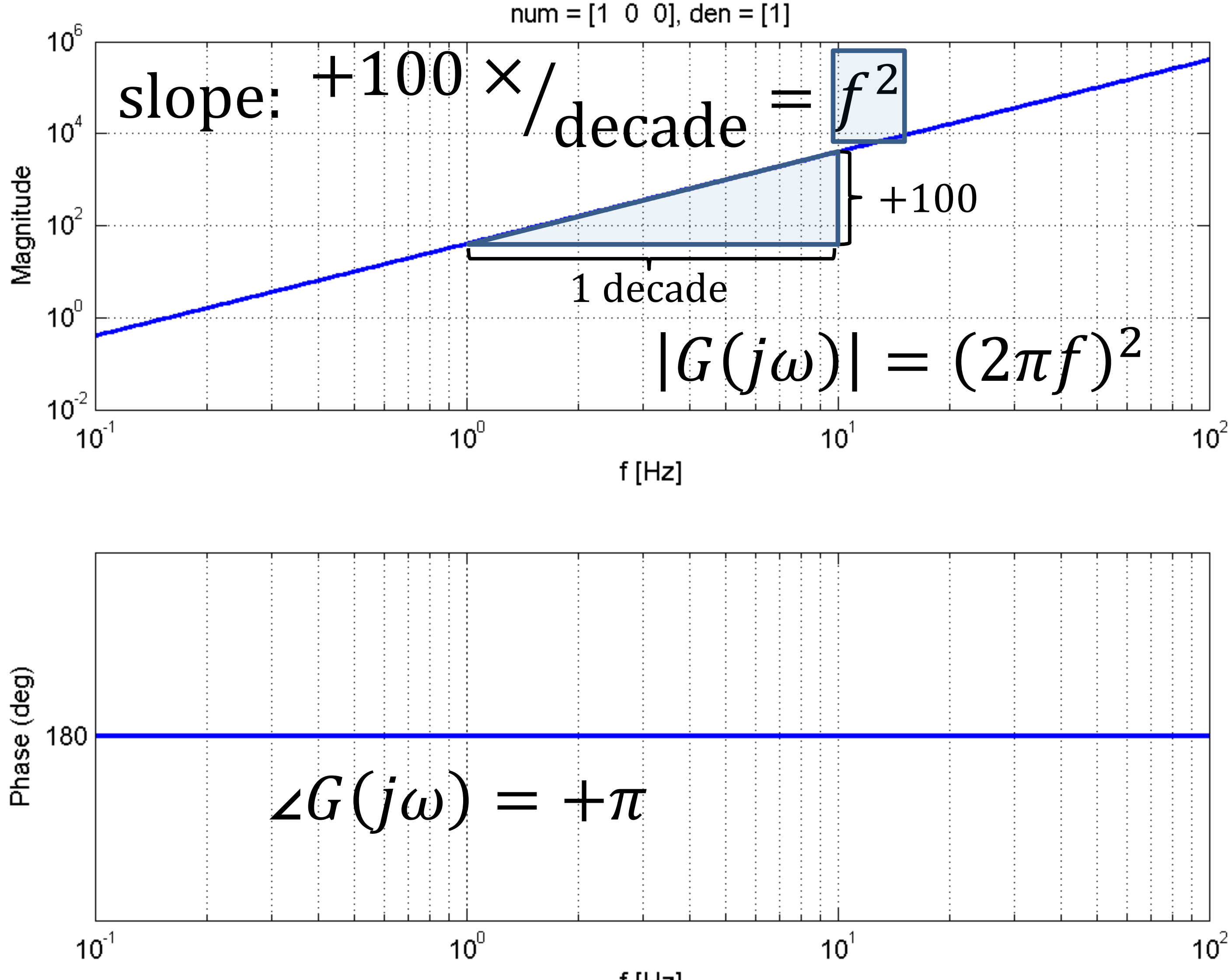
$$G(j\omega) = (j\omega)^2 = -\omega^2$$

$$|G(j\omega)| = GG^* = \omega^2$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathcal{Im}(G)}{\mathcal{Re}(G)}\right) = +\pi$$

$$|G(j\omega)| = \omega^2 \qquad \angle G(j\omega) = +\pi$$

Bode plot: $G(s) = s^2$



bodeexamples.m

Bode plot: $G(s) = {}^{a}/_{(s+a)}$

$$G(s) = \frac{a}{s+a}$$

$$G(j\omega) = \frac{a}{j\omega + a}$$

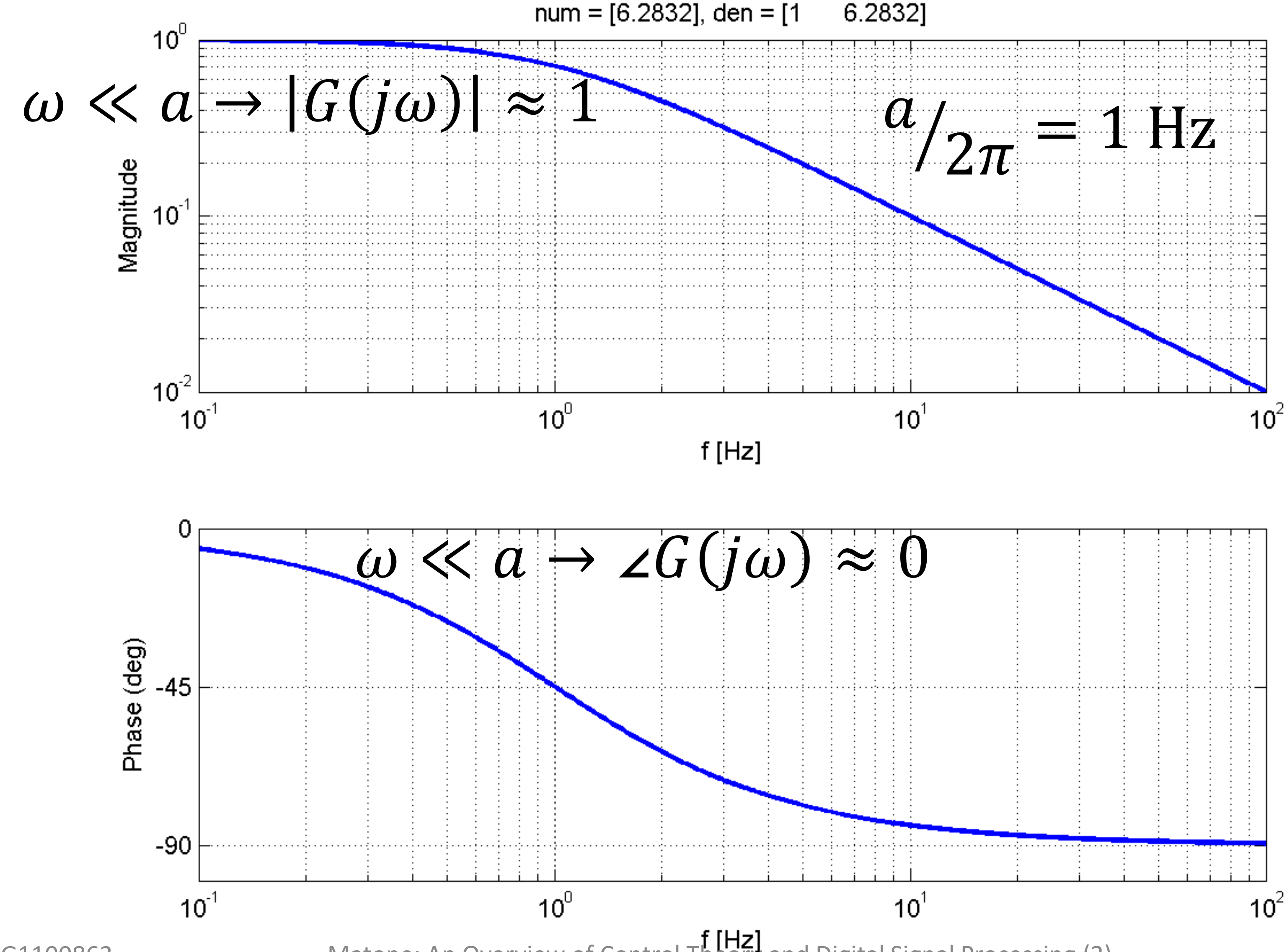
$$|G(j\omega)| = GG^* = \frac{a}{\sqrt{\omega^2 + a^2}}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathcal{Im}(G)}{\mathcal{Re}(G)}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$|G(j\omega)| = \frac{a}{\sqrt{\omega^2 + a^2}}$$

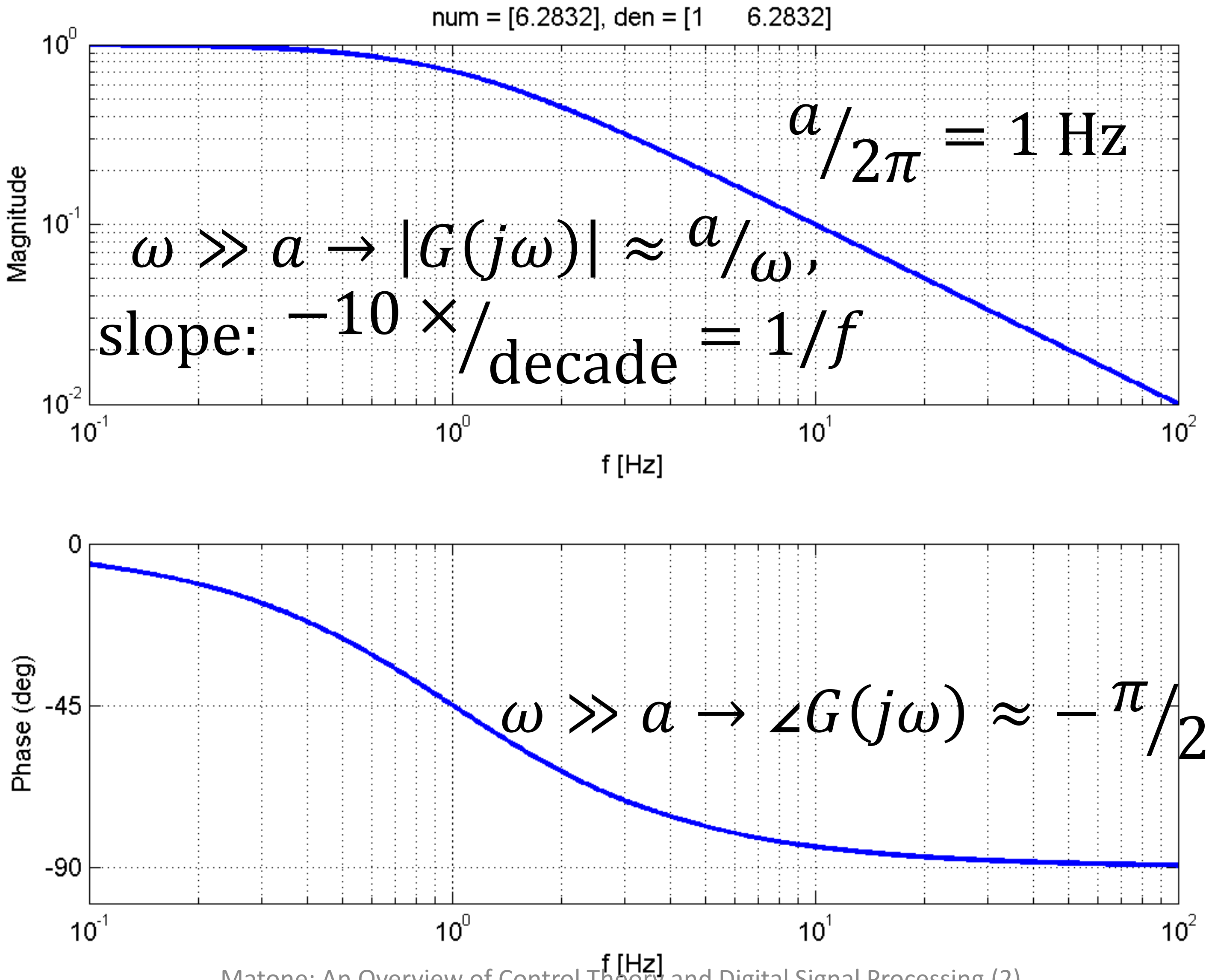
$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Bode plot: $G(s) = {}^{a}/_{(s+a)}$



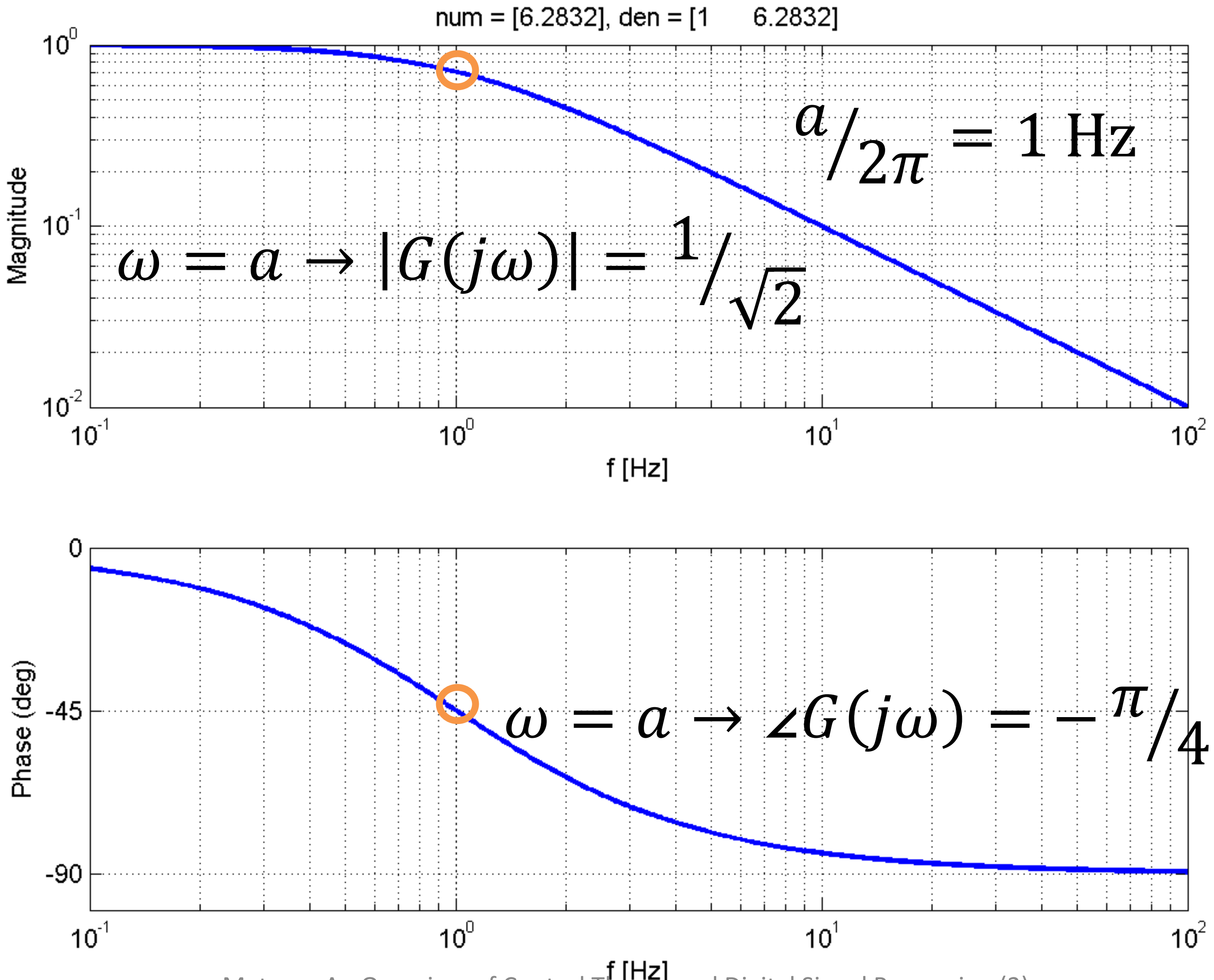
bodeexamples.m

Bode plot: $G(s) = {}^{a}/_{(s+a)}$



bodeexamples.m

Bode plot: $G(s) = {}^{a}/_{(s+a)}$



bodeexamples.m

Bode plot: SHO

$$G(s) = \frac{(\omega_0)^2}{(s^2 + 2\delta\omega_0 \cdot s + (\omega_0)^2)}$$

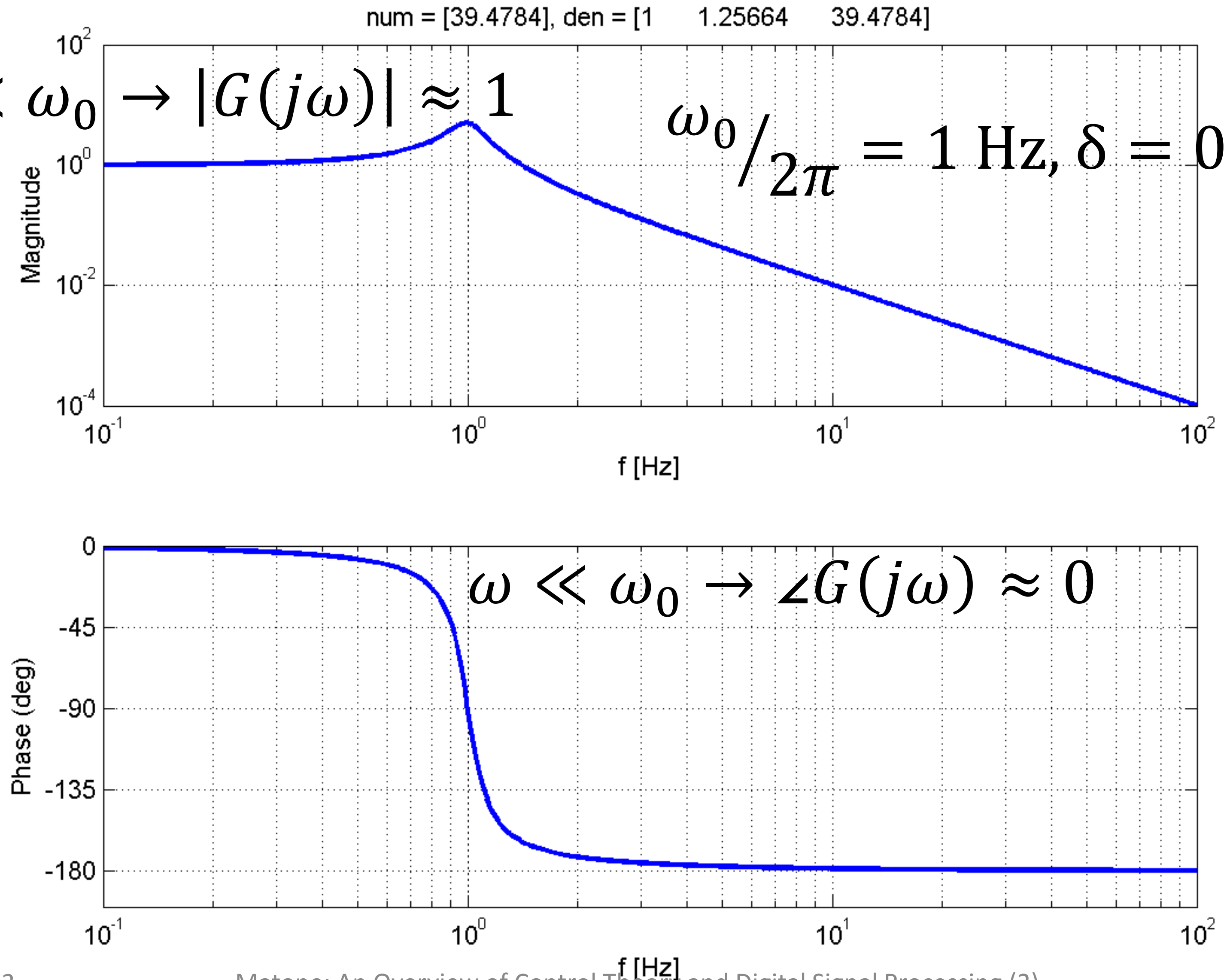
$$|G(j\omega)| = \frac{(\omega_0)^2}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (2\delta\omega_0\omega)^2}}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{2\delta\omega_0\omega}{(\omega_0)^2 - \omega^2}\right)$$

Bode plot: $G(s) = \frac{(\omega_0)^2}{(s^2+2\delta\omega_0 \cdot s+(\omega_0)^2)}$

$\omega \ll \omega_0 \rightarrow |G(j\omega)| \approx 1$

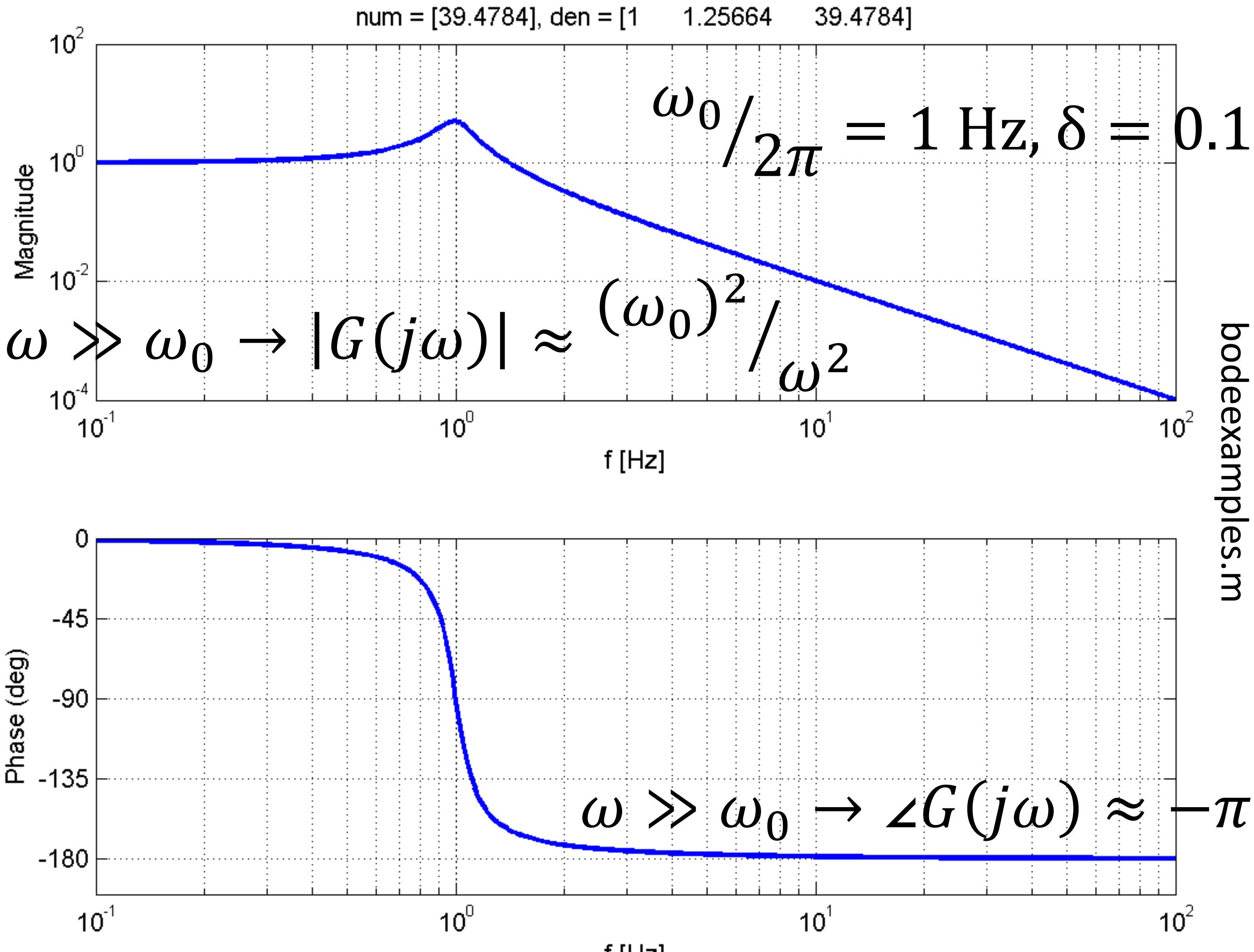
$\omega_0/_{2\pi} = 1\,\mathrm{Hz}, \delta = 0.1$



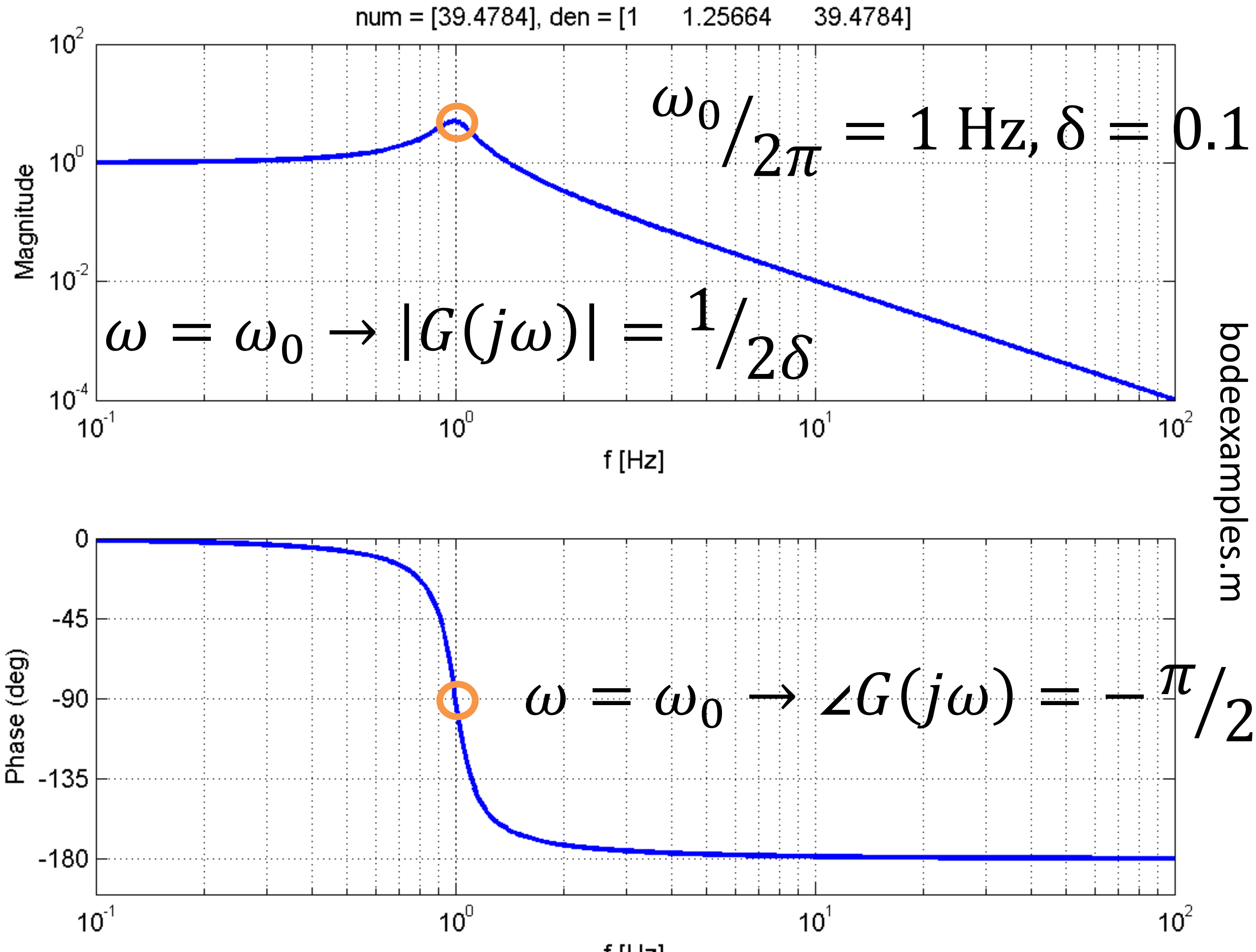
$\omega \ll \omega_0 \rightarrow \angle G(j\omega) \approx 0$

bodeexamples.m

Bode plot: $G(s) = \frac{(\omega_0)^2}{(s^2+2\delta\omega_0 \cdot s+(\omega_0)^2)}$



Bode plot: $G(s) = \frac{(\omega_0)^2}{(s^2+2\delta\omega_0 \cdot s+(\omega_0)^2)}$



bodeexamples.m

Bode plots for more complicated TFs? Break it into simpler parts

In general a transfer function $G(s)$ can be re-written in terms of simpler ones (of 2nd order at most)

$$G(s) = G_1(s) \cdot G_2\,(s) \cdots G_n\,(s)$$

then

$$|G(j\omega)| = |G_1(j\omega)| \cdot |G_2(j\omega)| \cdot \cdots \cdot |G_n(j\omega)|$$

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \cdots + \angle G_n(j\omega)$$

Example

Let's draw the bode plot for

$$G(s) = k\frac{(s+\omega_0)}{(s+\omega_1)(s+\omega_2)}$$

where

$$k = 500$$
$$\omega_0 = 2\pi \cdot 0.1 \text{ Hz}$$
$$\omega_1 = 2\pi \cdot 1 \text{ Hz}$$
$$\omega_2 = 2\pi \cdot 10 \text{ Hz}$$

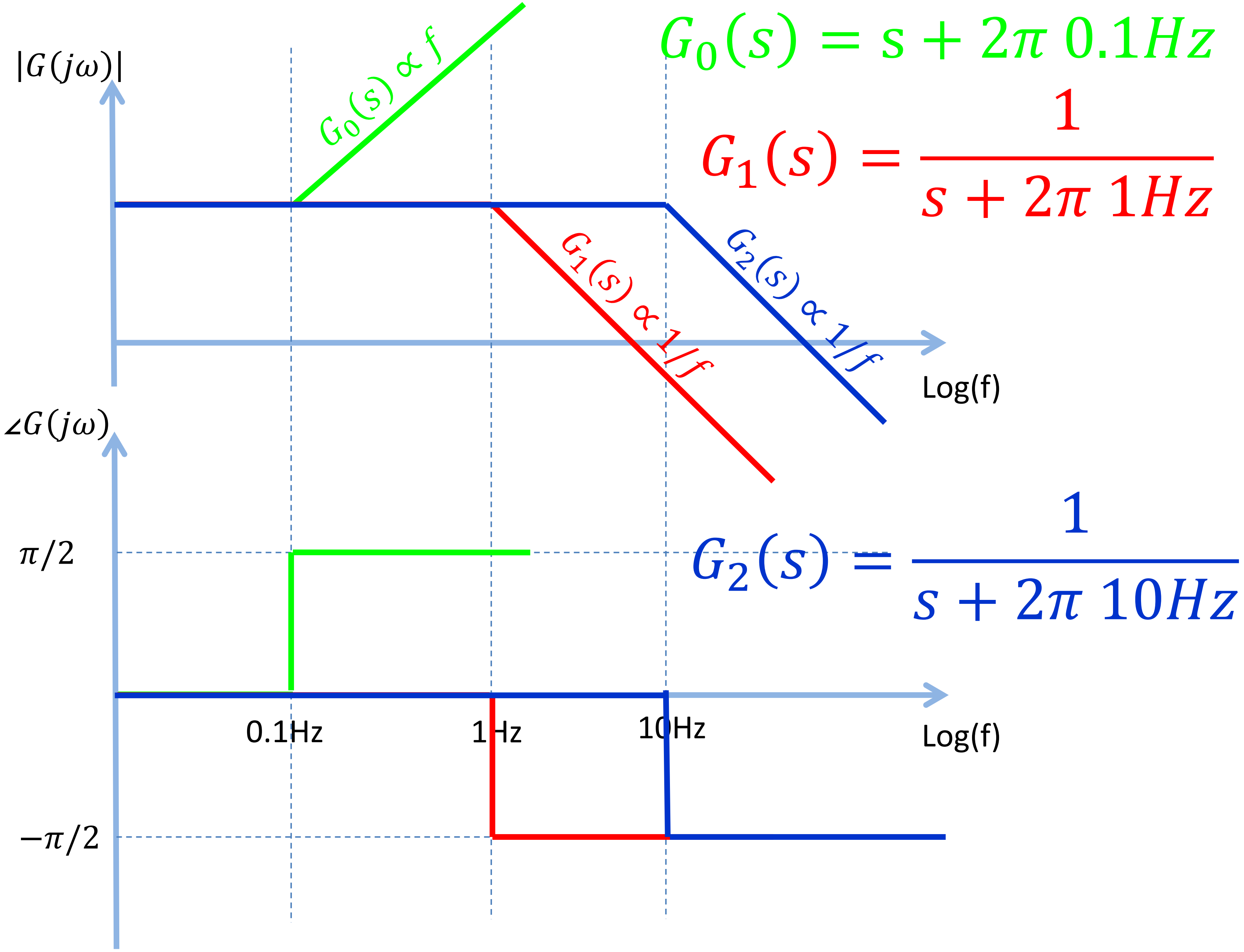
Example

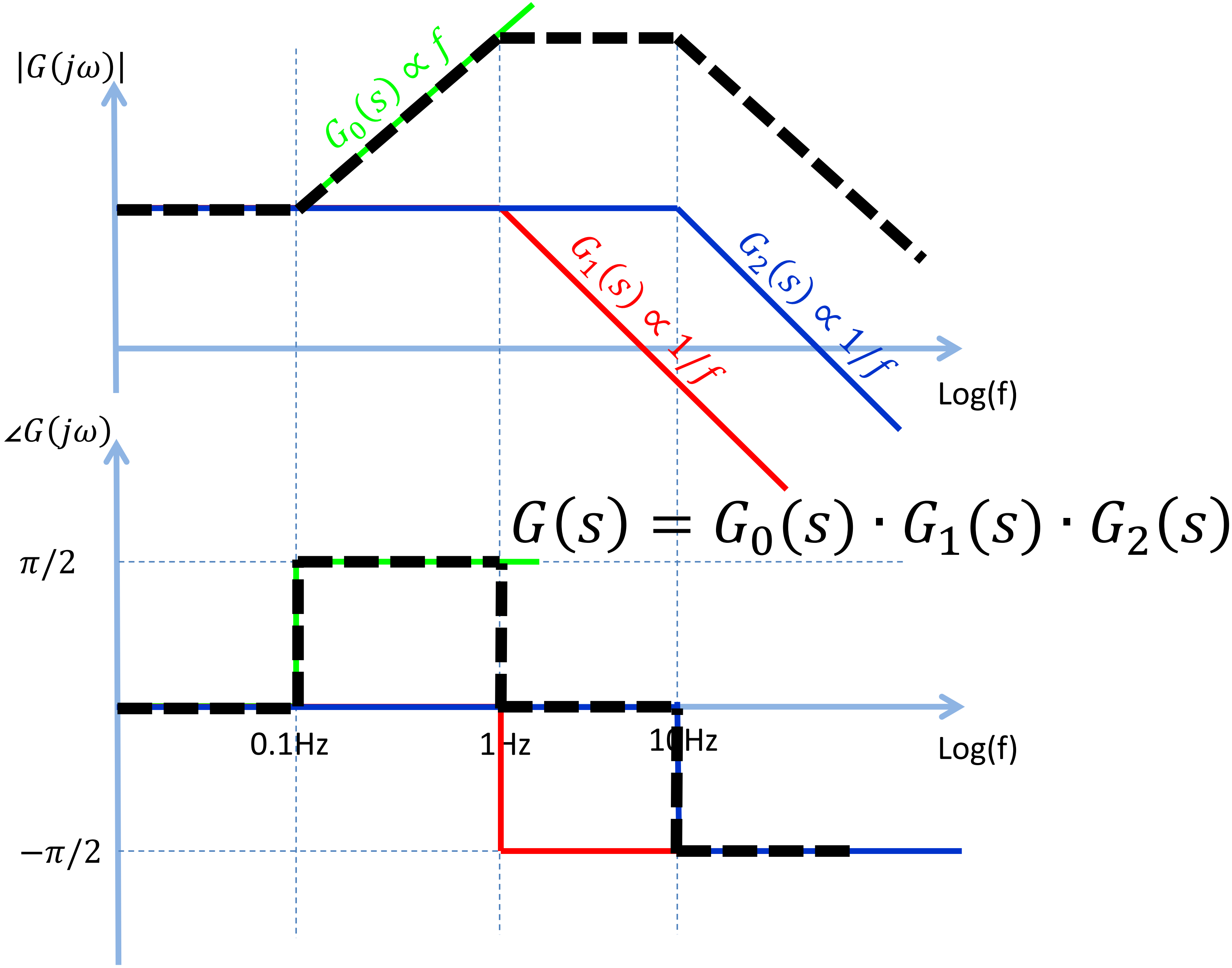
$$G(s) = k\frac{(s+\omega_0)}{(s+\omega_1)(s+\omega_2)}$$

$$= k \cdot G_0(s) \cdot G_1(s) \cdot G_2(s) \quad \begin{cases} G_0(s) = s + \omega_0 \\ G_1(s) = \dfrac{1}{s+\omega_1} \\ G_2(s) = \dfrac{1}{s+\omega_2} \end{cases}$$

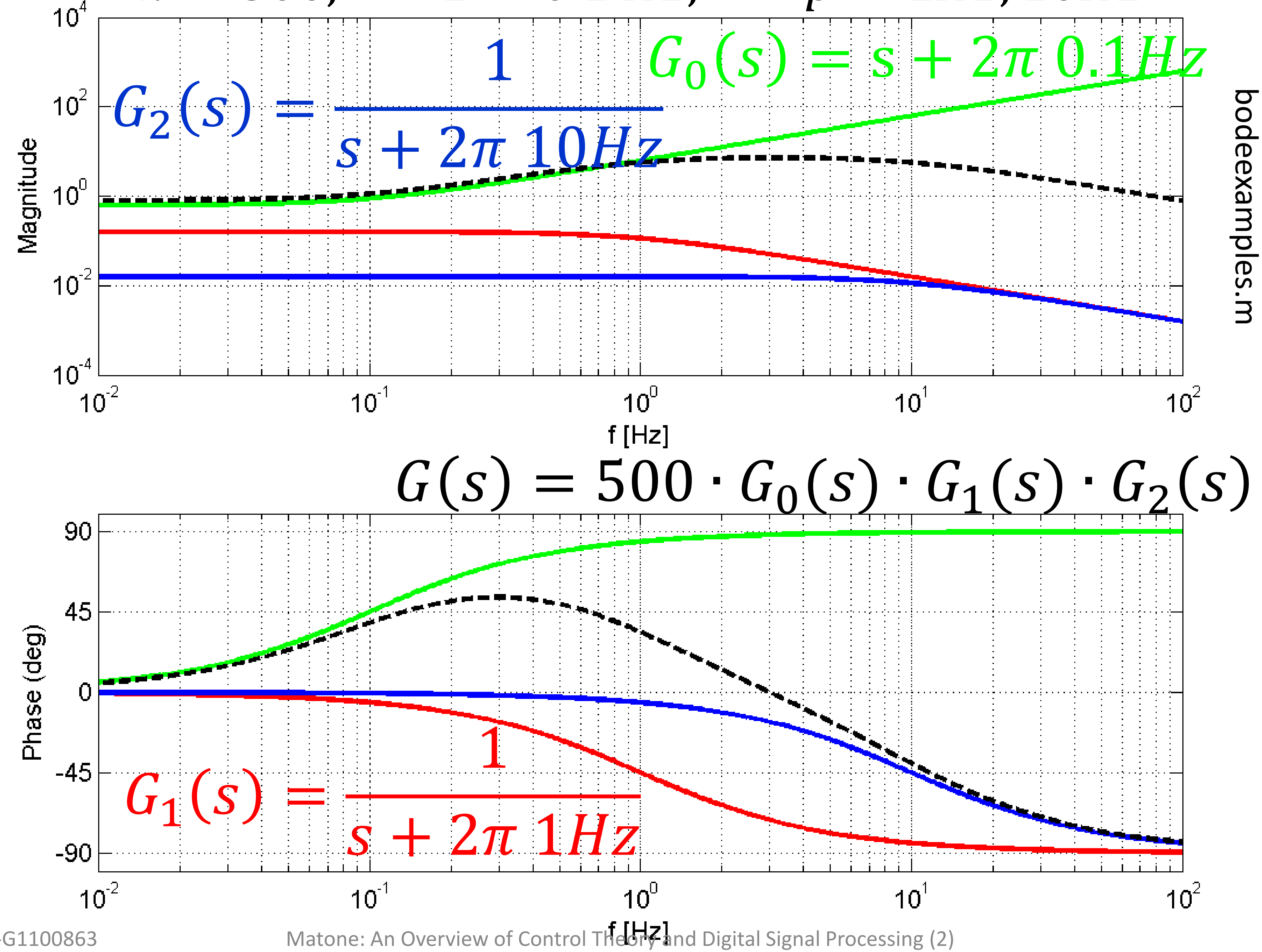
$$|G(j\omega)| = |G_0(j\omega)| \cdot |G_1(j\omega)| \cdot |G_2(j\omega)|$$

$$\angle G(j\omega) = \angle G_0(j\omega) + \angle G_1(j\omega) + \angle G_2(j\omega)$$

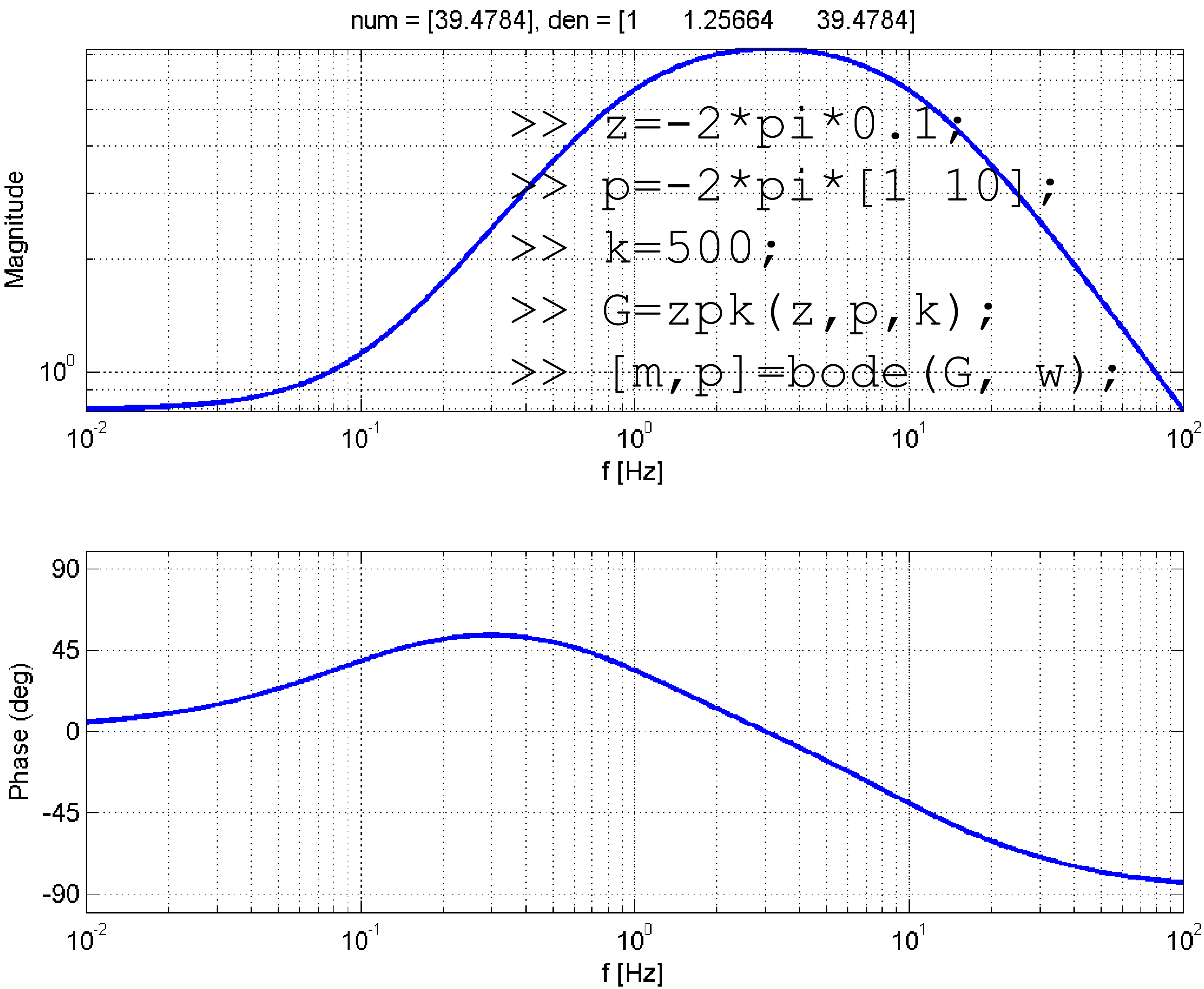

$|G(j\omega)|$
$G_0(s) \propto f$
$G_1(s) \propto 1/f$
$G_2(s) \propto 1/f$
Log(f)
$\angle G(j\omega)$
$\pi/2$
$-\pi/2$
0.1Hz
1 Hz
10Hz
Log(f)
$G_0(s) = s + 2\pi\, 0.1Hz$
$G_1(s) = \frac{1}{s + 2\pi\, 1Hz}$
$G_2(s) = \frac{1}{s + 2\pi\, 10Hz}$


$|G(j\omega)|$
$G_0(s) \propto f$
$G_1(s) \propto 1/f$
$G_2(s) \propto 1/f$
Log(f)
$\angle G(j\omega)$
$G(s) = G_0(s) \cdot G_1(s) \cdot G_2(s)$
$\pi/2$
$-\pi/2$
0.1Hz
1 Hz
10Hz
Log(f)

$k = 500, \quad z = 0.1\,Hz, \quad p = 1Hz, 10Hz$



$$k = 500, \qquad z = 0.1\,Hz, \qquad p = 1Hz, 10Hz$$



bodeexamples.m

Exercise

Sketch bode plot for the following TF

$$G(s) = 100\frac{s+50}{s+100}$$

What is the DC gain (gain for $\omega \to 0$)? What is the gain for $\omega \to \infty$? Confirm results with MATLAB

Exercise

Sketch bode plot for the following TF

$$G(s) = 100\frac{(s+1)}{(s+10)(s+20)(s+30)}$$

What is the DC gain (gain for $\omega \rightarrow 0$)? What is the gain for $\omega \rightarrow \infty$? Confirm results with MATLAB

Exercise

Sketch bode plot for the following TF

$$G(s) = 30\frac{s + 30}{s^2 + 2s + 40}$$

What is the DC gain (gain for $\omega \to 0$)? What is the gain for $\omega \to \infty$? Confirm results with MATLAB

So far…

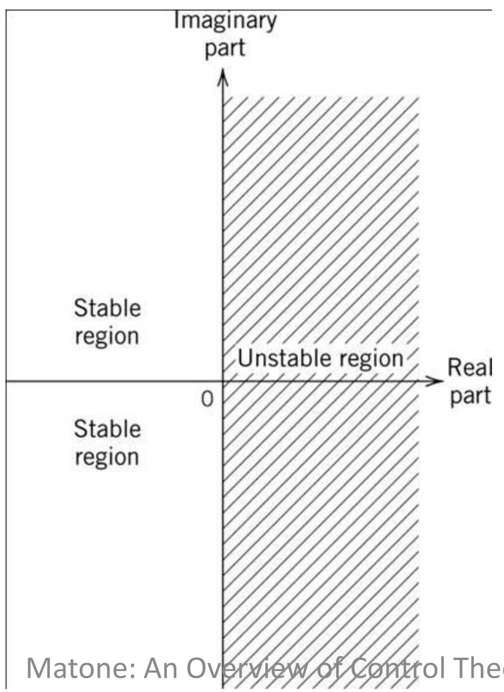
- A system's TF is a complex function
  - Can be represented in terms of its magnitude and phase
- Bode plots
  - Help visualize the TF
  - Plot of magnitude vs. frequency and phase vs. frequency.
  - Different conventions
- We have explored Bode plots of basic TFs
  - $\frac{1}{s}, \frac{1}{s^2}, s, s^2, \frac{a}{s+a}$ and SHO
- Bode plot of more complex TFs can be expressed in terms of simpler terms

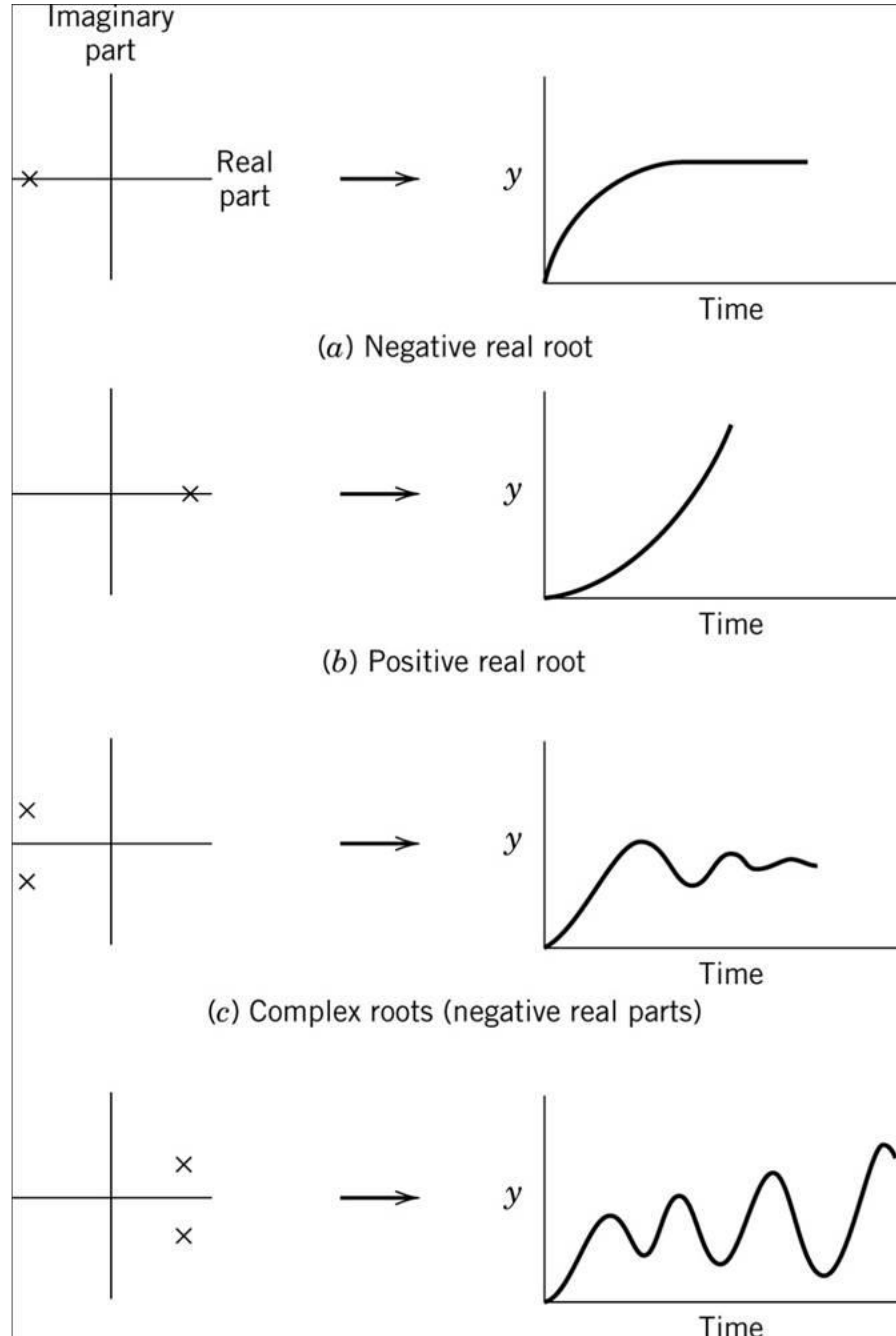
$$|G(j\omega)| = |G_1(j\omega)| \cdot |G_2(j\omega)| \cdot \cdots \cdot |G_n(j\omega)|$$

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \cdots + \angle G_n(j\omega)$$

General Stability Criterion

The feedback control system is <u>stable</u> if and only if all the *poles of the closed loop transfer function* G_{CL} have a negative real part. Otherwise the system is unstable.



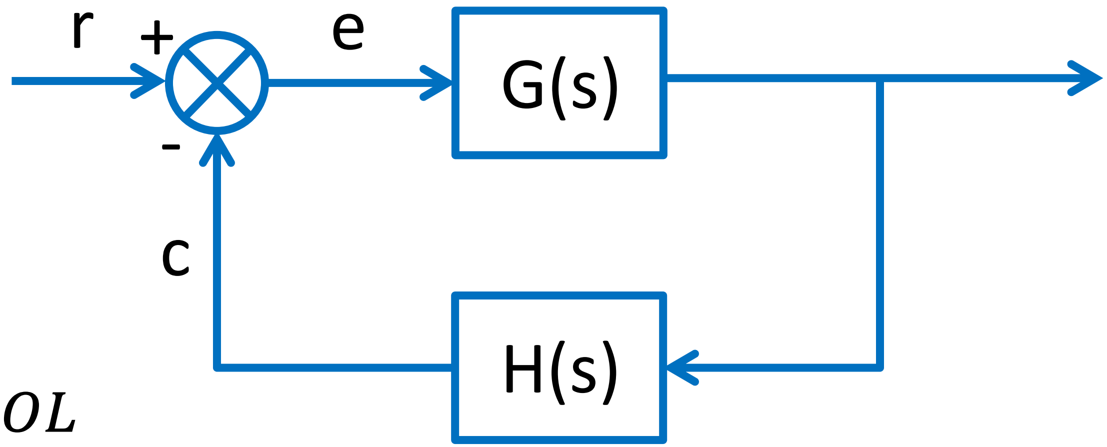


(a) Negative real root
(b) Positive real root
(c) Complex roots (negative real parts)
(d) Complex roots (positive real parts)

In general

$$\frac{e}{r} = \frac{1}{1 + GH}$$

Open loop gain G_{OL}



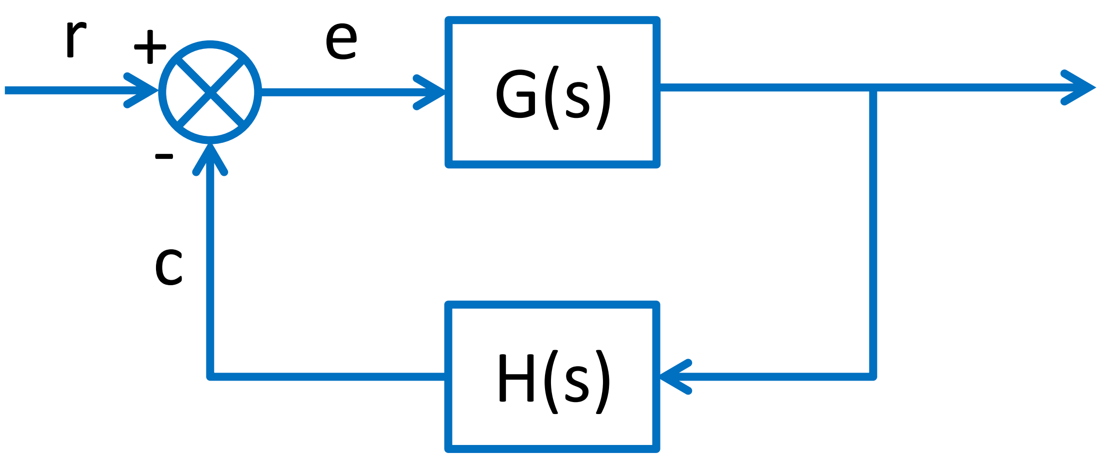
Stability: the poles' real part of G_{CL} must be negative

$$\frac{c}{r} = \frac{GH}{1 + GH}$$

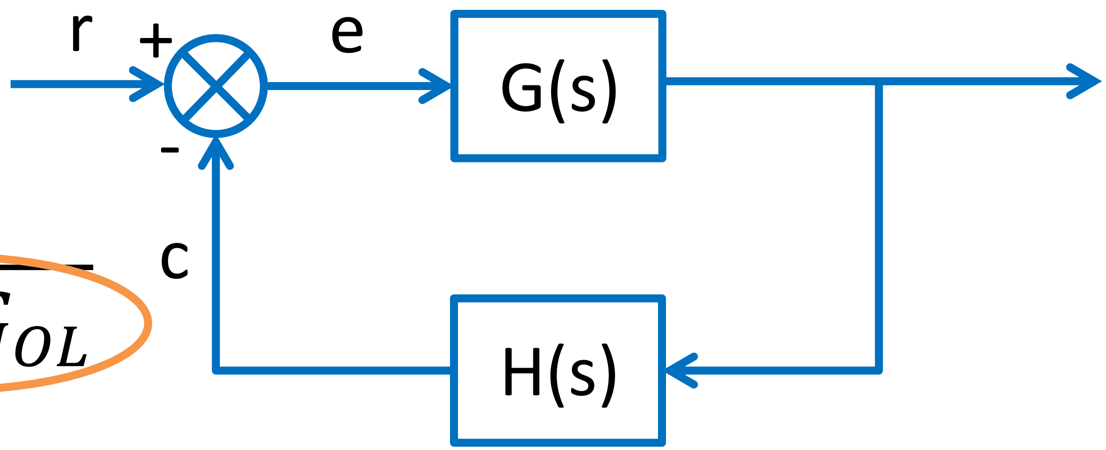
Closed loop gain G_{CL}

Loop stability and design

- If the system is unstable,
  - We can't change $G(s)$ but
  - We can design a different controller H so as to make the system stable
- But how should we change H? Let's look closely at the root of the problem



The problem



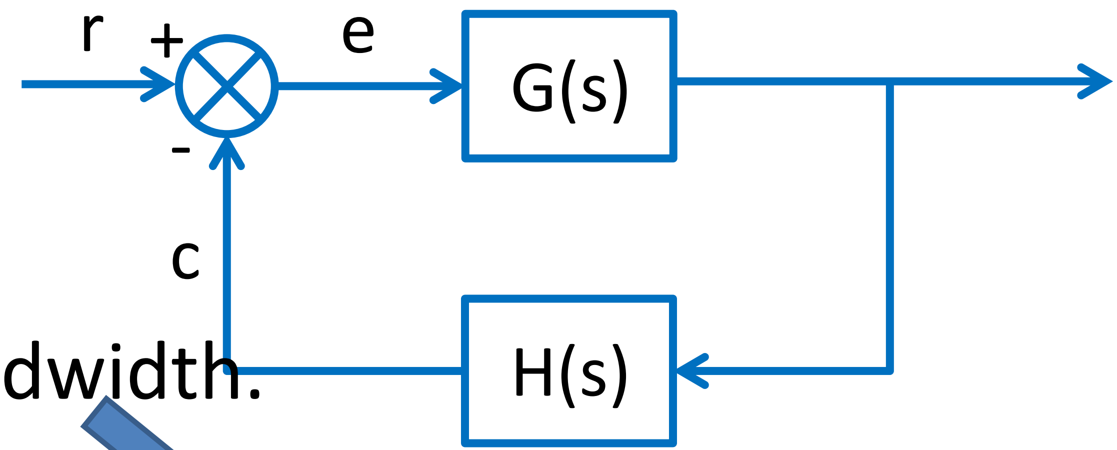
$$\frac{e}{r} = \frac{1}{1+GH} = \frac{1}{1+G_{OL}}$$

$$\frac{c}{r} = \frac{GH}{1+GH} = \frac{G_{OL}}{1+G_{OL}} = G_{CL}$$

If G_{OL} ever becomes -1 then system is unstable

The general shape of G_{OL}

$$\frac{e}{r} = \frac{1}{1 + G_{OL}}$$



G_{OL} has a limited bandwidth.

Within bandwidth:

$$G_{OL} \gg 1$$

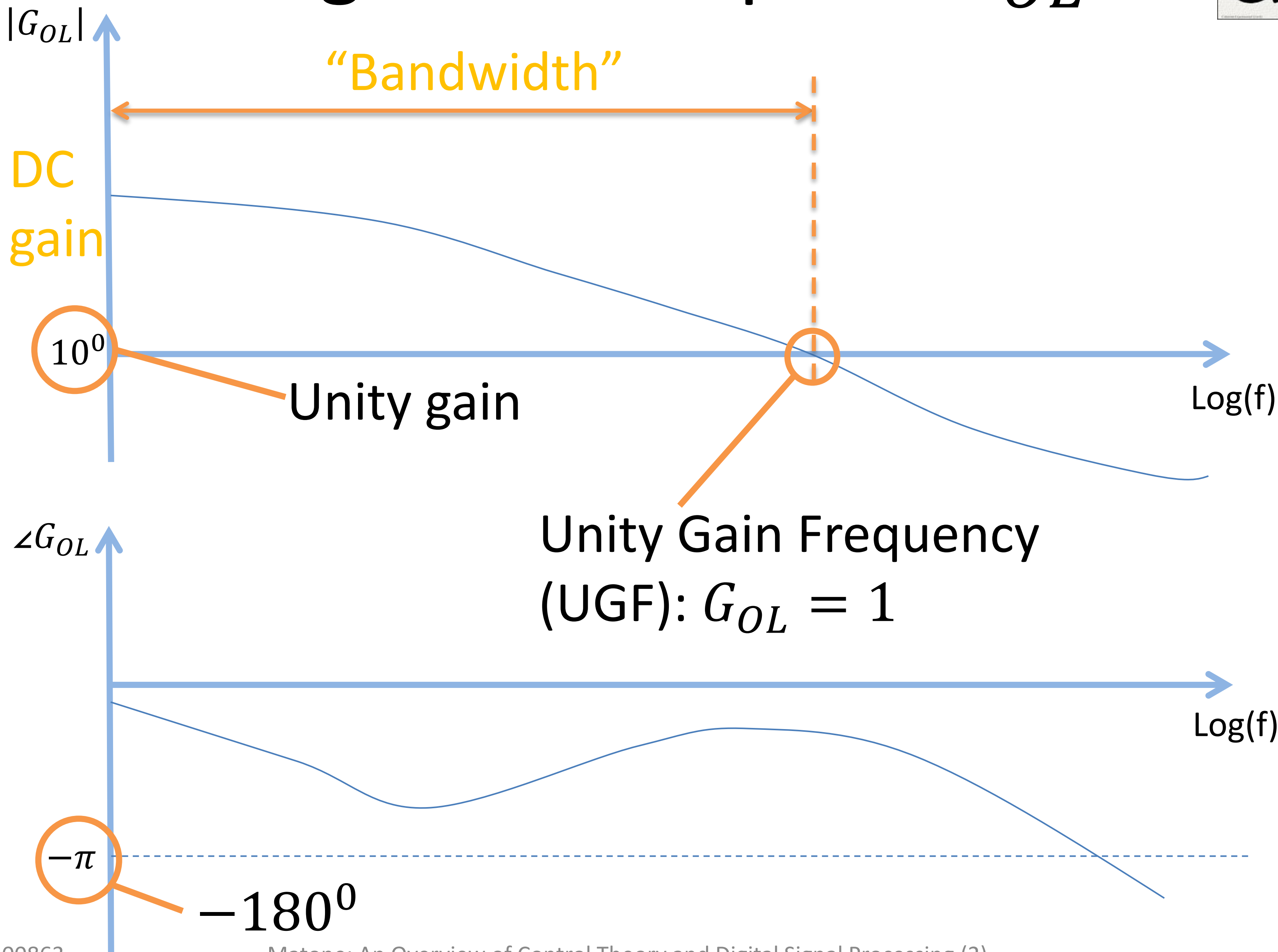
$$\frac{e}{r} \cong 0$$

Outside bandwidth:

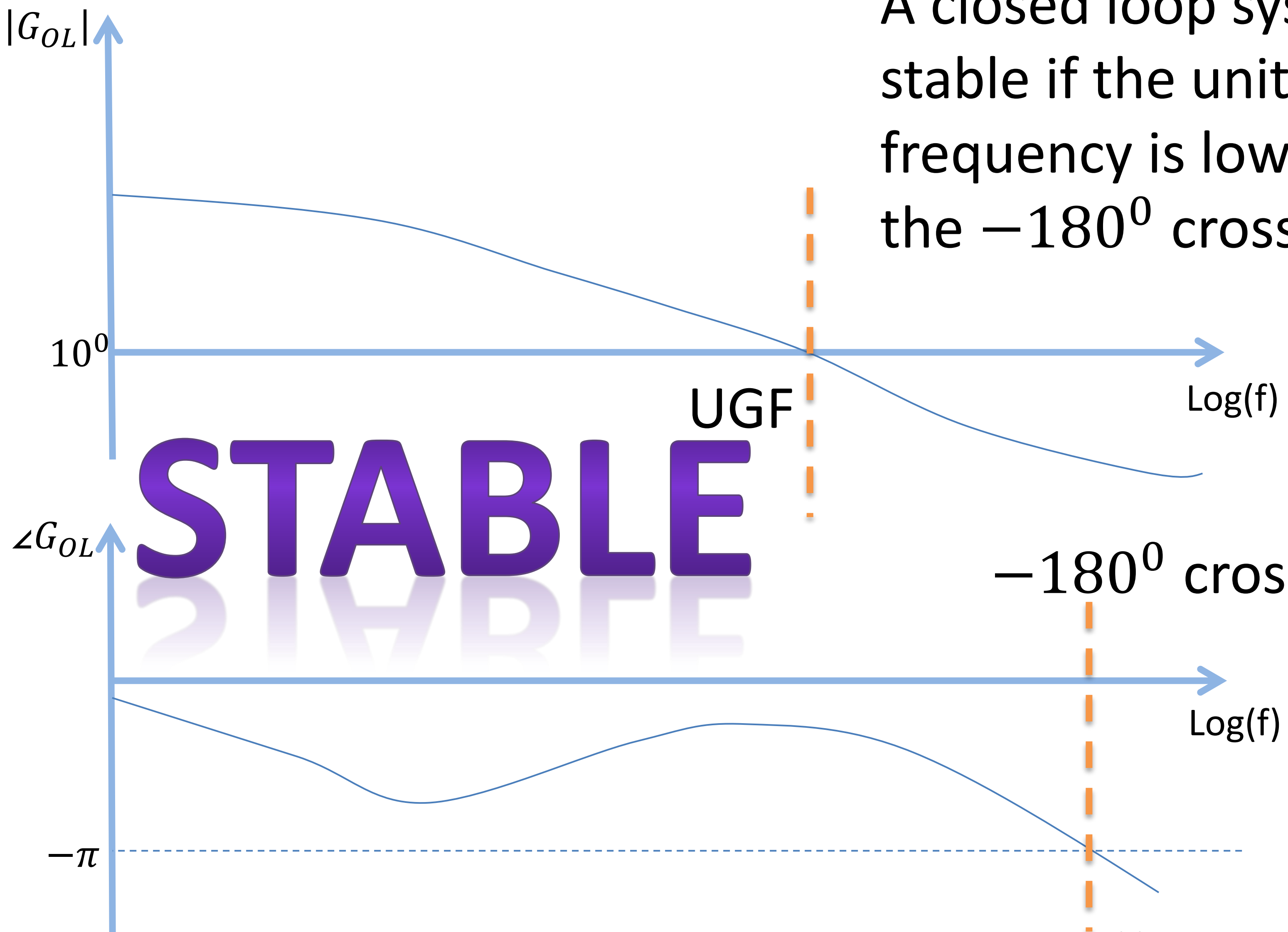
$$G_{OL} \ll 1$$

$$\frac{e}{r} \cong 1$$

The general shape of G_{OL}



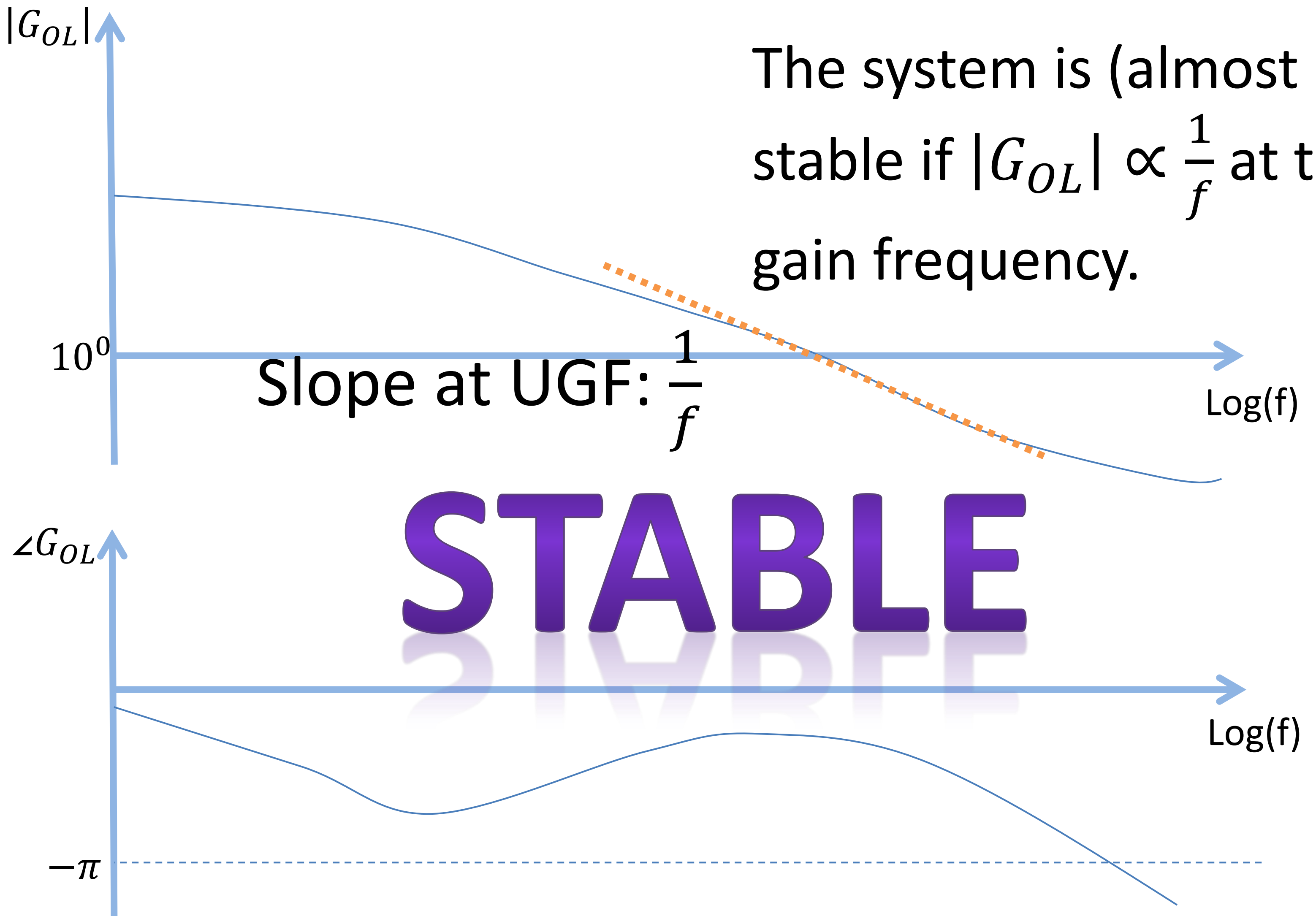
Stability Criteria



A closed loop system is stable if the unity gain frequency is lower than the -180^0 crossing.

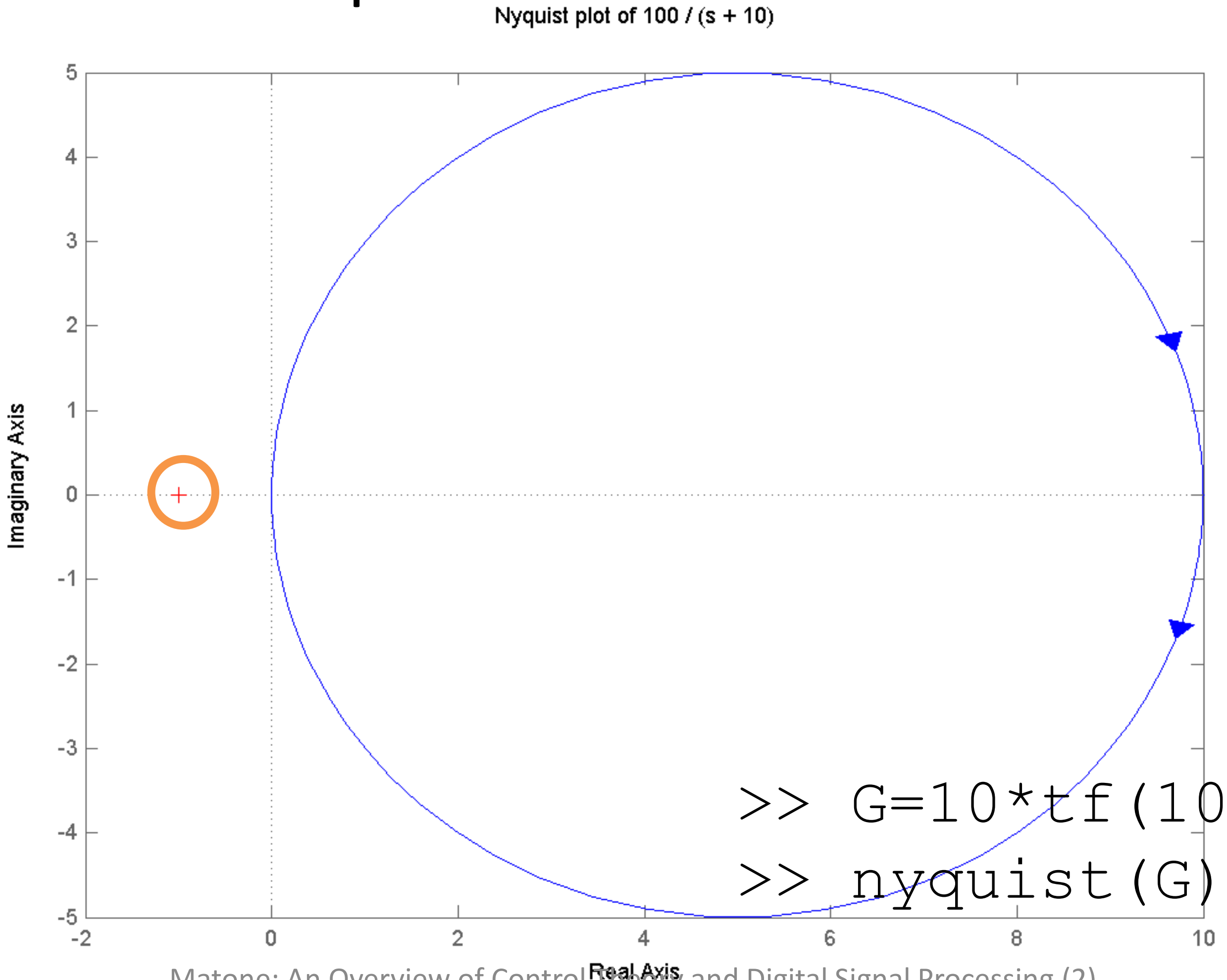
Stability Criteria: Rule of Thumb

The system is (almost always) stable if $|G_{OL}| \propto \frac{1}{f}$ at the unity gain frequency.



Nyquist stability criterion

The closed loop system is stable if the polar plot of the open loop transfer function $\left(\mathcal{Im}(G_{OL}) \text{ vs } \mathcal{Re}(G_{OL})\right)$ does not encircle the -1 point.

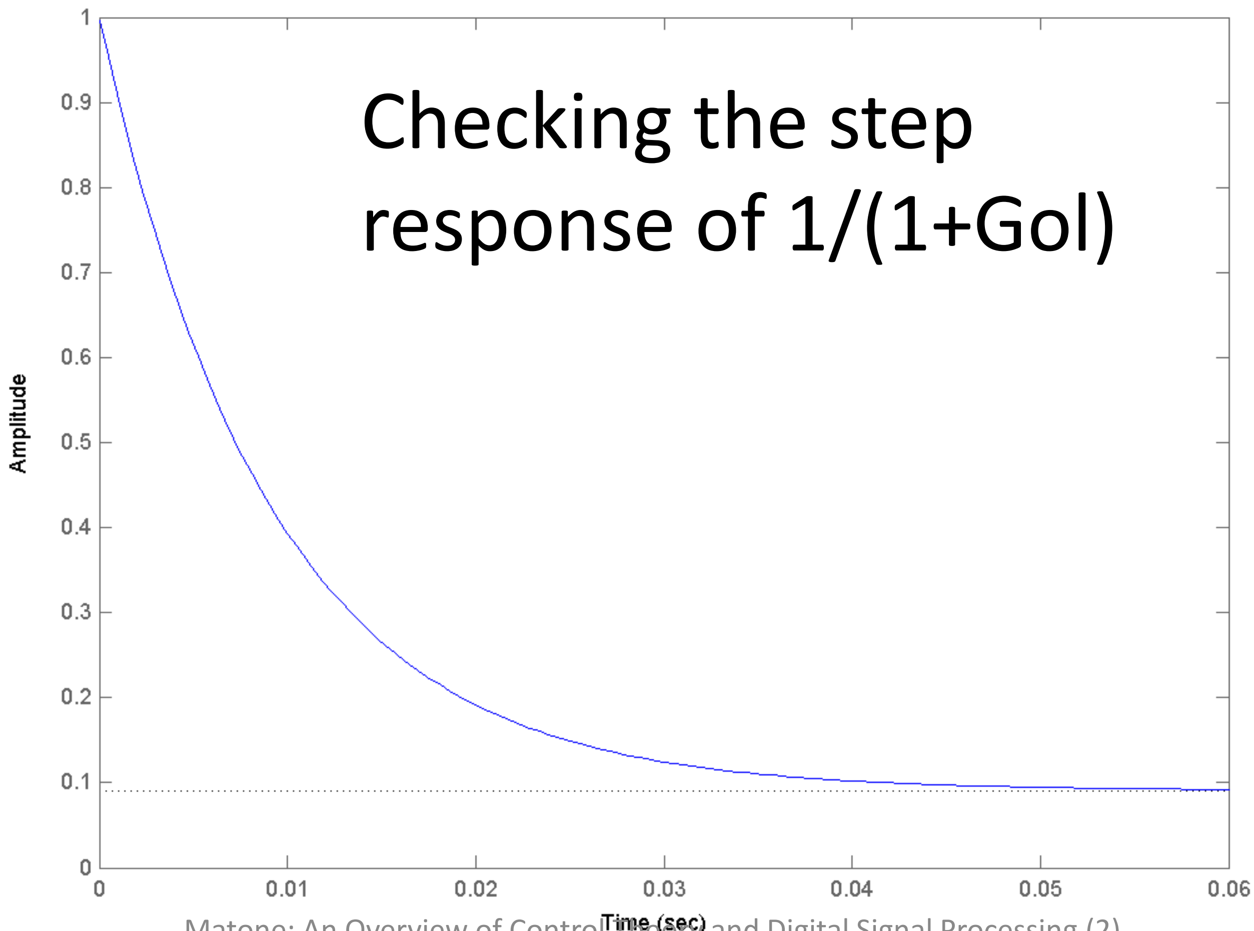



```
>> G=10*tf(10,[1 10];
>> nyquist(G)
```

nyquist_example1.m

Nyquist stability criterion

The closed loop system is stable if the polar plot of the open loop transfer function $\left(\mathcal{Im}(G_{OL}) \text{ vs } \mathcal{Re}(G_{OL})\right)$ does not encircle the -1 point.



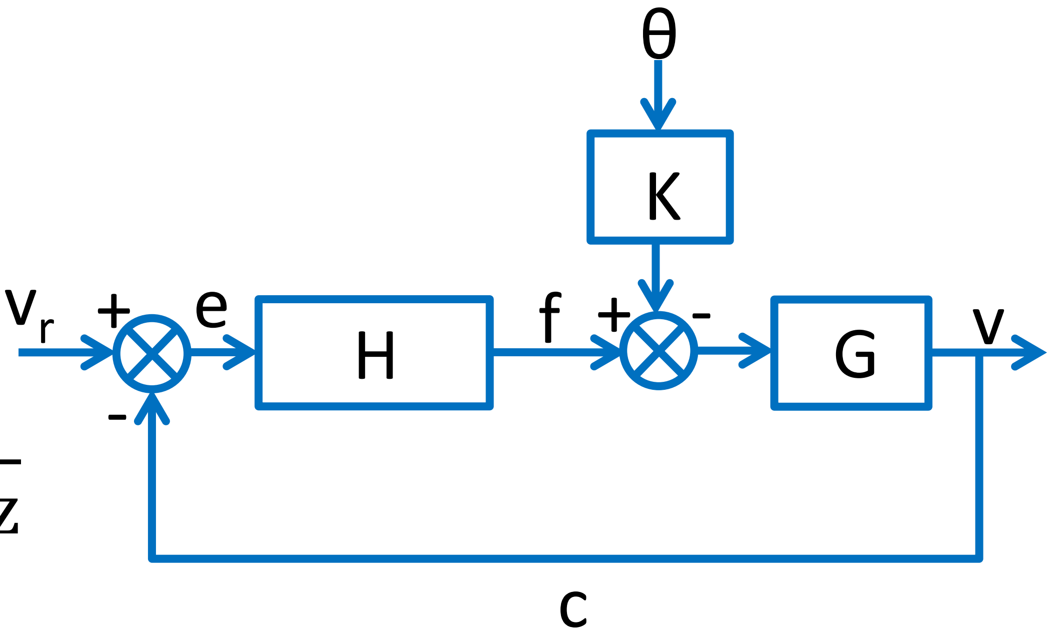
nyquist_example1.m

Back to cruise control

Let's inspect the system's loop stability. Recall

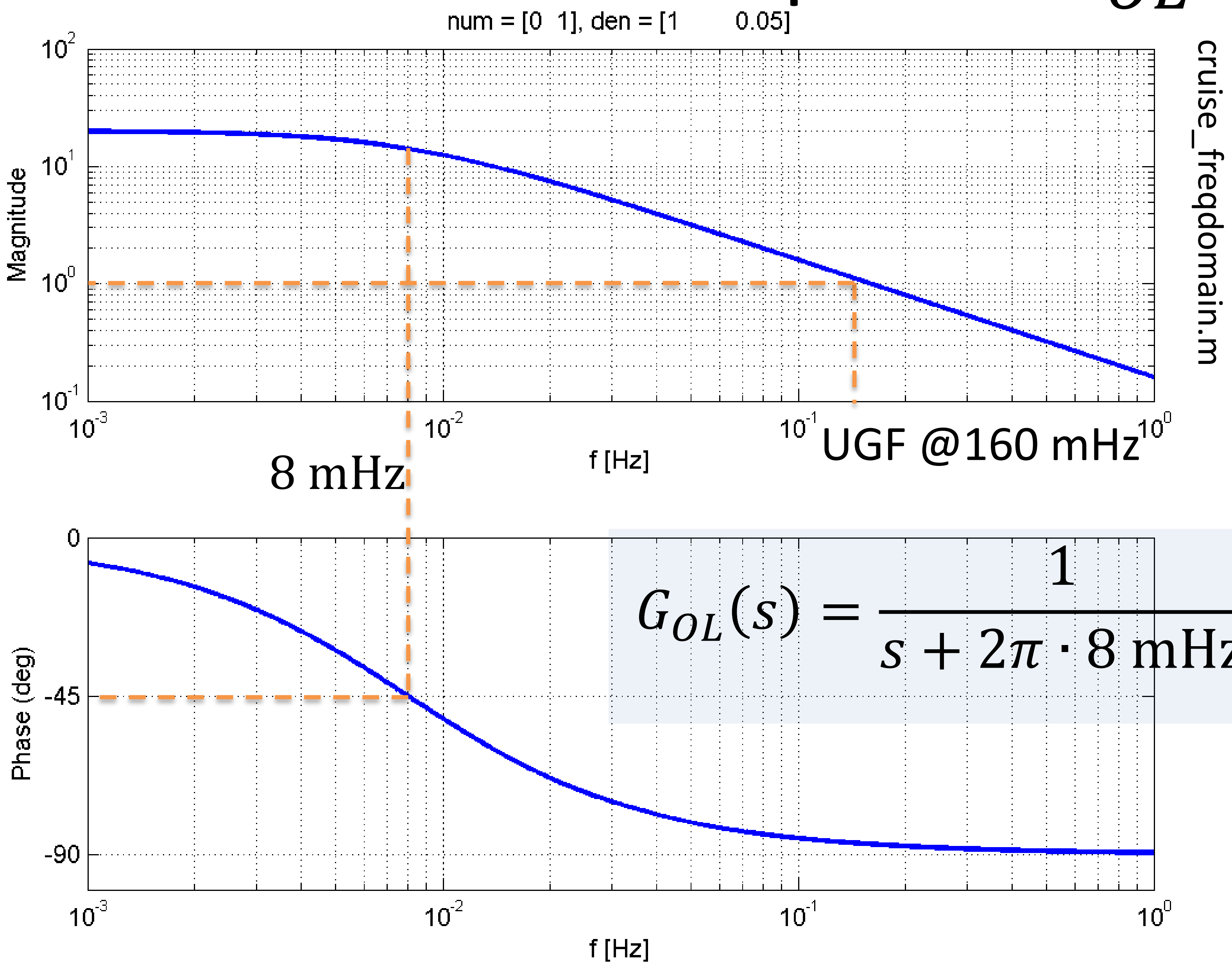
- $H = 1000 \, {}^{N}\!/_{(m/s)}$
- $G = \frac{1/m}{s+b/m}$
- Mass m = 1000 kg
- Coefficient for air friction b = 50 kg/s

$$e = \frac{1}{1 + G \cdot H} v_r + \frac{K \cdot G}{1 + G \cdot H} \theta$$



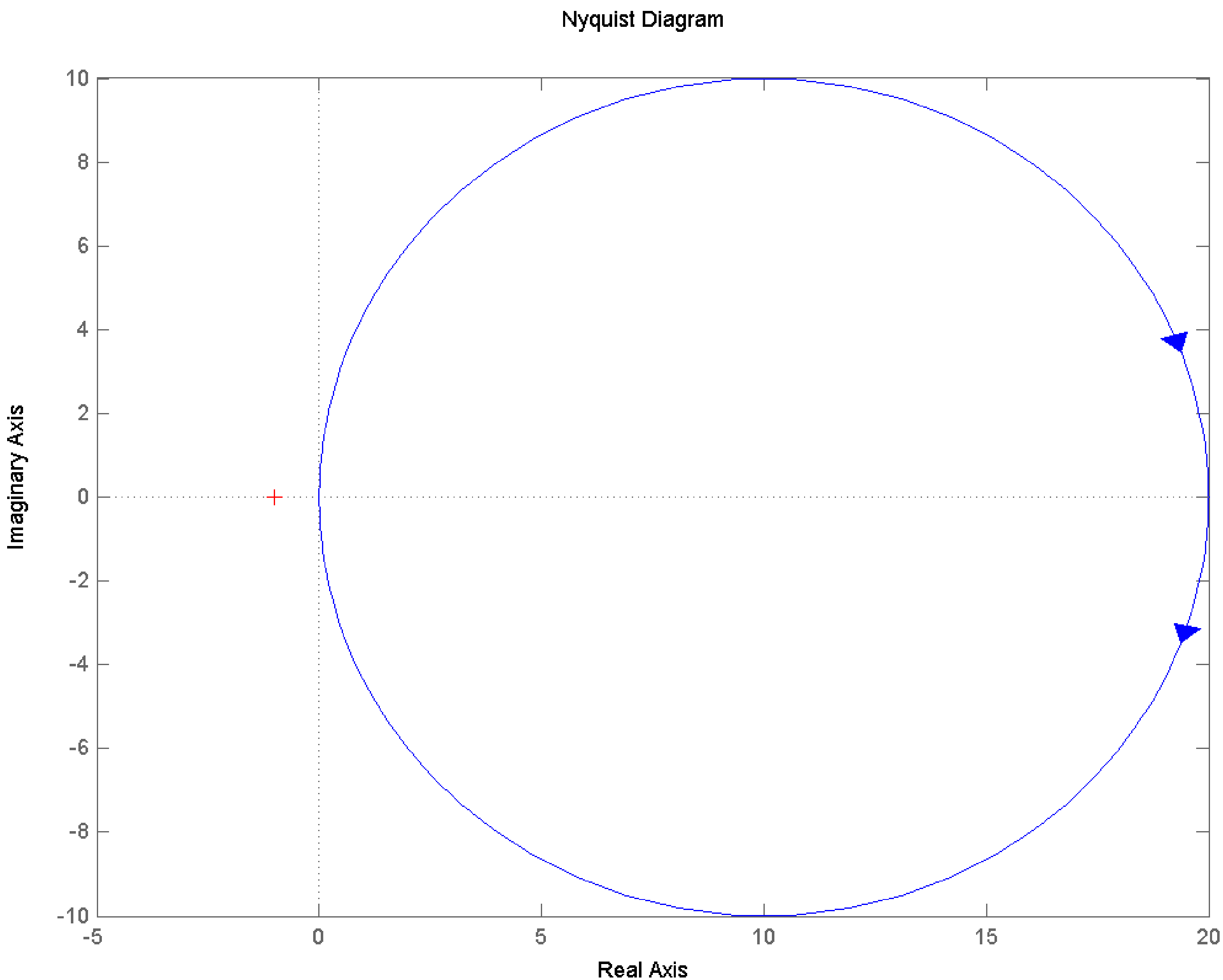
$$G_{OL}(s) = \frac{1}{s + 2\pi \cdot 8 \text{ mHz}}$$

Cruise control: Bode plot of G_{OL}



$$G_{OL}(s) = \frac{1}{s + 2\pi \cdot 8\ \text{mHz}}$$

Cruise control: Nyquist plot of G_{OL}



cruise_freqdomain.m

G_{CL}

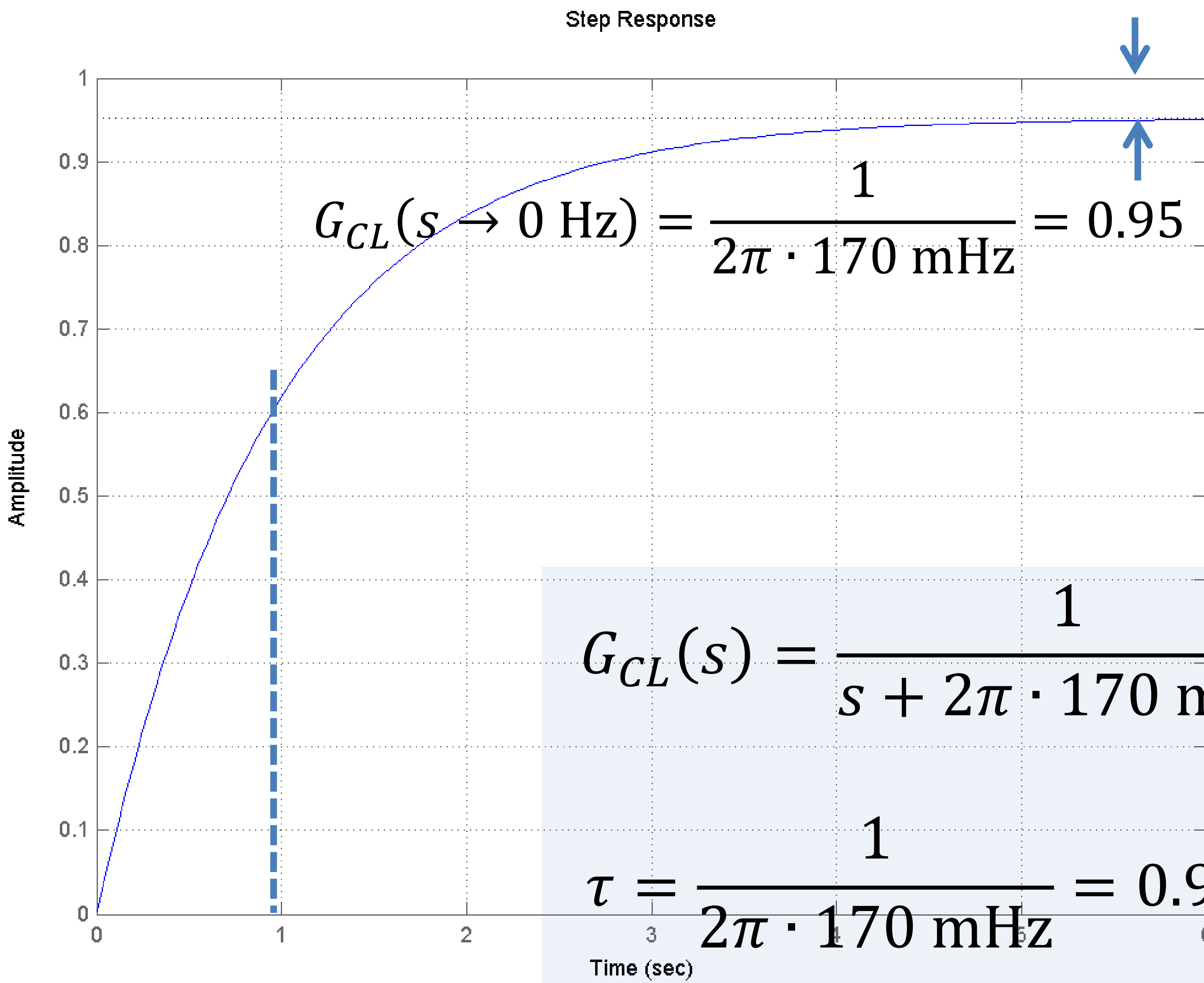
With a little algebra:

$$G_{CL} = \frac{G_{OL}}{1 + G_{OL}} = \frac{{}^{1}\!/_{(s+a)}}{1 + {}^{1}\!/_{(s+a)}} = \frac{1}{s + a + 1}$$

$$= \frac{1}{s + 2\pi\, 8\mathrm{mHz} + 1} = \frac{1}{s + 2\pi\, 170\mathrm{mHz}}$$

$$G_{CL}(s) = \frac{1}{s + 2\pi \cdot 170\ \mathrm{mHz}}$$

Let's check step response of G_{CL}



$$G_{CL}(s \to 0\ \text{Hz}) = \frac{1}{2\pi \cdot 170\ \text{mHz}} = 0.95$$

$$G_{CL}(s) = \frac{1}{s + 2\pi \cdot 170\ \text{mHz}}$$

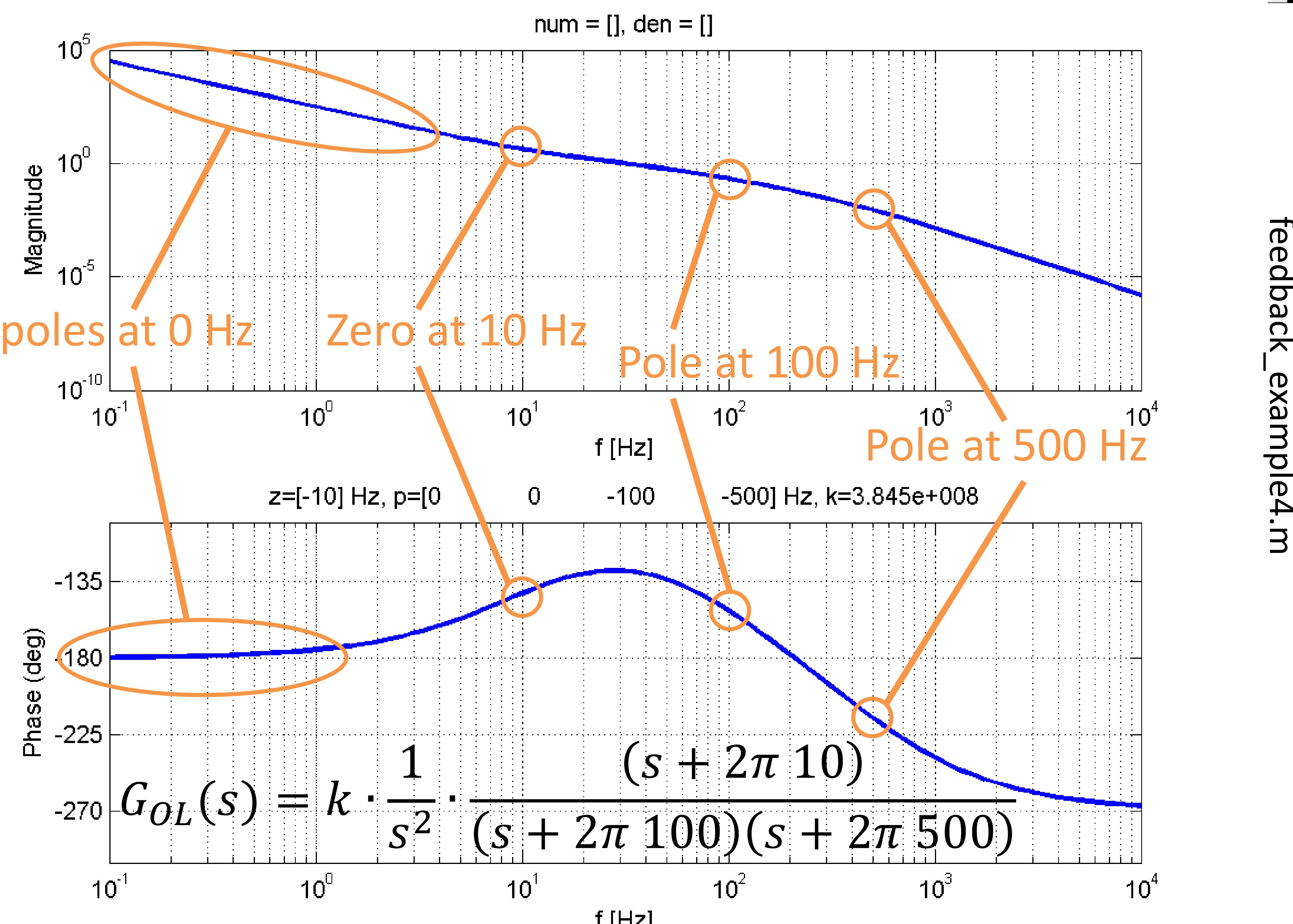
$$\tau = \frac{1}{2\pi \cdot 170\ \text{mHz}} = 0.95\ \text{s}$$

Example

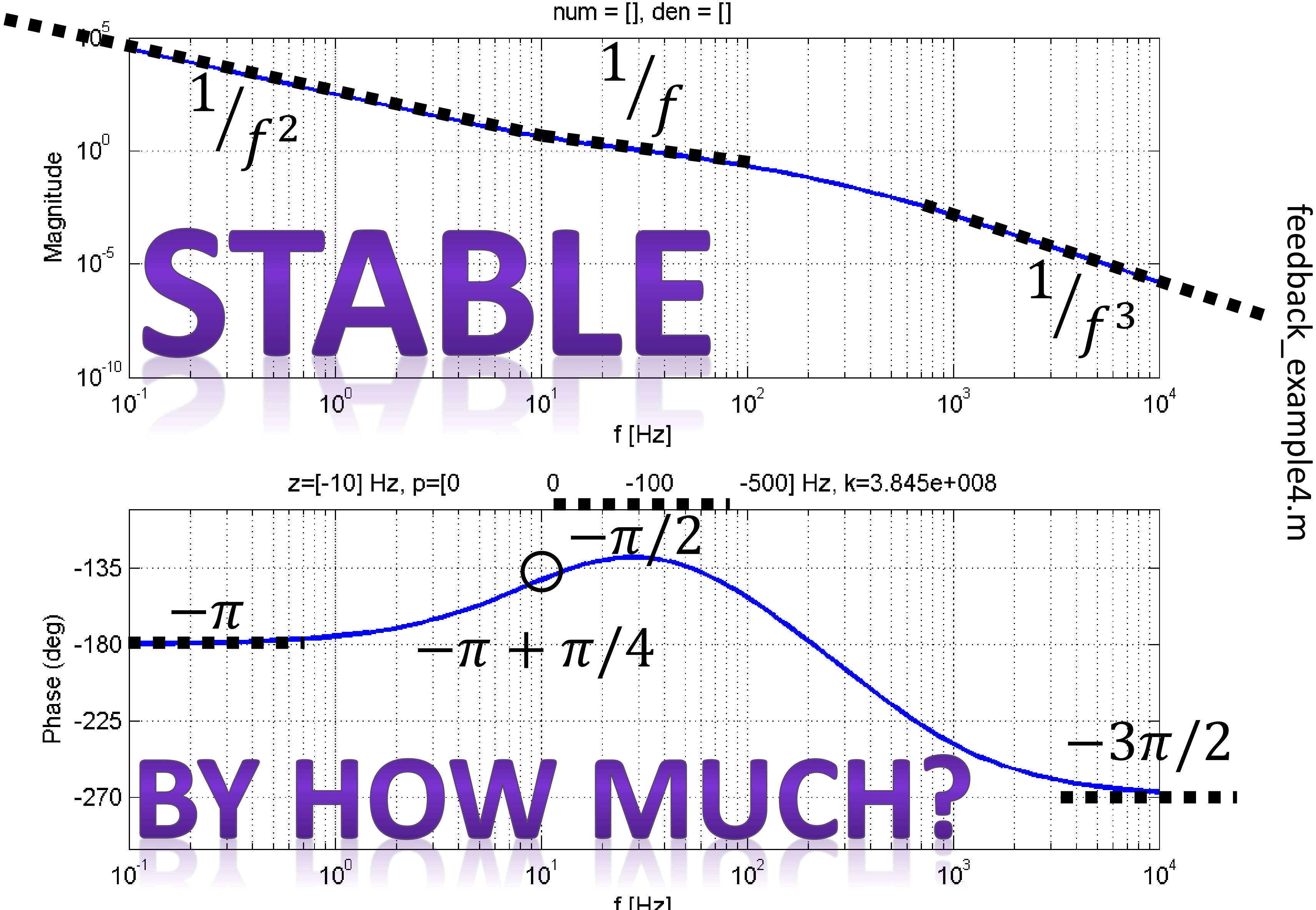
Is the system with open loop transfer function G_{OL} stable?

$$G_{OL}(s) = k \cdot \frac{1}{s^2} \cdot \frac{(s + 2\pi\,10)}{(s + 2\pi\,100)(s + 2\pi\,500)}$$

$$k = 3.8 \times 10^8$$


num = [], den = []
Magnitude
10^{5}
10^{0}
10^{-5}
10^{-10}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
f [Hz]
Two poles at 0 Hz
Zero at 10 Hz
Pole at 100 Hz
Pole at 500 Hz
z=[-10] Hz, p=[0 0 -100 -500] Hz, k=3.845e+008
Phase (deg)
-135
-180
-225
-270
f [Hz]
$G_{OL}(s) = k \cdot \frac{1}{s^2} \cdot \frac{(s + 2\pi\, 10)}{(s + 2\pi\, 100)(s + 2\pi\, 500)}$
feedback_example4.m

$$G_{OL}(s) = k \cdot \frac{1}{s^2} \cdot \frac{(s + 2\pi\, 10)}{(s + 2\pi\, 100)(s + 2\pi\, 500)}$$



Problem

If a system has an open loop transfer function

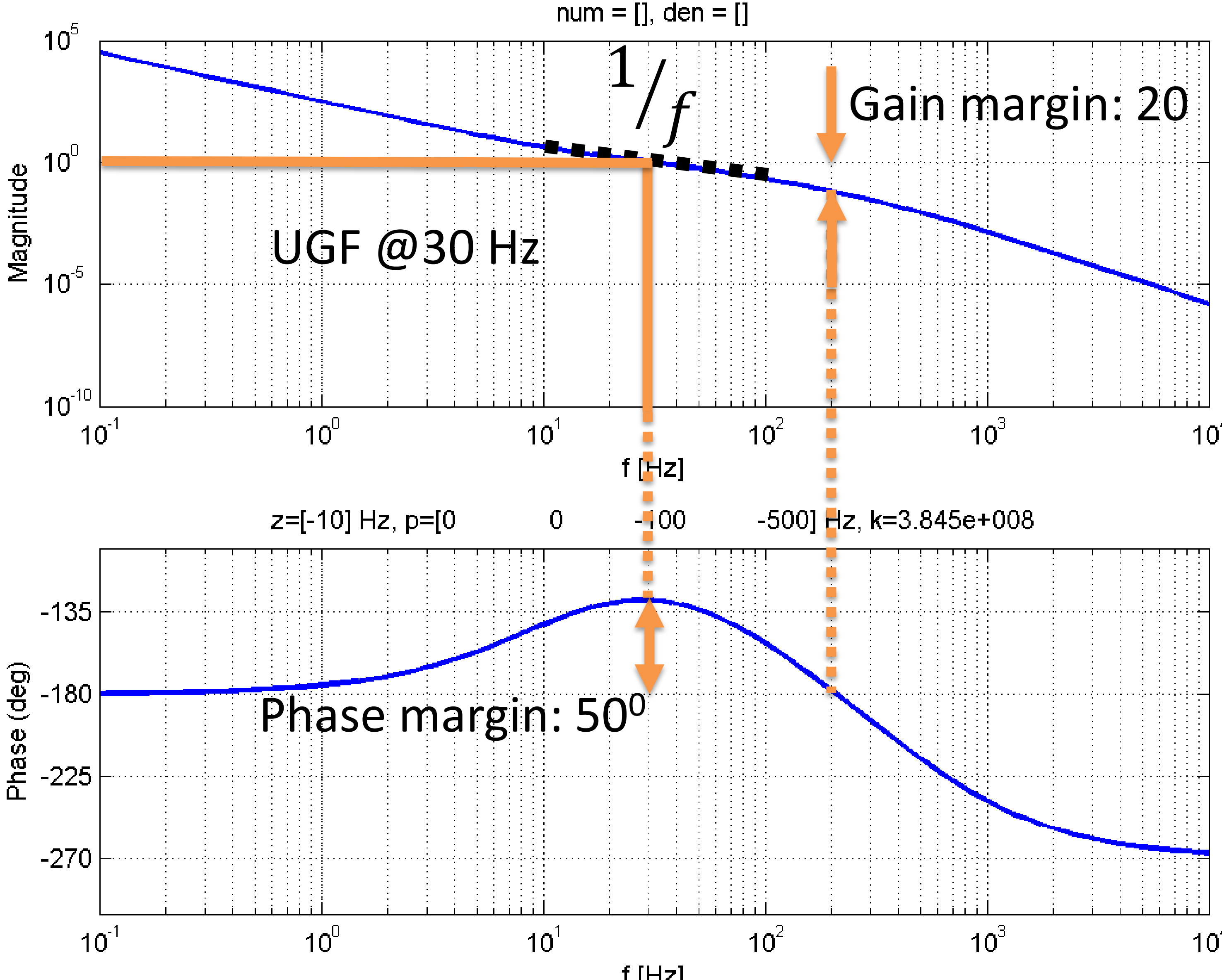
$$G_{OL} = \frac{k}{(s+10)(s+100)}$$

what values of k make it stable? Use MATLAB to confirm this.

Relative stability

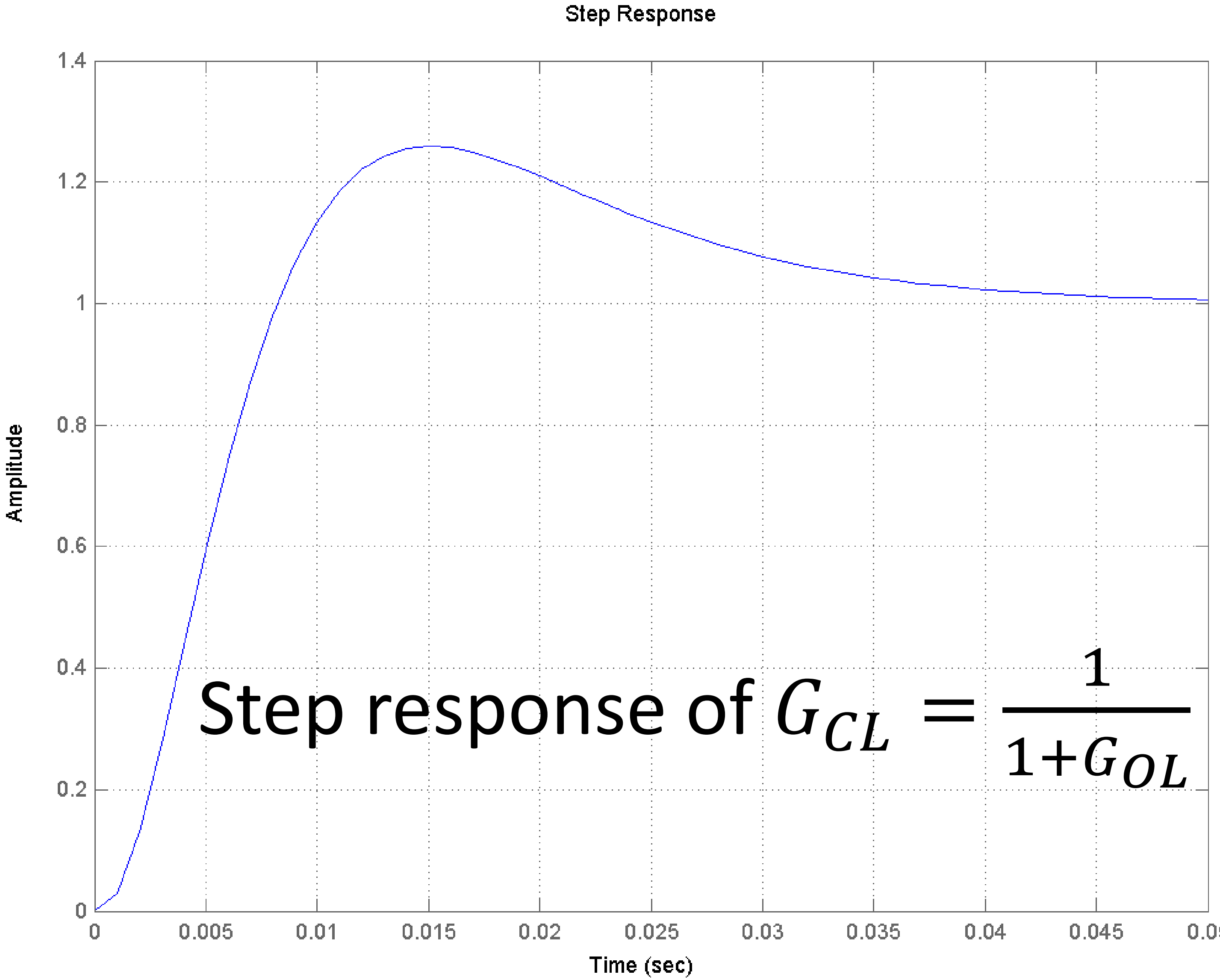
- Gain and phase margin
  - Measure of “relative” stability
  - The larger they are → the “safer we are”
- Gain margin
  - By how much can the gain increase until the system becomes unstable?
  - Defined as $GM = \frac{1}{|G_{OL}(\omega_\pi)|}$
- Phase margin
  - By how much can the system tolerate a phase change at UGF?
  - Defined as $\mathrm{PM} = \angle G_{OL}(\omega_{UGF}) + 180$
  - Rule of thumb: keep the phase margin above $40°$

$$G_{OL}(s) = k \cdot \frac{1}{s^2} \cdot \frac{(s + 2\pi\, 10)}{(s + 2\pi\, 100)(s + 2\pi\, 500)}$$



feedback_example4.m

$$G_{OL}(s) = k \cdot \frac{1}{s^2} \cdot \frac{(s + 2\pi\ 10)}{(s + 2\pi\ 100)(s + 2\pi\ 500)}$$

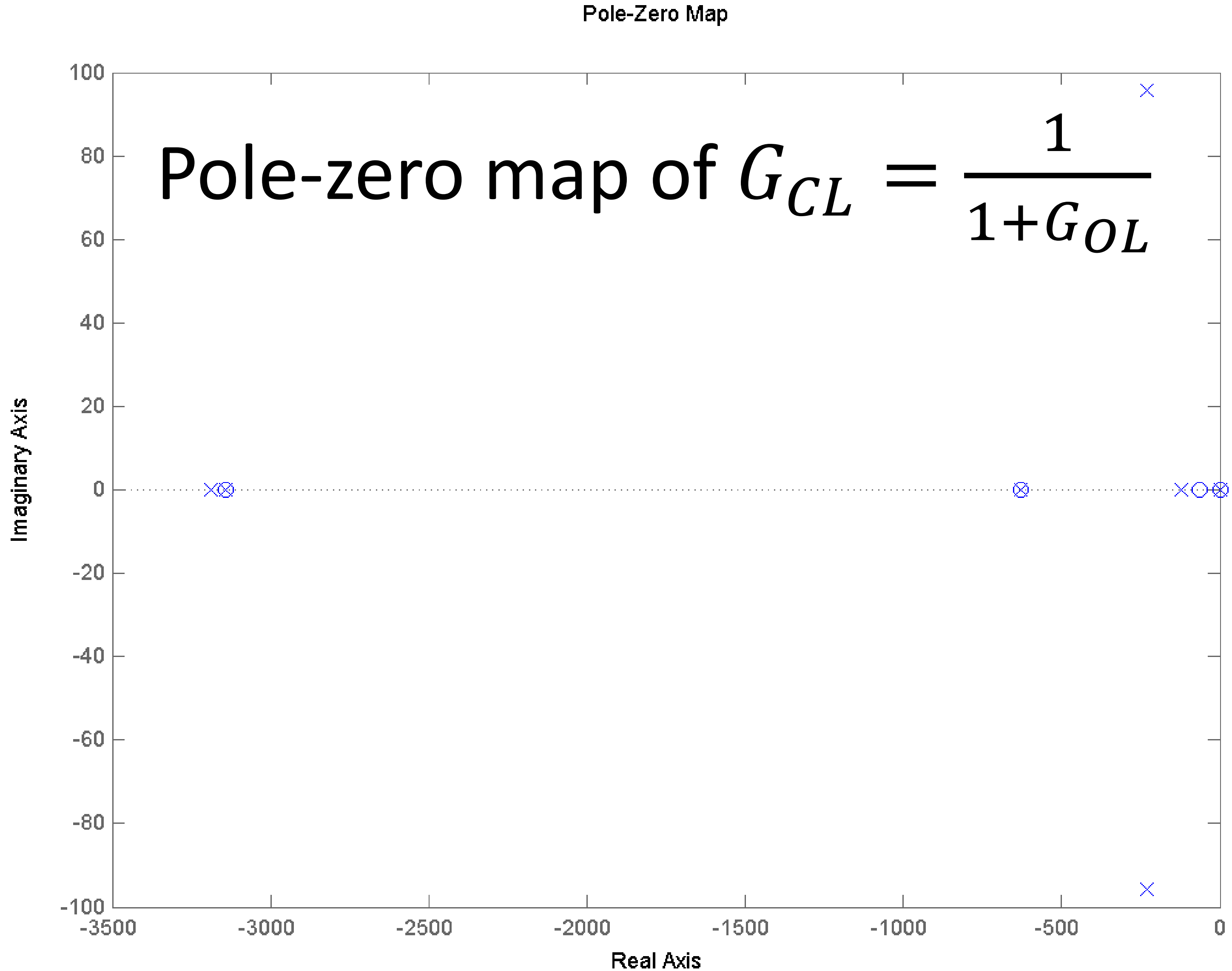


Step response of $G_{CL} = \frac{1}{1+G_{OL}}$

feedback_example4.m

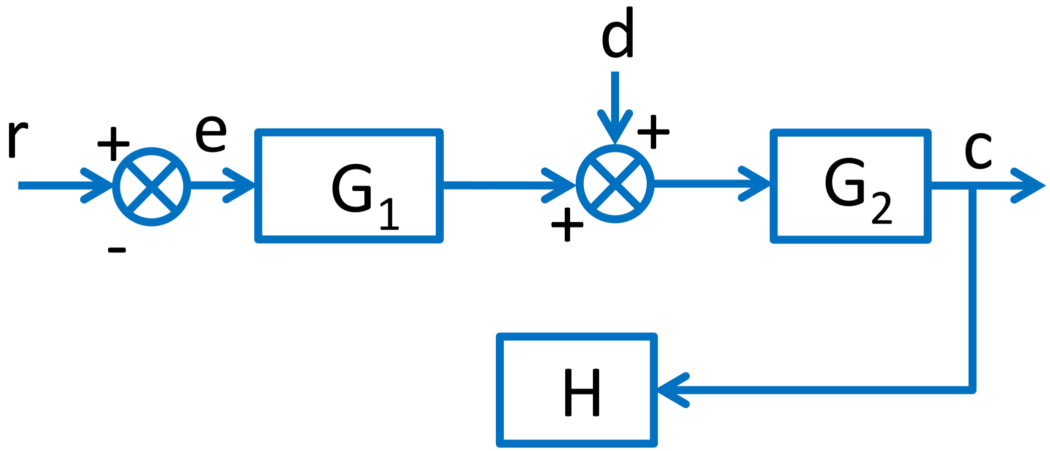
$$G_{OL}(s) = k \cdot \frac{1}{s^2} \cdot \frac{(s + 2\pi\,10)}{(s + 2\pi\,100)(s + 2\pi\,500)}$$

Pole-zero map of $G_{CL} = \frac{1}{1+G_{OL}}$



feedback_example4.m

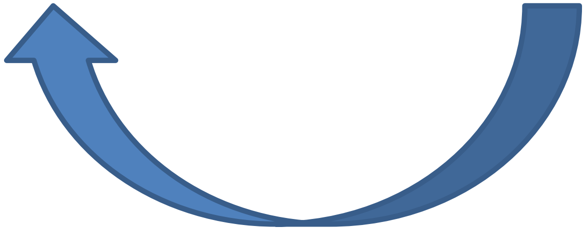
Performance to noise input d: with no feedback



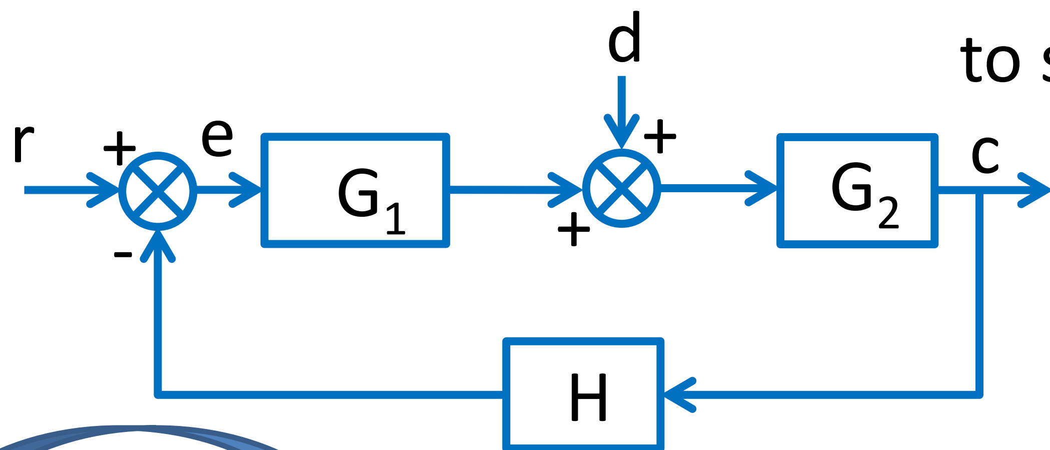
$$c = (G_1 G_2)r + (G_2)d$$

$$e = r$$

Noise contribution to signal c



Performance to noise input d: with feedback



Noise contribution to signal c

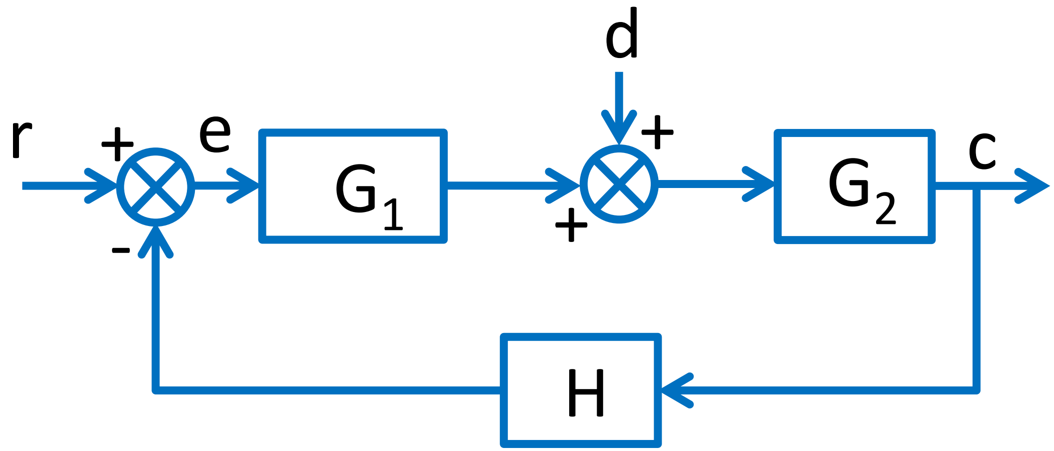
Controlled signal

$$c = \left(\frac{G_1 G_2}{1 + G_{OL}}\right) r + \left(\frac{G_2}{1 + G_{OL}}\right) d$$

$$e = \left(\frac{1}{1 + G_{OL}}\right) r + \left(\frac{H G_2}{1 + G_{OL}}\right) d$$

Suppression factor

Setting the parameters

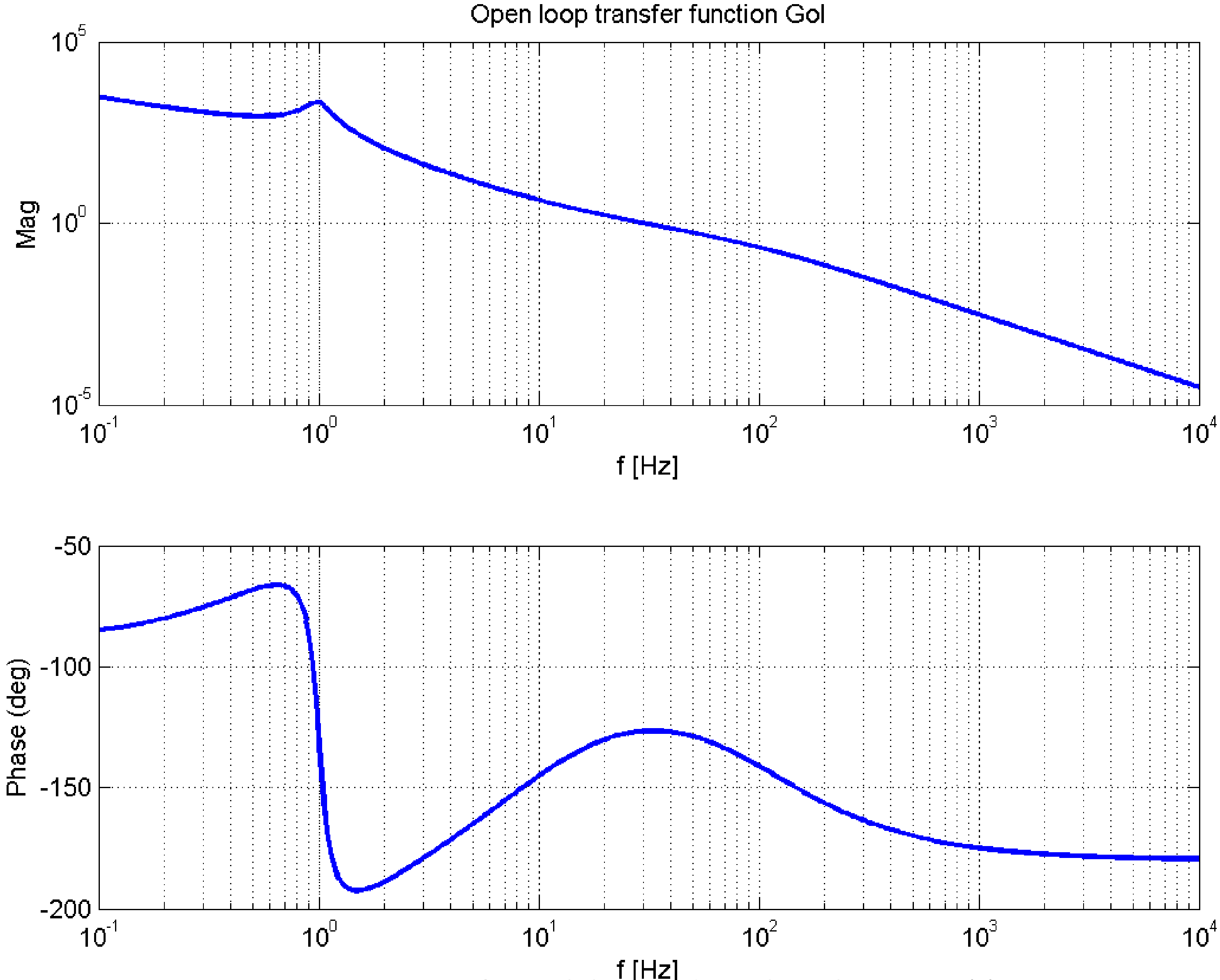


G_1 = zeros at 1, 10Hz; poles at 0, 100 Hz, k = 300

$$G_2 = \frac{\widetilde{\omega}}{s^2 + 2\delta\widetilde{\omega}s + \widetilde{\omega}^2} \text{ with } \widetilde{\omega} = 2\pi \text{ 1Hz}, \delta = 0.1$$

$$H = 1$$

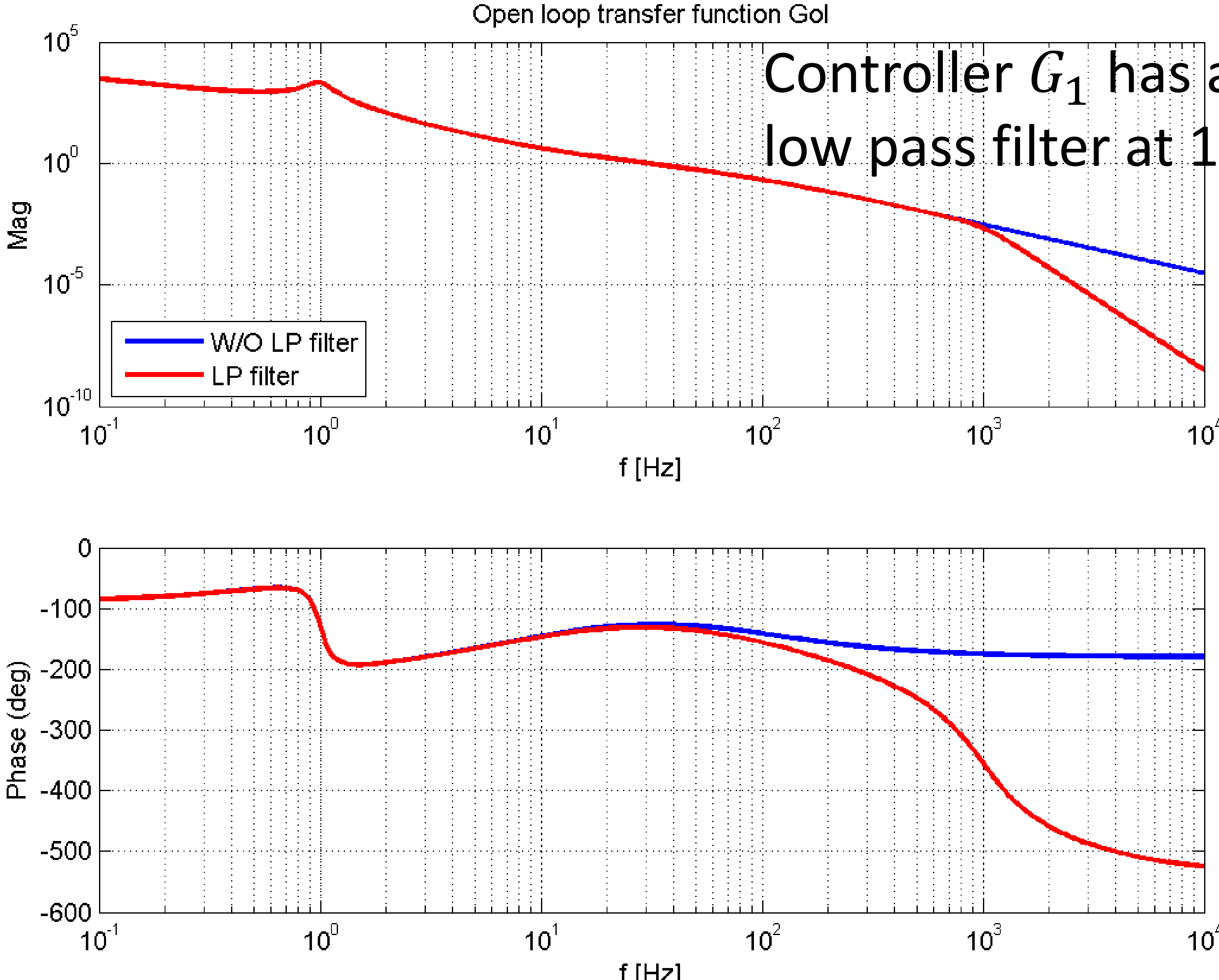
$$G_{OL} = G_1 \cdot G_2 \cdot H$$



feedback_example6.m

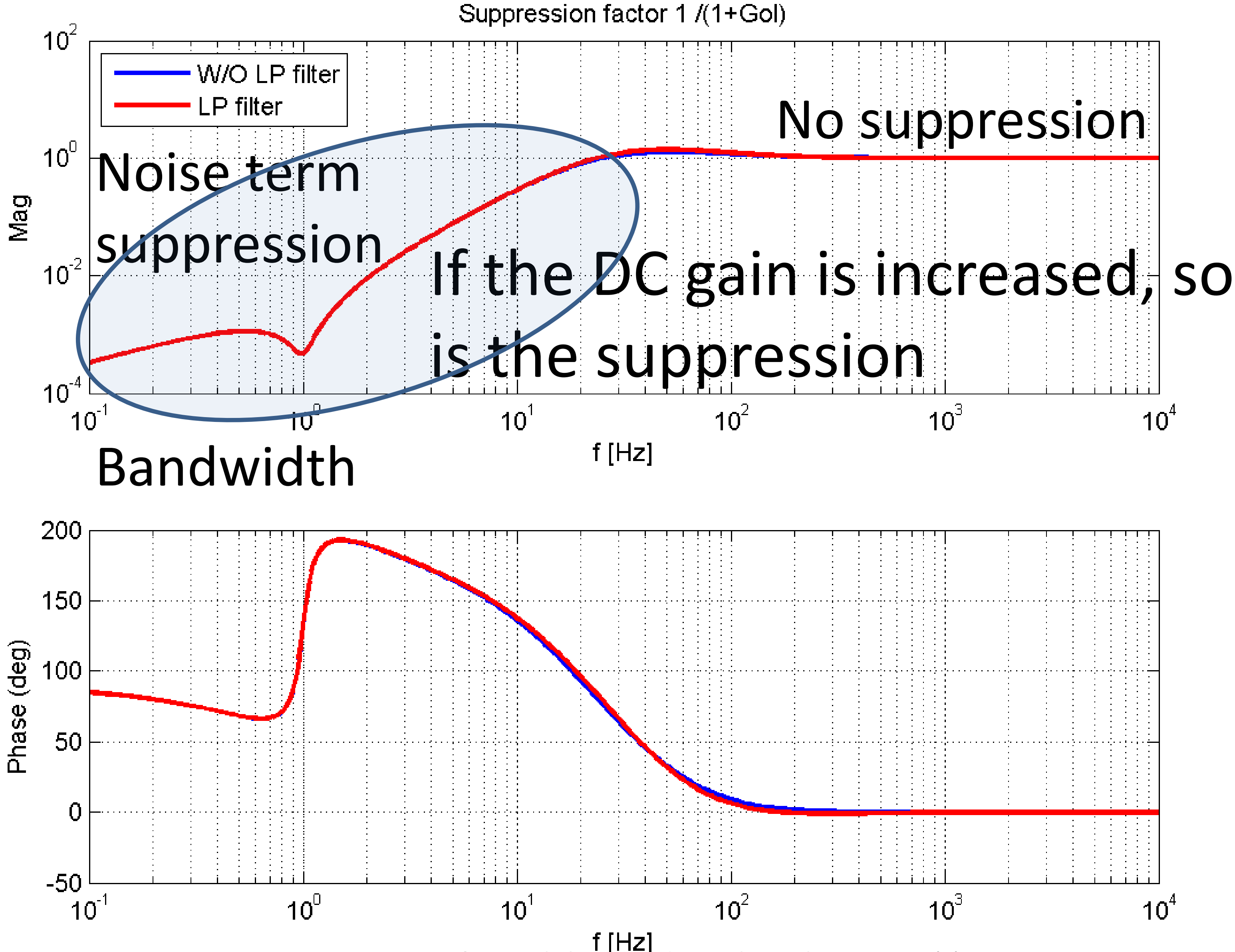
$$G_{OL} = G_1 \cdot G_2 \cdot H$$

Controller G_1 has a low pass filter at 1kHz



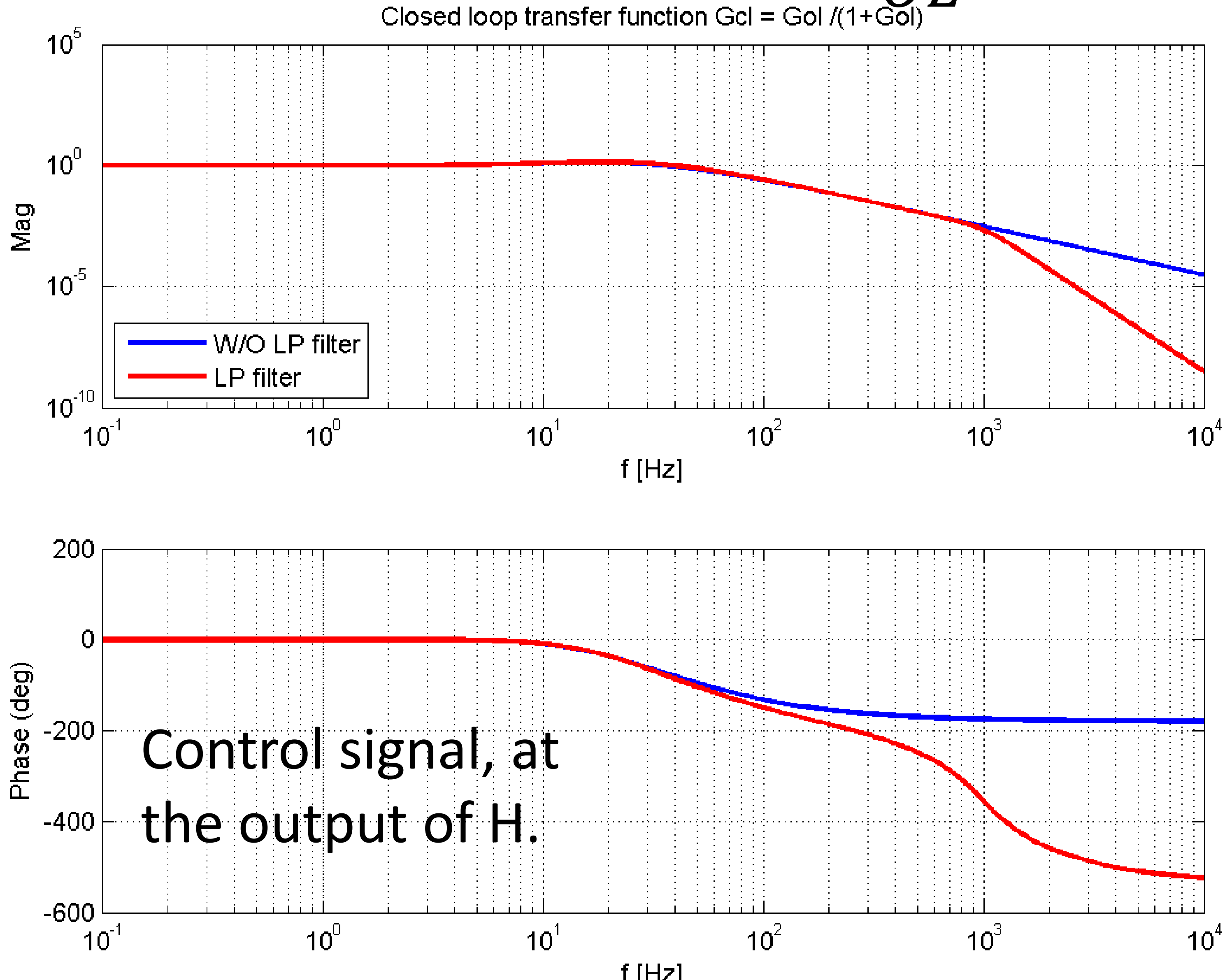
feedback_example6.m

Suppression factor $^1/_{1+G_{OL}}$



feedback_example6.m

$$G_{CL} = {G_{OL}}/{1 + G_{OL}}$$



feedback_example6.m

How can we correct an unstable loop? Typical compensators

- Integral controller

$$G(s) = \frac{1}{s}$$

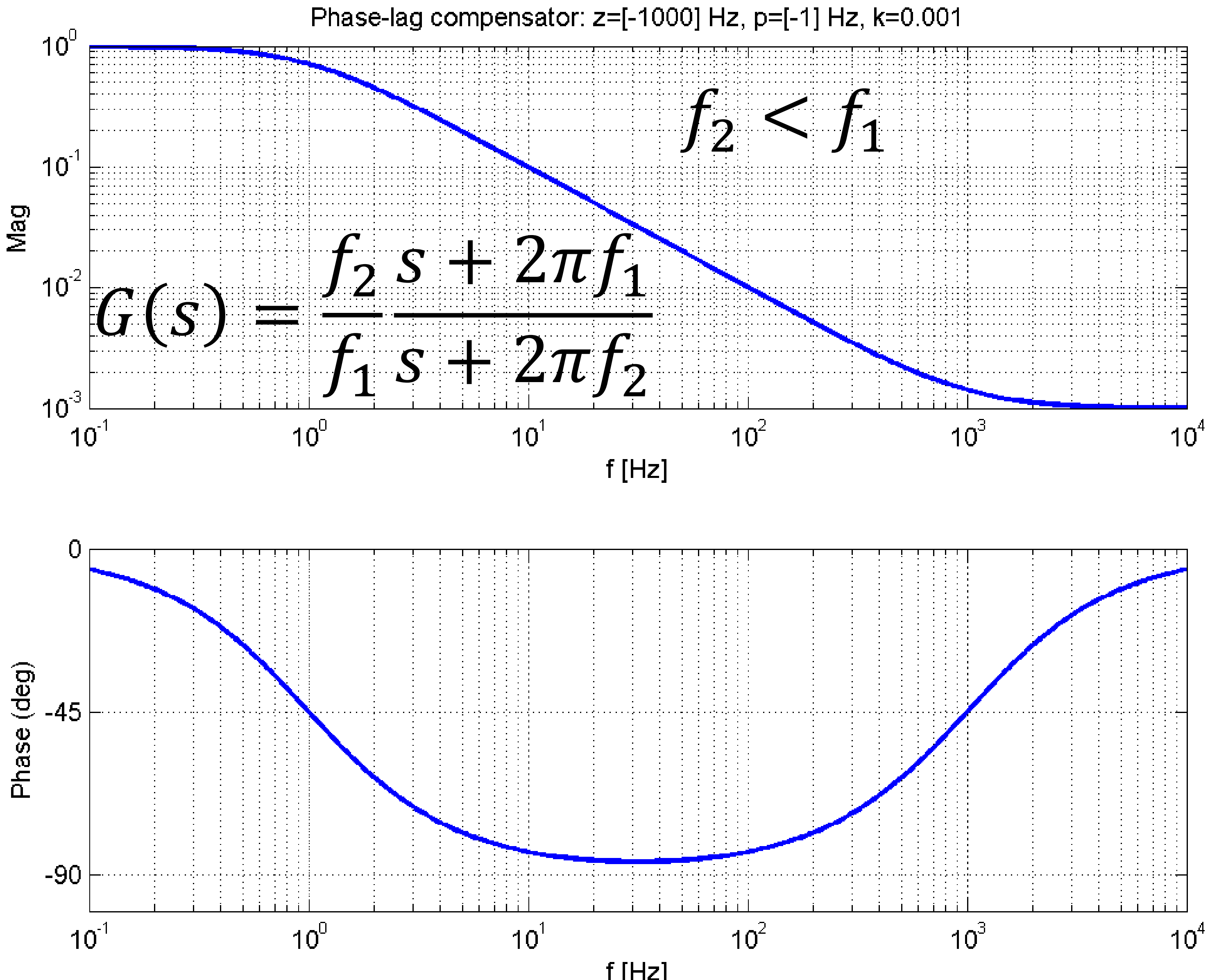
The output is proportional to the time integral of the input

- Derivative controller

$$G(s) = s$$

The output is proportional to the time derivative of the input

Phase-lag compensator

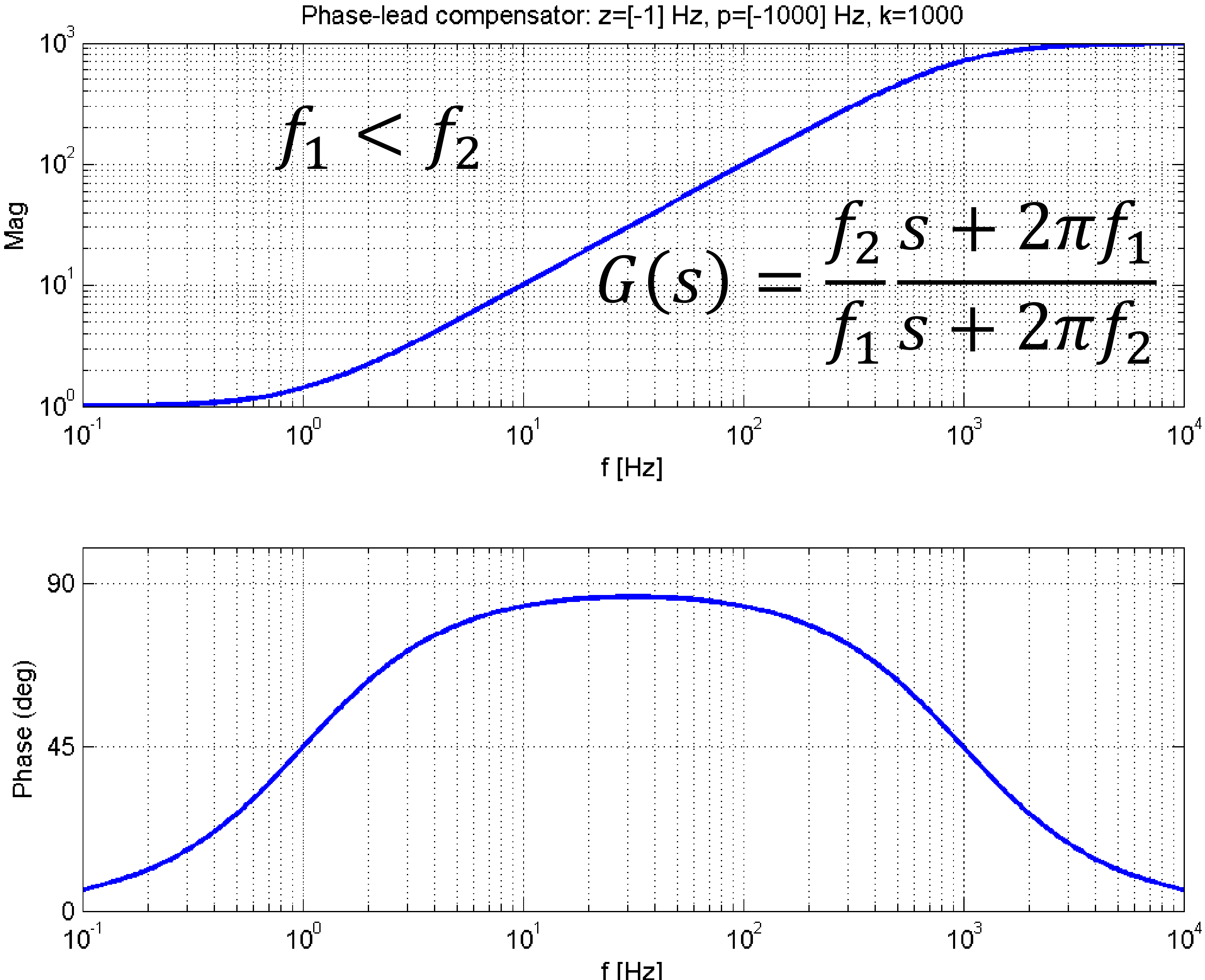


$$f_2 < f_1$$

$$G(s) = \frac{f_2}{f_1}\frac{s + 2\pi f_1}{s + 2\pi f_2}$$

compensator_phaselag.m

Phase-lead compensator

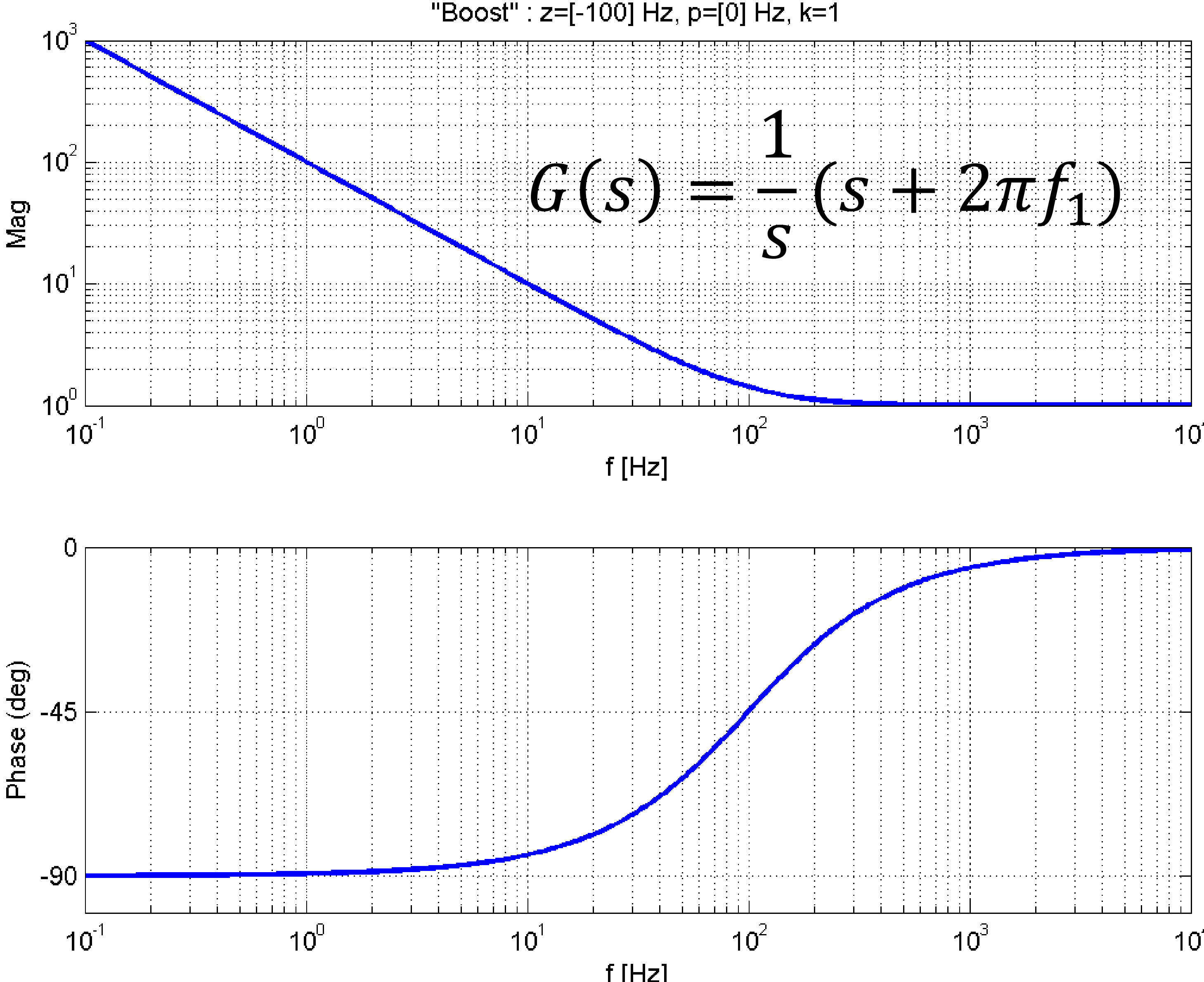


$$f_1 < f_2$$

$$G(s) = \frac{f_2}{f_1}\frac{s + 2\pi f_1}{s + 2\pi f_2}$$

compensator_phaselead.m

"Boost"



$$G(s) = \frac{1}{s}(s + 2\pi f_1)$$

compensator_boost.m

Problem

If a system has an open loop transfer function

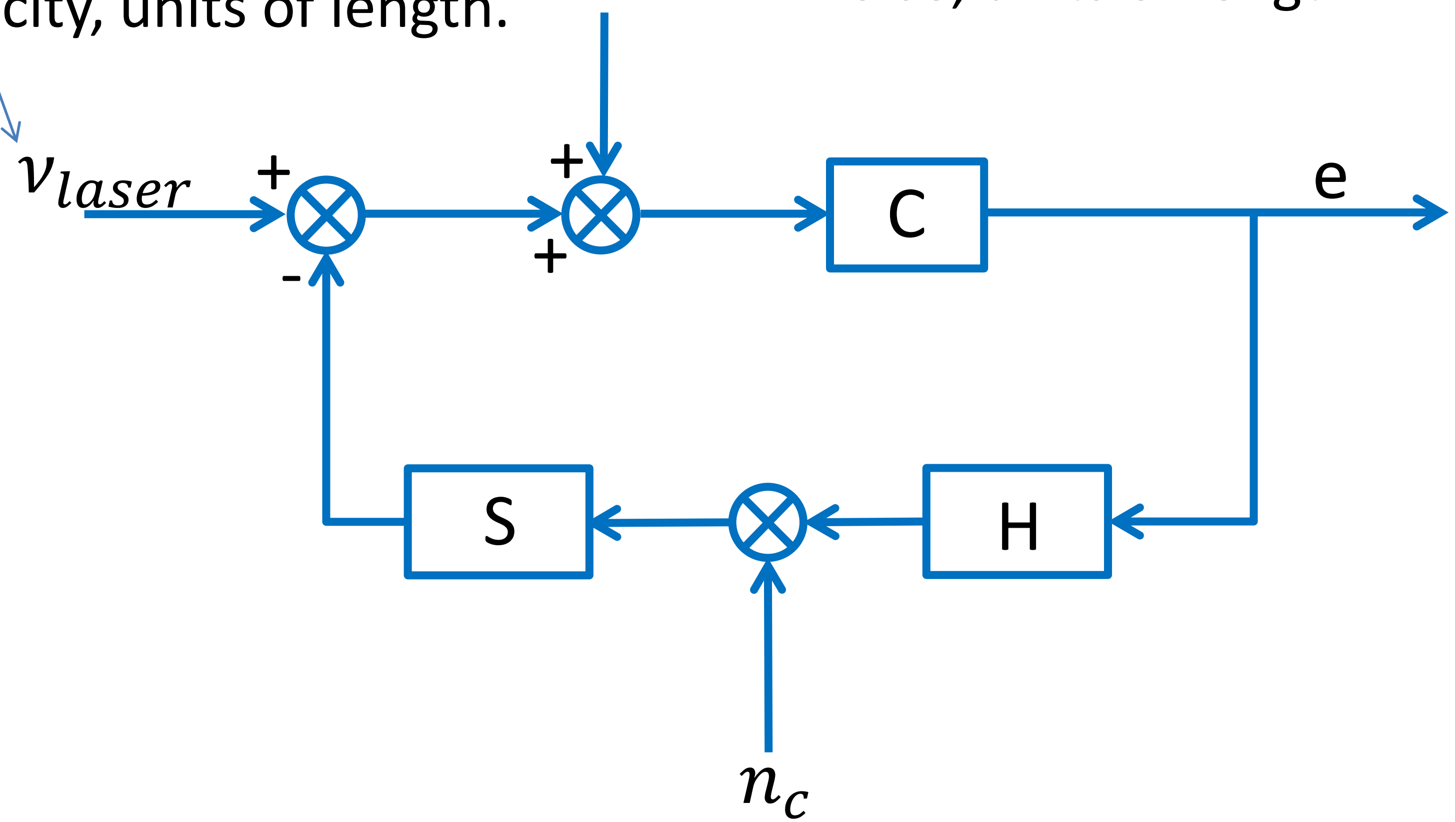
$$G_{OL} = \frac{10^3}{(s+10)^3}$$

design a compensator that would make the system stable with an UGF at 100 Hz. Use MATLAB to confirm this.

Example: locking one LIGO arm

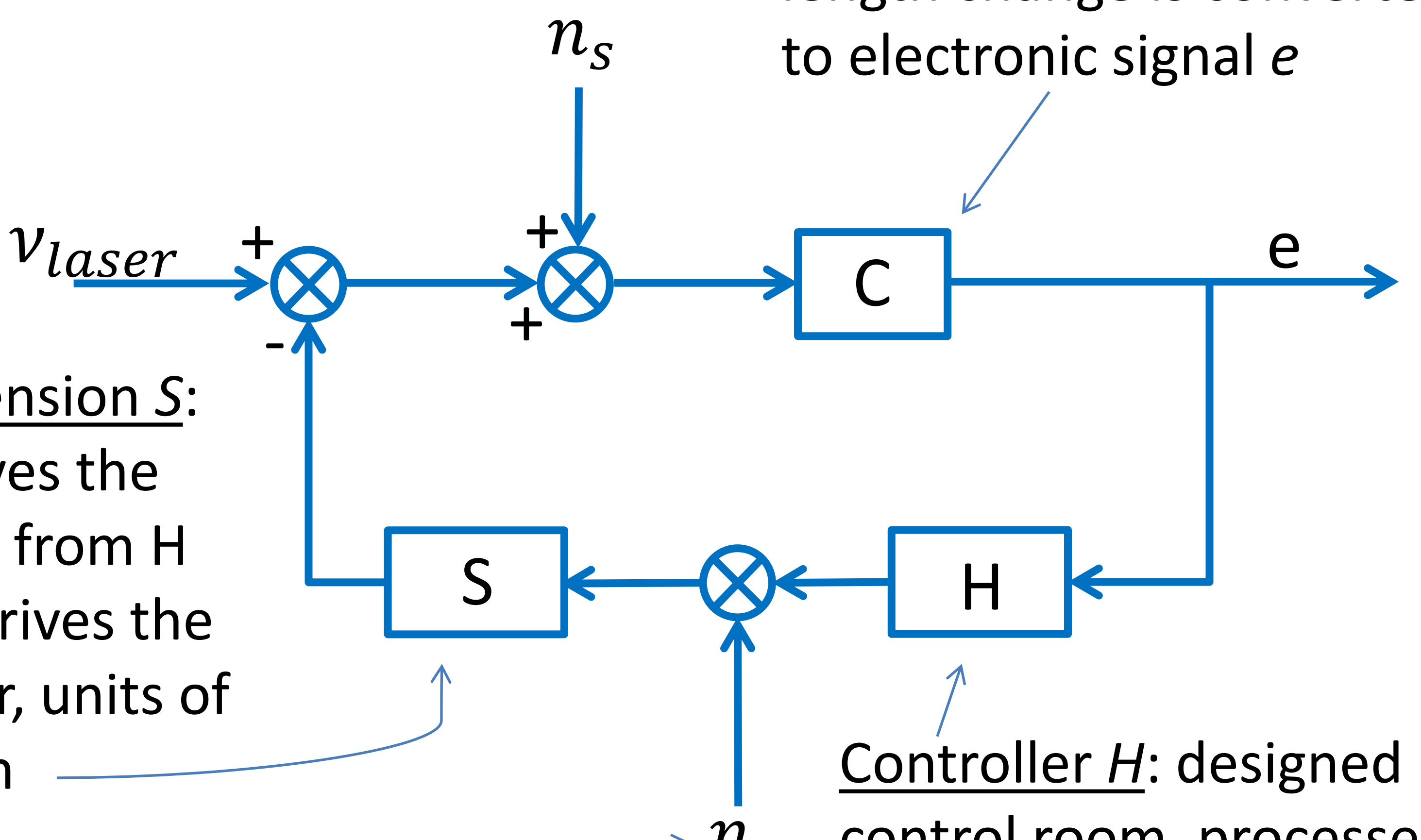
Reference signal ν_{laser}: the cavity "locks" to the laser frequency. For simplicity, units of length.

Noise term n_s: Mirror motion due to seismic noise, units of length



Example: locking one LIGO arm

<u>Ligo arm cavity *C*</u>: a cavity length change is converted to electronic signal *e*



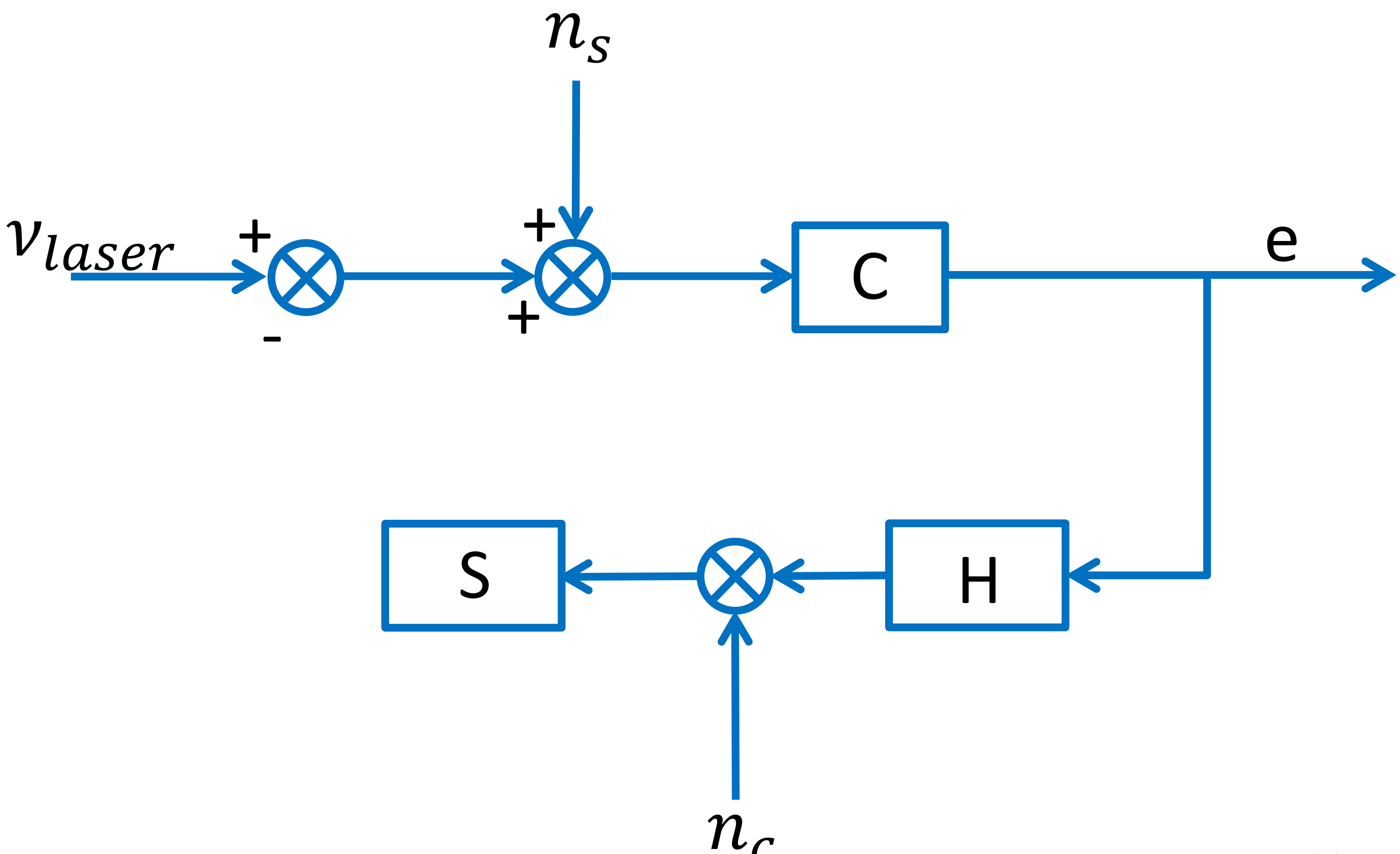
<u>Suspension *S*</u>: receives the signal from H and drives the mirror, units of length

<u>Noise term n_c</u>: Noise from coil drivers

<u>Controller *H*</u>: designed in the control room, processes signal *e* and sends commands to the suspension *S*

Example: no lock

$$e = C\, n_s + C\, \nu_{laser}$$

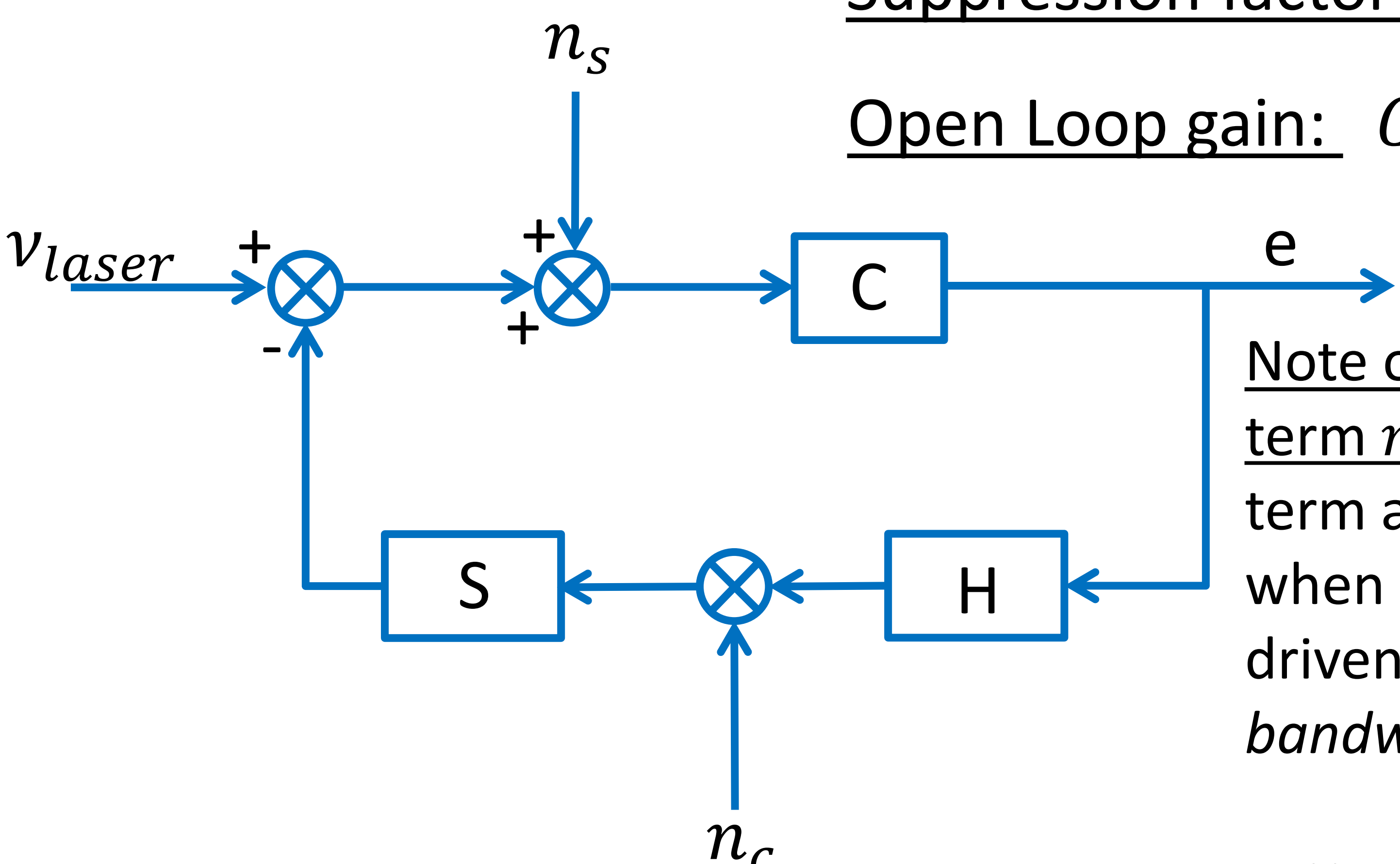


Example: locking one LIGO arm

$$e = \left(\frac{1}{1 + CHS}\right) \cdot (C \cdot n_s + C \cdot \nu_{laser} - S \cdot C \cdot n_c)$$

Suppression factor : $\frac{1}{1+CHS}$

Open Loop gain: $C\ H\ S$



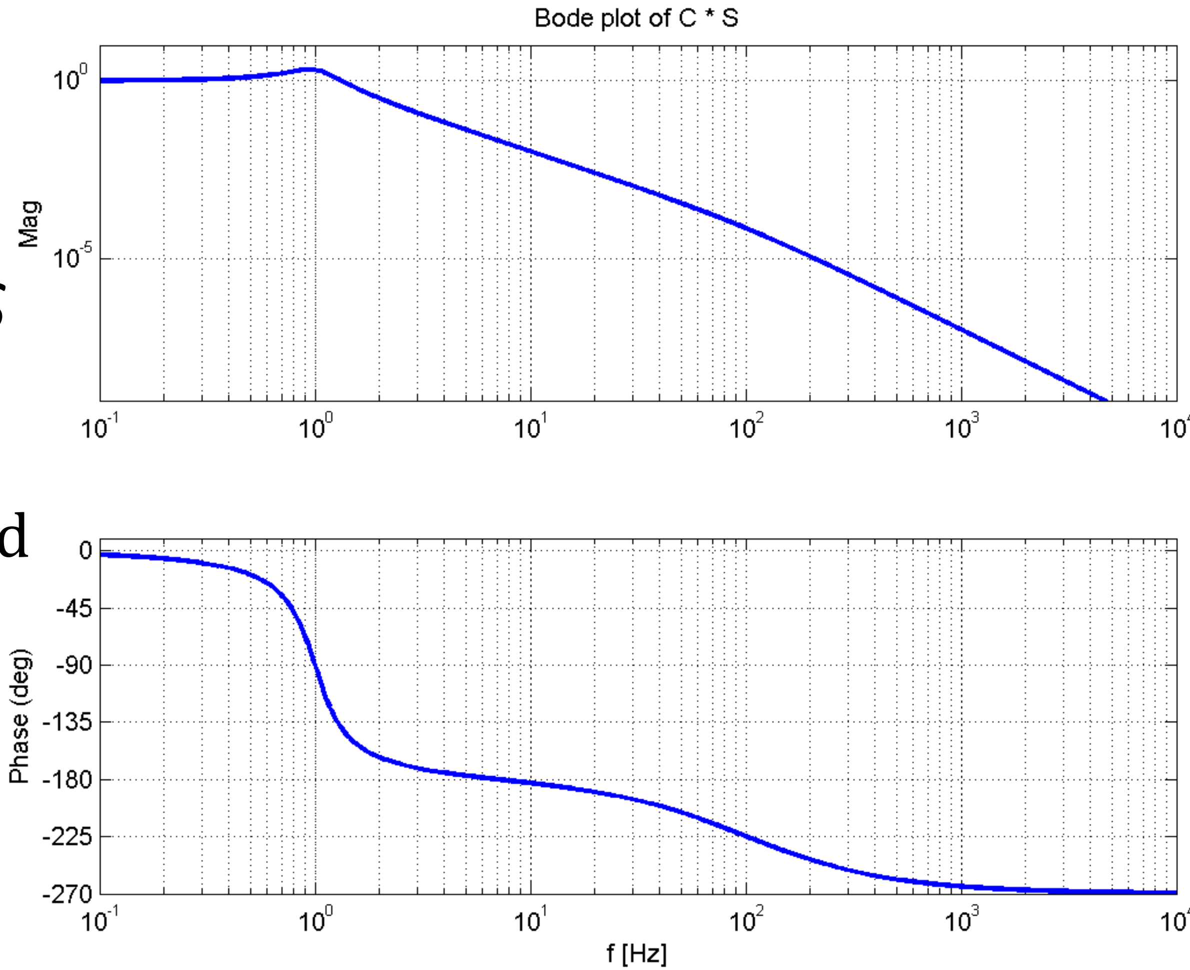
Note on noise term n_c: Noise term appears only when mirrors are driven → *limited bandwidth desired*

Example: locking one LIGO arm

- Need to design H so as to have
  - Enough suppression of noise terms
  - Stable
  - “Small” bandwidth

Example: locking one LIGO arm

- Cavity transfer function *C*:
  - Pole at 100 Hz
- Suspension transfer function *S*
  - Simple harmonic oscillator (SHO) with $f_0 = 1Hz$ and quality factor $Q = 2$
- Shown is $C \cdot S$
- What controller *H* can we use?



cavityfeedback_exampleA.m

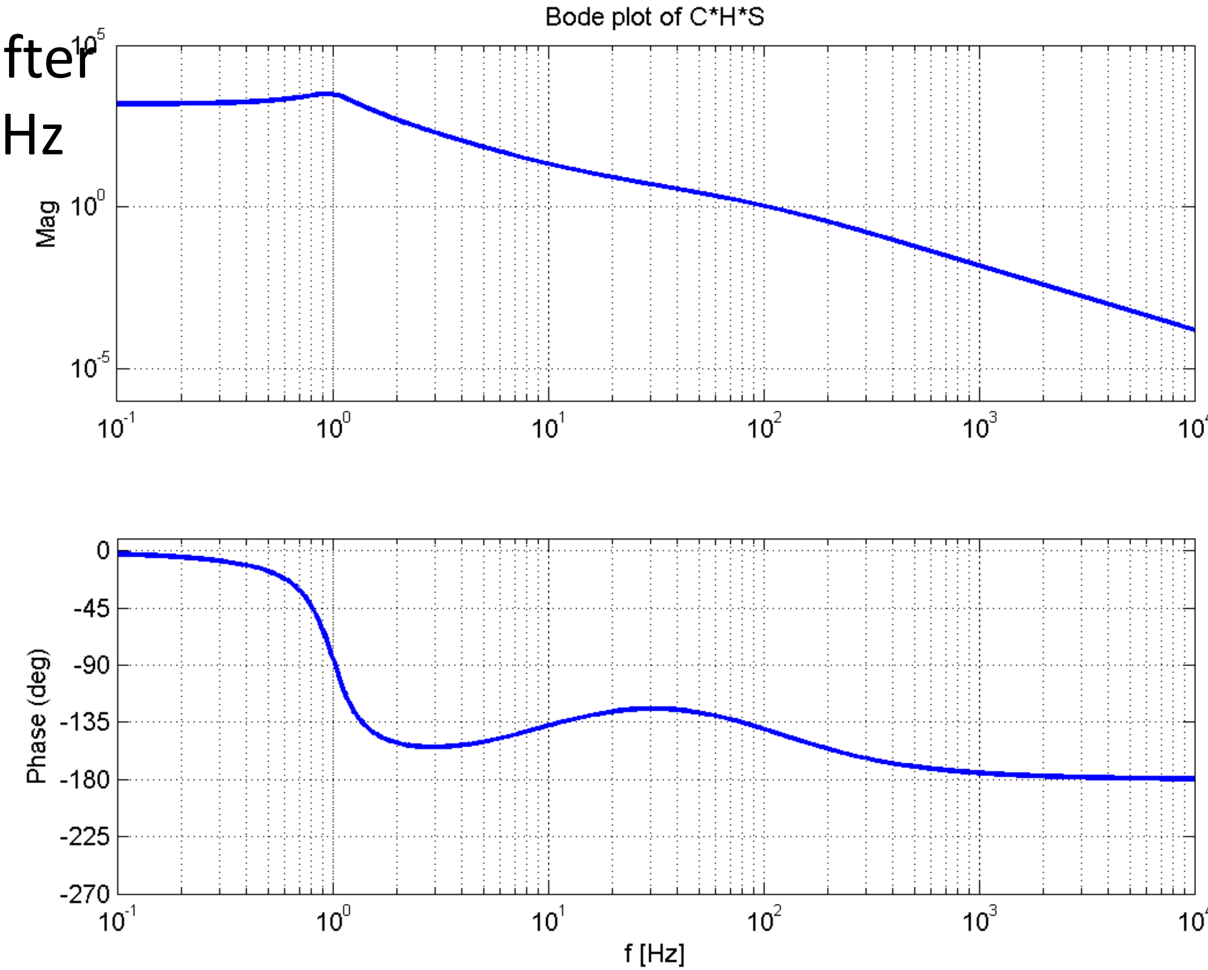
Example: locking one LIGO arm

- Set UGF at 100 Hz
- Need *H* with a zero after 1 Hz and before 100 Hz
- Try phase lead

$$H(s) = k \cdot \frac{s + \omega_1}{\omega_1}$$

with $k = 1500$ and $\omega_1 = 2\pi\ 10\mathrm{Hz}$.

- Bode plot of OL
  - UGF at 100 Hz
  - PM ~40 deg
  - Stable



cavityfeedback_exampleA.m

Example: locking one LIGO arm

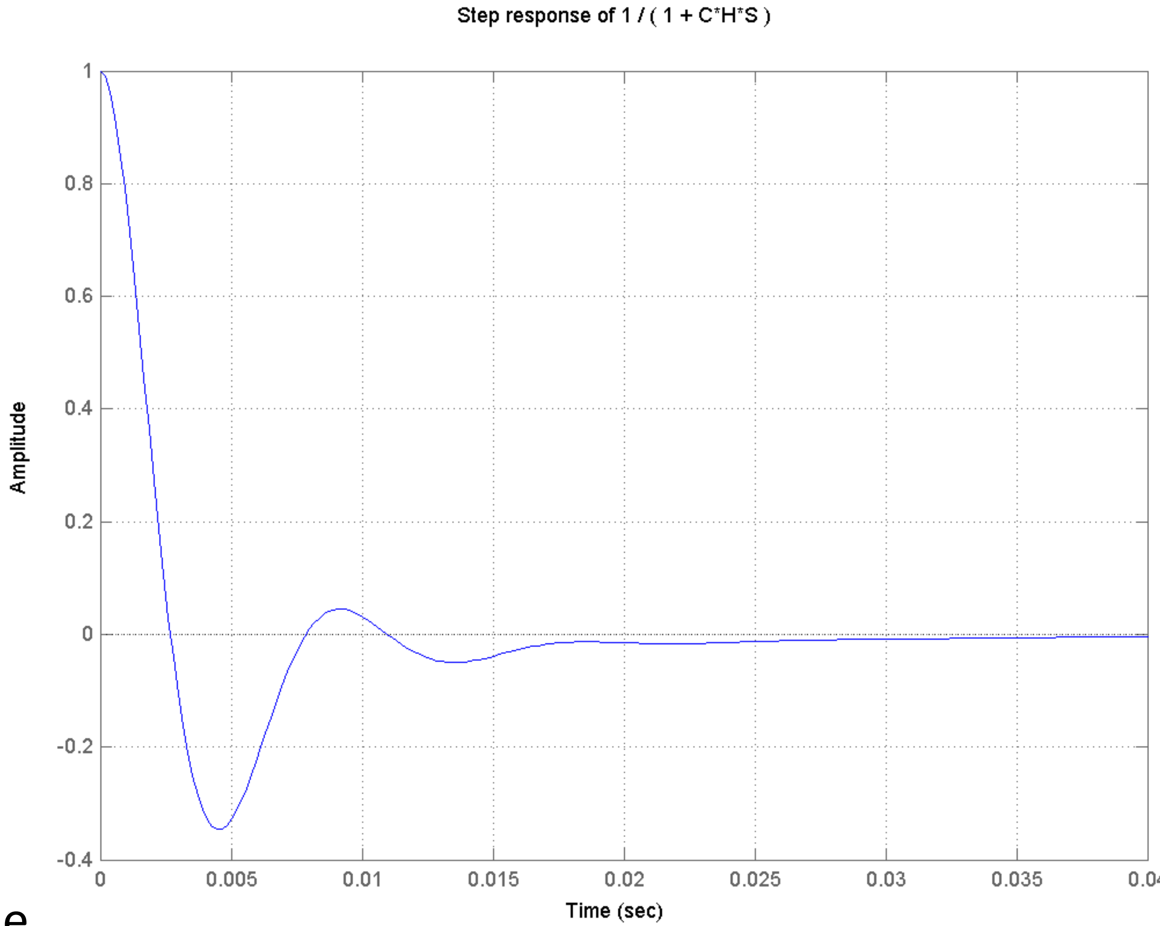
Double check stability

- Step response plot of
$$\frac{1}{1+CHS}$$

- Step is driven to zero as it should (it is a suppression factor)

- In about 10 ms (~1/UGF) response is close to zero

- Two oscillation cycles – little ringing



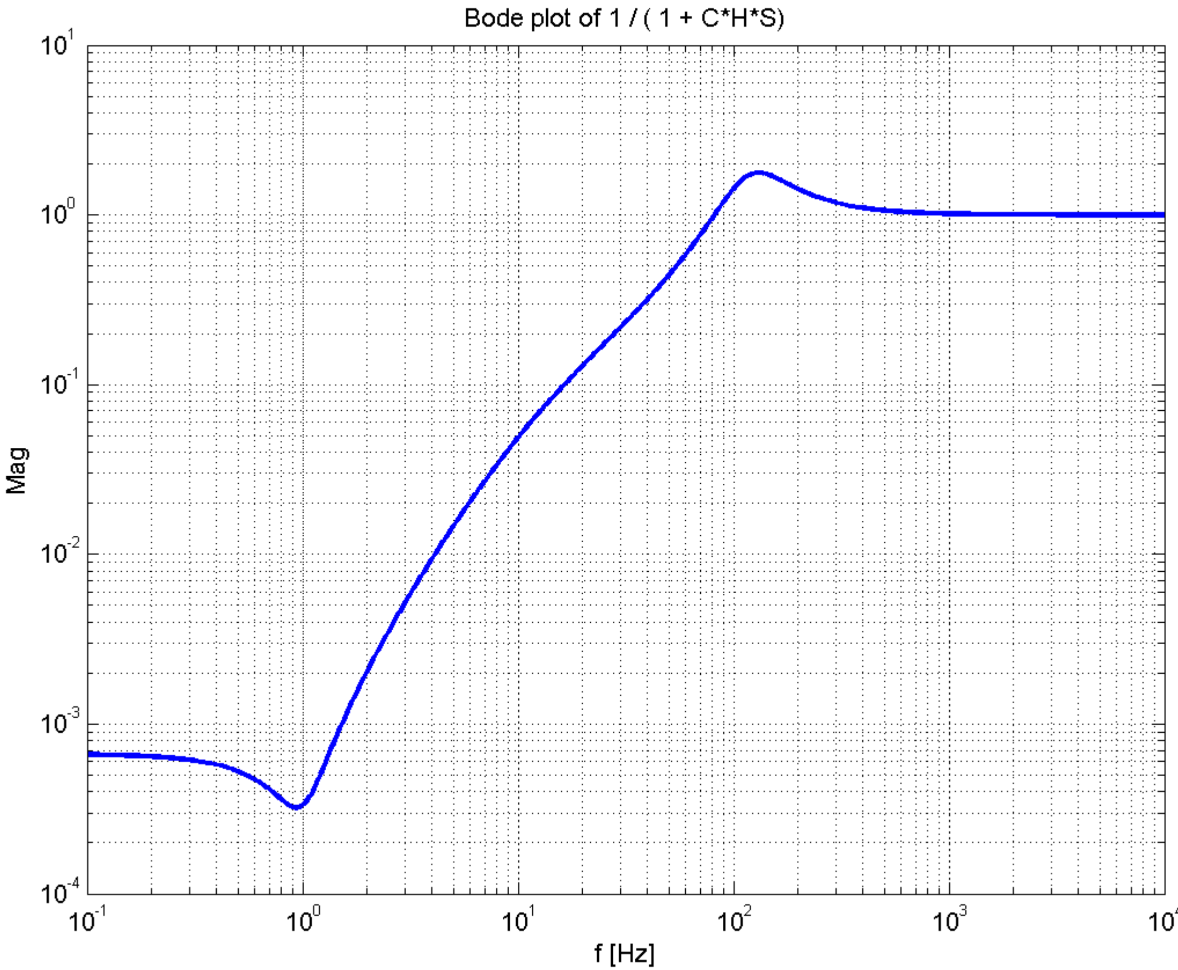
cavityfeedback_exampleA.m

Example: locking one LIGO arm

- Bode plot of the suppression factor

$$\frac{1}{1 + CHS}$$

- Suppression of ~1500x at 100 mHz
- No suppression above 100 Hz
- Notice spike at 100 Hz
  - This spike is responsible of the ringing in the step response
  - decreased if phase margin is increased



cavityfeedback_exampleA.m

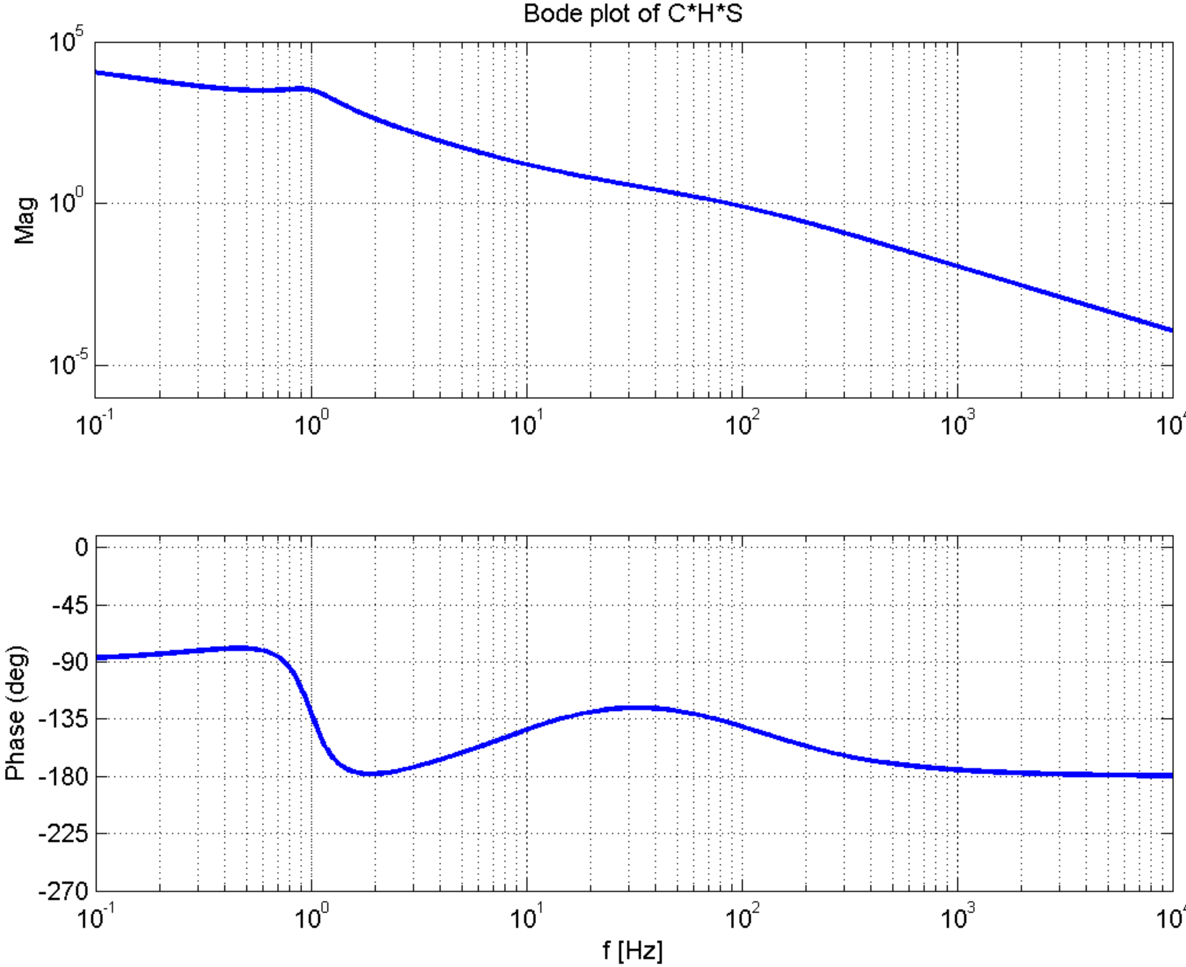
Example: locking one LIGO arm

- Let's increase the low frequency gain with a "boost"
- Try H

$$H(s) = k \cdot \frac{1}{s} \cdot \frac{s+\omega_1}{\omega_1} \cdot \frac{s+\omega_2}{\omega_2}$$

with $k = 7000$, $\omega_1 = 2\pi\ 10\text{Hz}$ and $\omega_2 = 2\pi\ 1\ \text{Hz}$

- OL bode plot
  - UGF at 100 Hz
  - PM ~40 deg
  - Stable

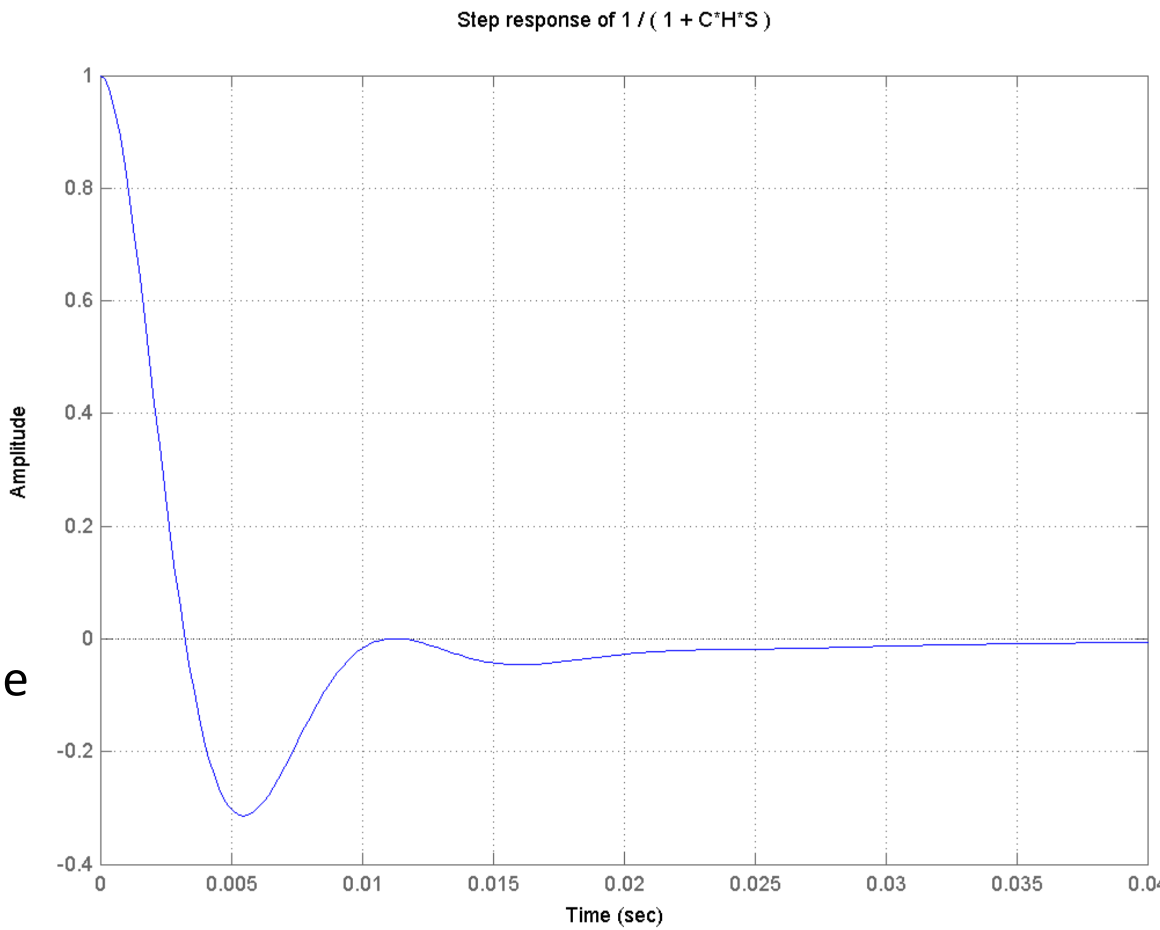


cavityfeedback_exampleB.m

Example: locking one LIGO arm

Double check stability

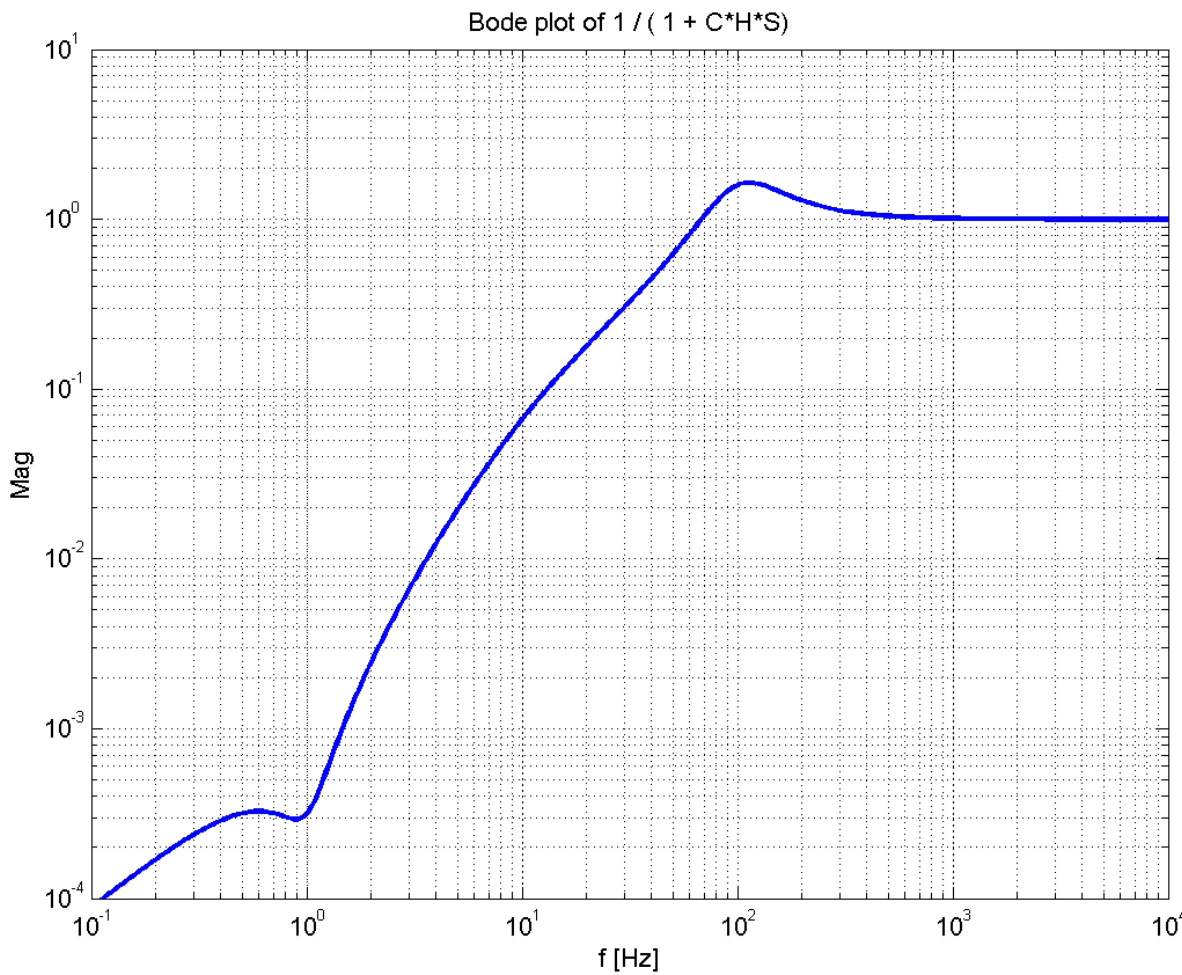
- Step response plot of
$$\frac{1}{1 + CHS}$$
- Very similar response
- Step is driven to zero as it should (it is a suppression factor)
- In about 10 ms (~1/UGF) response is close to zero
- Two oscillation cycles – little ringing



cavityfeedback_exampleB.m

Example: locking one LIGO arm

- Bode plot of the suppression factor
$$\frac{1}{1 + CHS}$$
- More suppression at low frequencies: $\sim 10^4$ at 100 mHz
- No suppression above 100 Hz
- Notice spike at 100 Hz
  - Similar ringing



cavityfeedback_exampleB.m

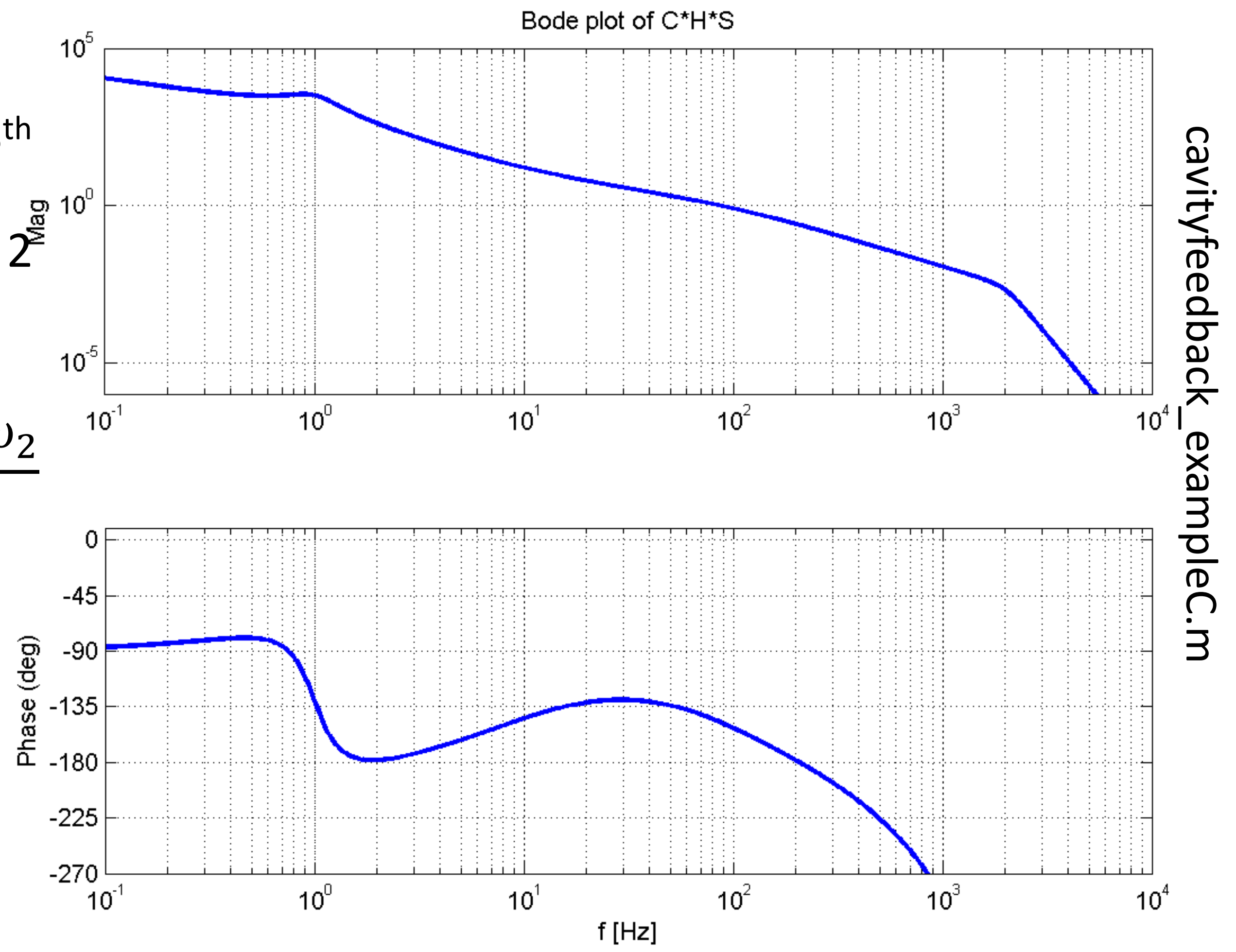
Example: locking one LIGO arm

- Let's also "cut-off" the drive to the coils at high frequency
- Introduce a low pass LP 6th order Butterworth filter with cut-off frequency at 2 kHz.
- Try H

$$H(s) = k \cdot \frac{1}{s} \cdot \frac{s+\omega_1}{\omega_1} \cdot \frac{s+\omega_2}{\omega_2} \cdot \mathrm{LP}$$

with $k = 7000$, $\omega_1 = 2\pi\ 10\mathrm{Hz}$ and $\omega_2 = 2\pi\ 1\ \mathrm{Hz}$

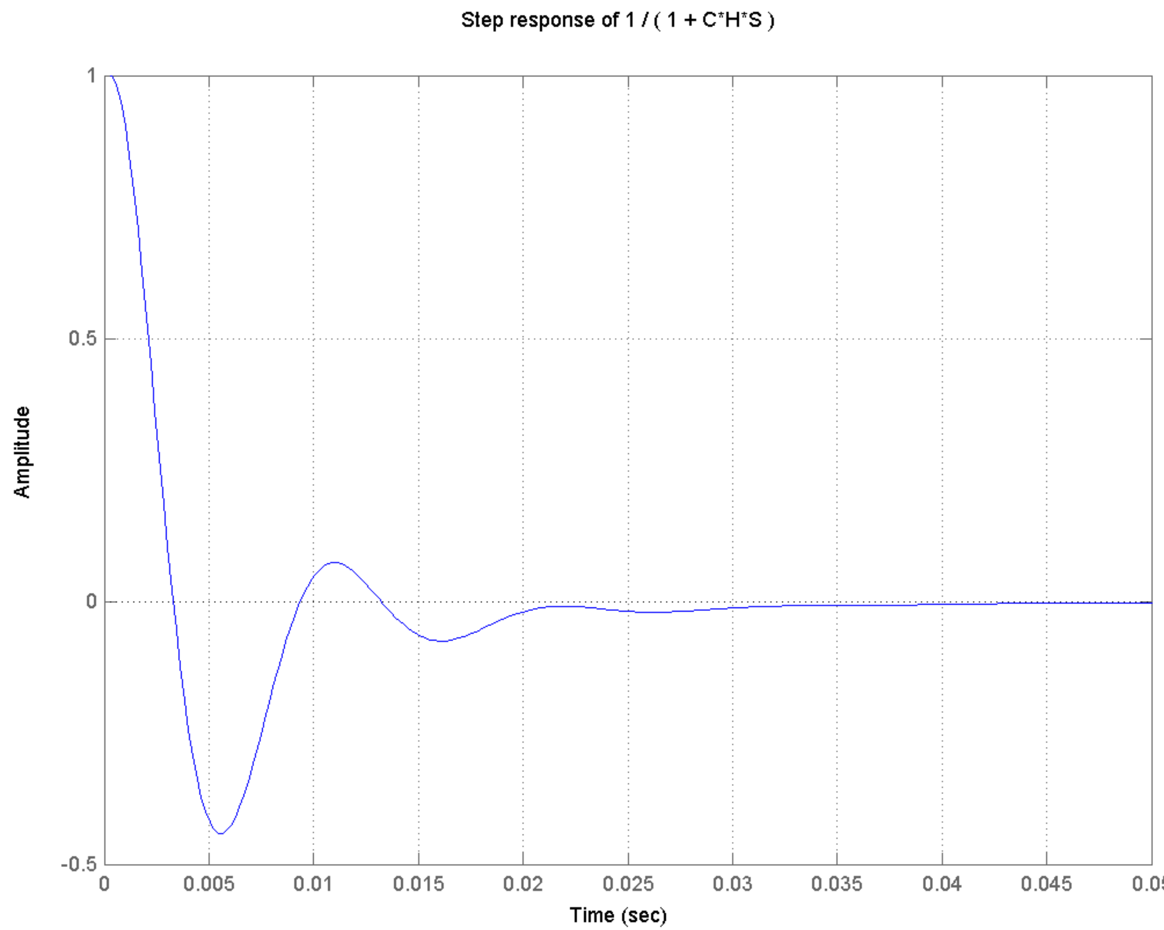
- OL bode plot
  - UGF at 100 Hz
  - PM ~30 deg
  - Stable



Example: locking one LIGO arm

Double check stability

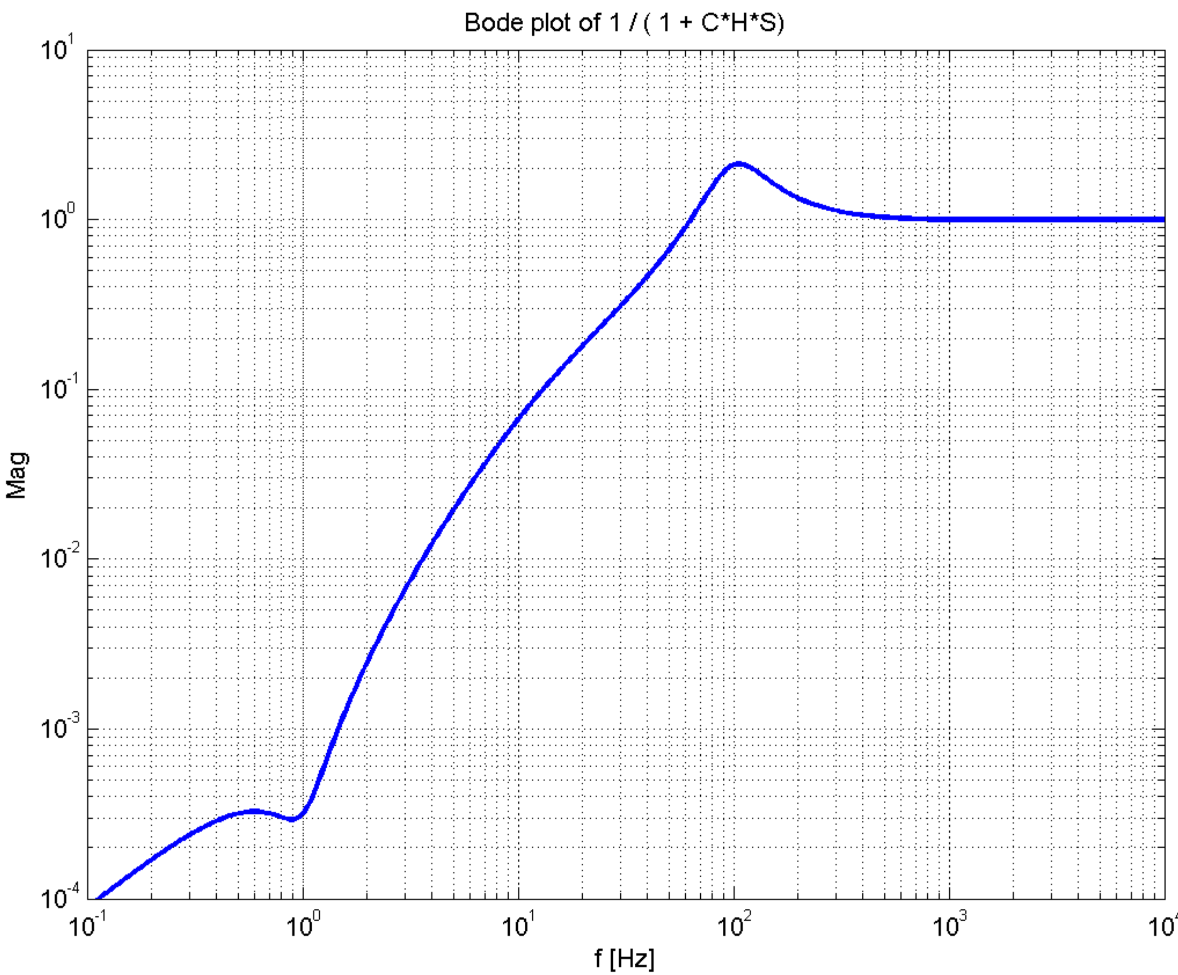
- Step response plot of
$$\frac{1}{1+CHS}$$
- Very similar response
- Step is driven to zero as it should (it is a suppression factor)
- In about 10 ms (~1/UGF) response is close to zero
- ~Two oscillation cycles



cavityfeedback_exampleC.m

Example: locking one LIGO arm

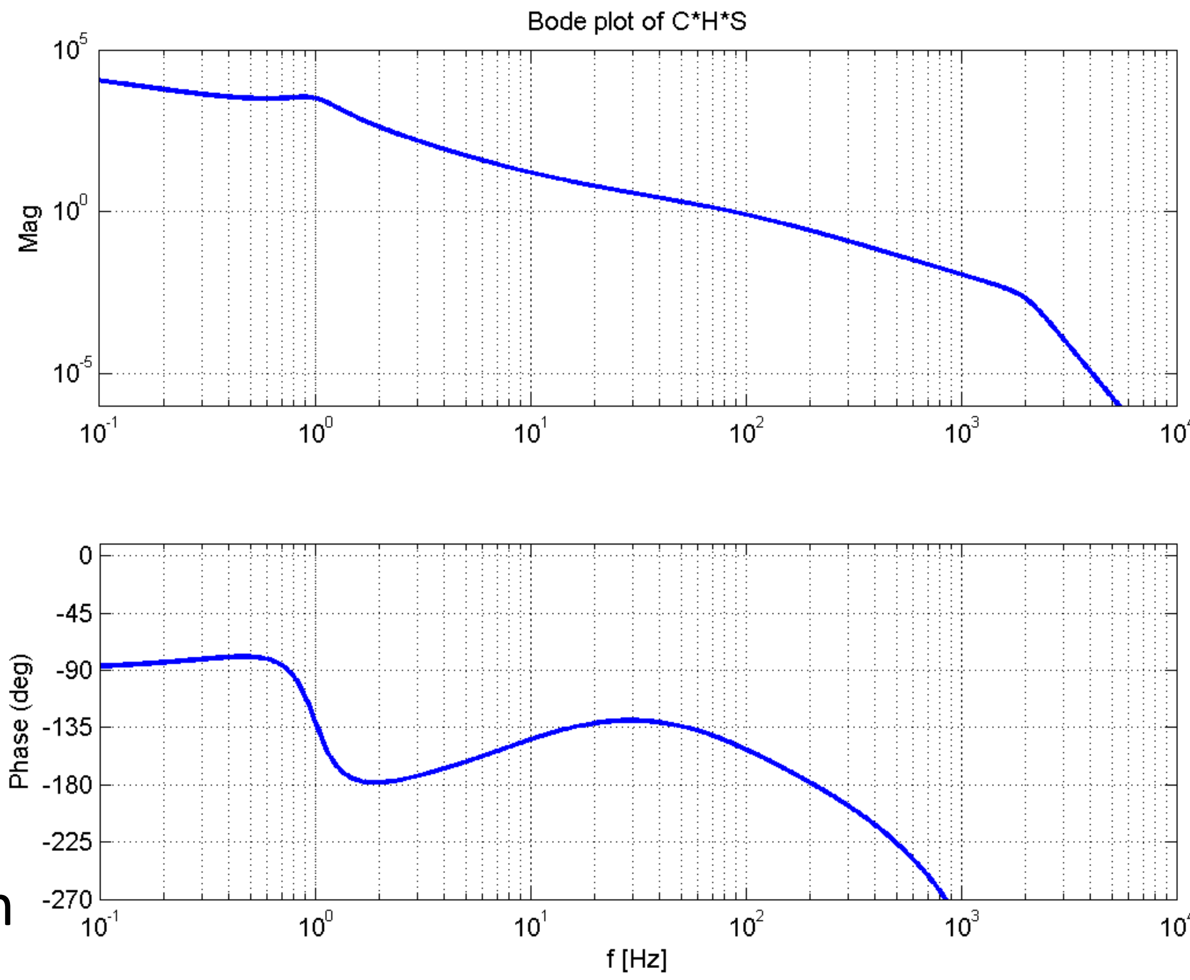
- Bode plot of the suppression factor
$$\frac{1}{1+CHS}$$
- Same suppression: $\sim 10^4$ at 100 mHz
- No suppression above 100 Hz
- Notice spike at 100 Hz
  - A little higher than before
  - Similar ringing



cavityfeedback_exampleC.m

Example: locking one LIGO arm

- Increasing the gain by 2x: $k = 14000$
- OL bode plot
  - UGF at ~133 Hz
    - Should have gone to 200 Hz but the slope is not 1/f (because of the cavity pole at 100 Hz)
  - PM ~20 deg
  - Stable but with little phase margin left



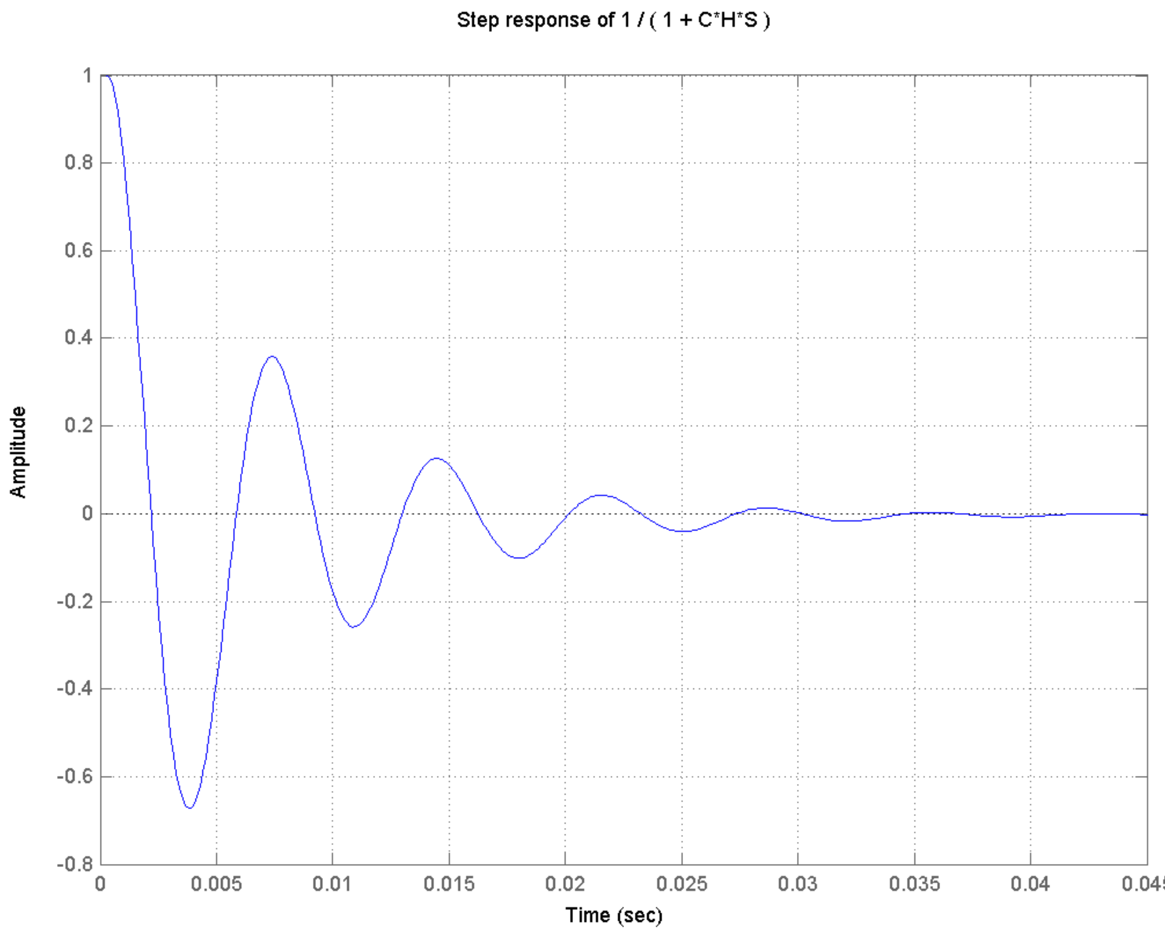
cavityfeedback_exampleD.m

Example: locking one LIGO arm

- Step response plot of

$$\frac{1}{1 + CHS}$$

- Ringing has increased



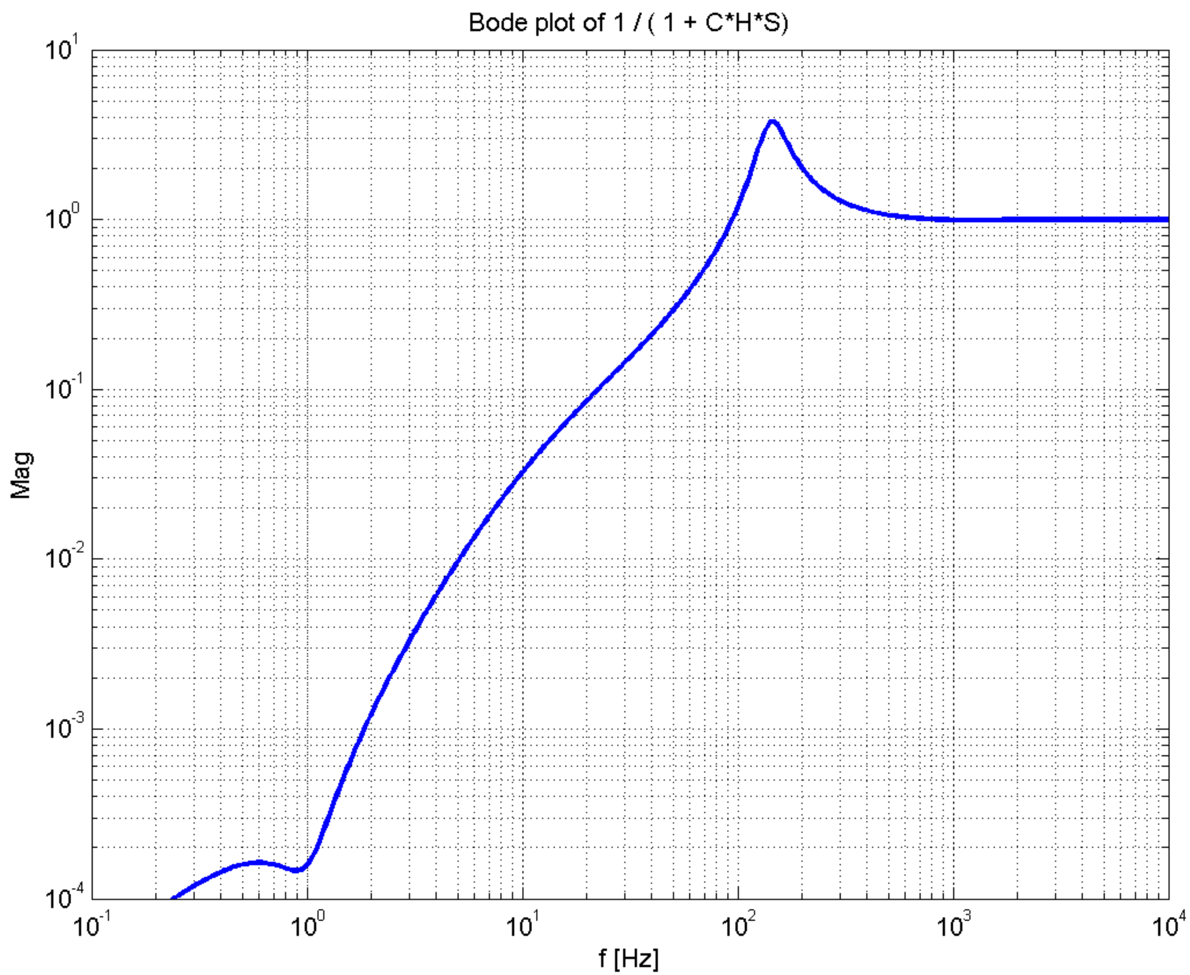
cavityfeedback_exampleD.m

Example: locking one LIGO arm

- Bode plot of the suppression factor

$$\frac{1}{1 + CHS}$$

- Suppression has increased
  - Gain was increased by factor 2
- Notice spike at 100 Hz is more pronounced



cavityfeedback_exampleD.m

Summary

- We have explored the stability criteria
  - The feedback control system is <u>stable</u> if and only if all the *poles of the* ***closed loop transfer function*** G_{CL} have a negative real part. Otherwise the system is unstable.
- Stability in terms of the **open loop gain**
  - A closed loop system is stable if the unity gain frequency is lower than the -180^0 crossing.
  - Rule of thumb: the system is (almost always) stable if $|G_{OL}| \propto \frac{1}{f}$ at the unity gain frequency

Summary

- Noise suppression
- How close to instability is a system? Gain and phase margin
  - Measure of “relative” stability
  - The larger they are → the “safer we are”
  - Rule of thumb: keep the phase margin to more than $40°$
- Typical compensators
  - Phase-lag
  - Phase-lead
  - “Boost”
- Cavity lock example

Problem for the afternoon

Identify a (single-input-single-output) control system at LIGO – its plant TF along with its controller TF (LSC, ASC, SUS, MC, PSL, …)

1. Sketch the block diagram and model the system with MATLAB. Generate the corresponding bode plot.
2. Can you measure its OL TF? Where is the UGF and how does it compare with the model?
3. For what range of frequencies can the UGF be placed at by simply adjusting the systems' gain? What DC gain does it have, what suppression?

Problem for the afternoon

opticallever.m

Optical levers are/were used to damp the fundamental mode of the suspensions. The controller has no DC gain (check this).

1. Sketch the block diagram and model the system with MATLAB. Generate the corresponding bode plot.
2. Can you measure its OL TF? Where is the UGF and how does it compare with the model?
3. For what range of frequencies can the UGF be placed at by simply adjusting the systems' gain? What DC gain does it have, what suppression?

Solutions to problems

What is the transfer function of a system whose input u and output y are related by the following differential equation?

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u + \frac{du}{dt}$$

Sol: Taking the Laplace transform of the equation

$$s^2\, Y(s) + 3\, s\, Y(s) + 2\, Y(s) = U(s) + s\, U(s)$$

Which can be re-written as

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2 + 3\, s + 2}$$

Given $P(s) = \frac{2\,s+1}{s^2+s+1}$, determine the system's differential equation to input $u(t)$.

Sol:

$$y = \left(\frac{2\,D + 1}{D^2 + D + 1}\right)\,u$$

or

$$D^2 y + Dy + y = 2\,D\,u + u$$

or

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 2\frac{du}{dt} + u$$

Determine which of the following transfer functions represent stable systems and which represent unstable systems. Use MATLAB's `step` to verify your answer.

a) $P(s) = \frac{s-1}{(s+2)(s^2+4)}$, unstable

b) $P(s) = \frac{s-1}{(s+2)(s+4)}$, stable

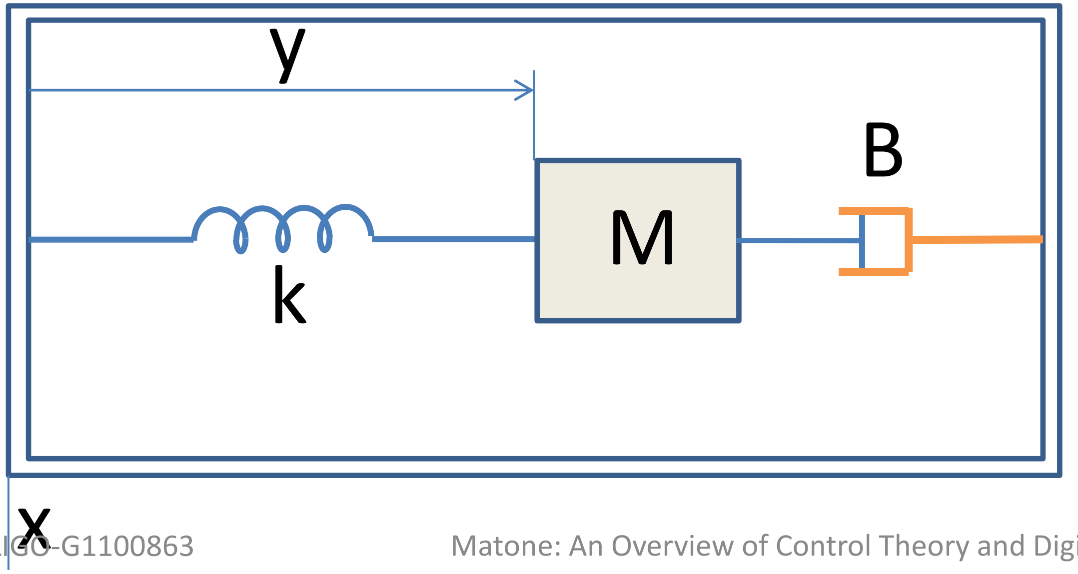
c) $P(s) = \frac{(s+2)(s-2)}{(s+1)(s-1)(s+4)}$, unstable

d) $P(s) = \frac{6}{(s^2+s+1)(s+1)^2}$, stable

e) $P(s) = \frac{5\,(s+10)}{(s^2-s+10)(s+5)}$, unstable

A simple mechanical accelerometer is shown below. The position *y* is with respect of the case, the case's position is *x*. What is the transfer function between the input acceleration A $(a = d^2x/dt^2)$ and the output *Y*?

$$-B\frac{dy}{dt} - ky = M\frac{d^2}{dt^2}(y - x)$$



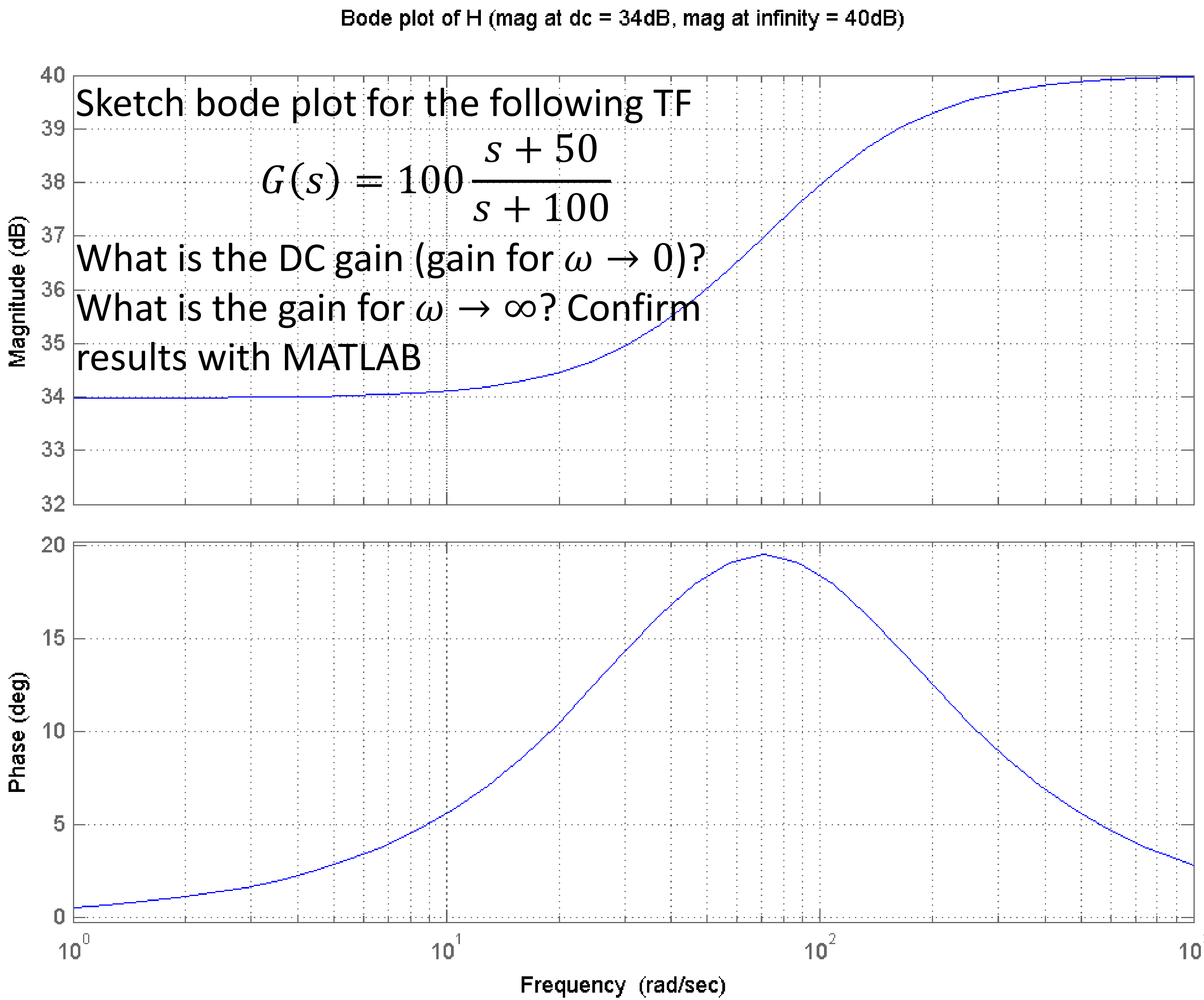
Sol:

$$\frac{Y}{A} = \frac{1}{s^2 + (B/M)\,s + K/M}$$

Sketch bode plot for the following TF

$$G(s) = 100\frac{s+50}{s+100}$$

What is the DC gain (gain for $\omega \to 0$)? What is the gain for $\omega \to \infty$? Confirm results with MATLAB

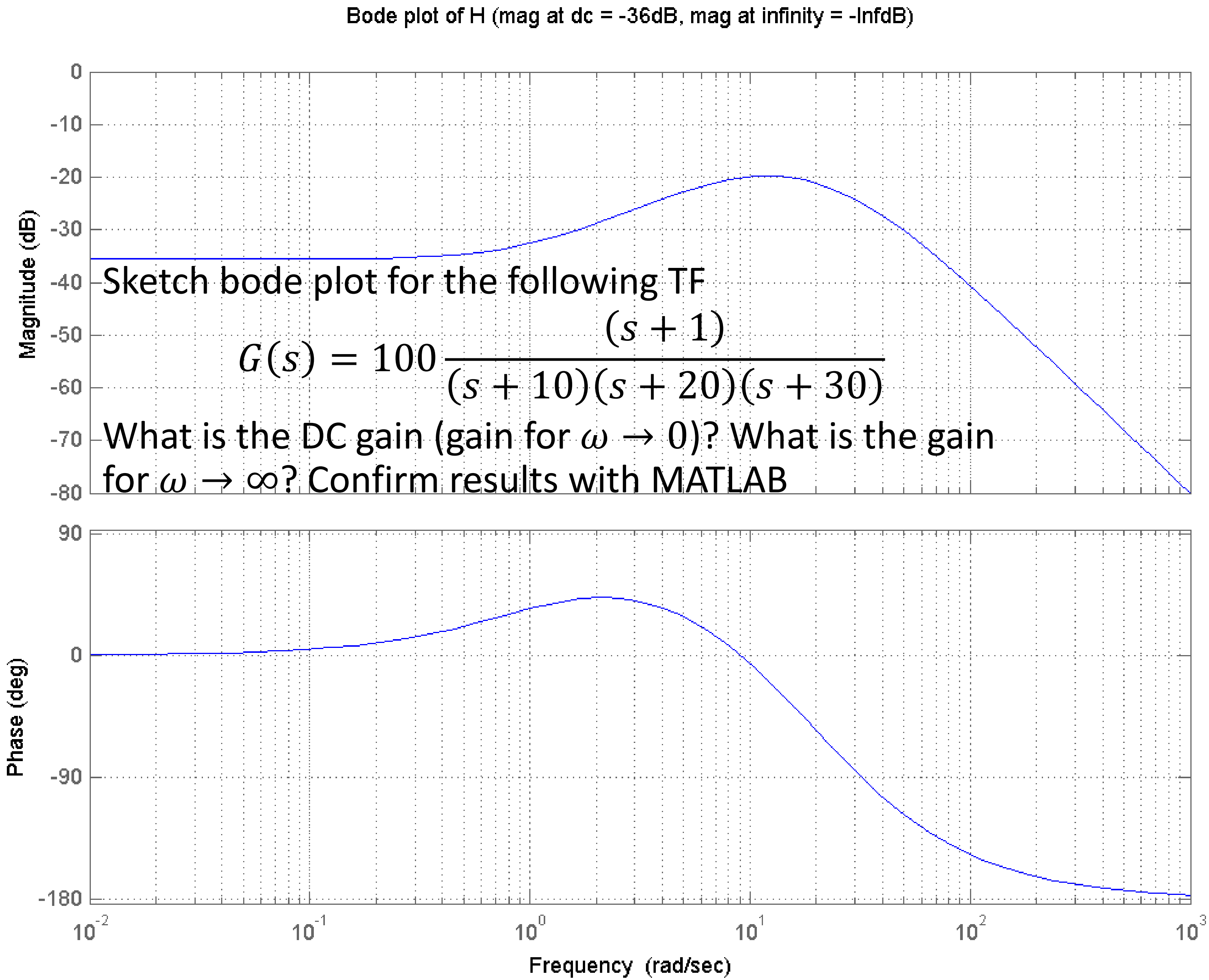


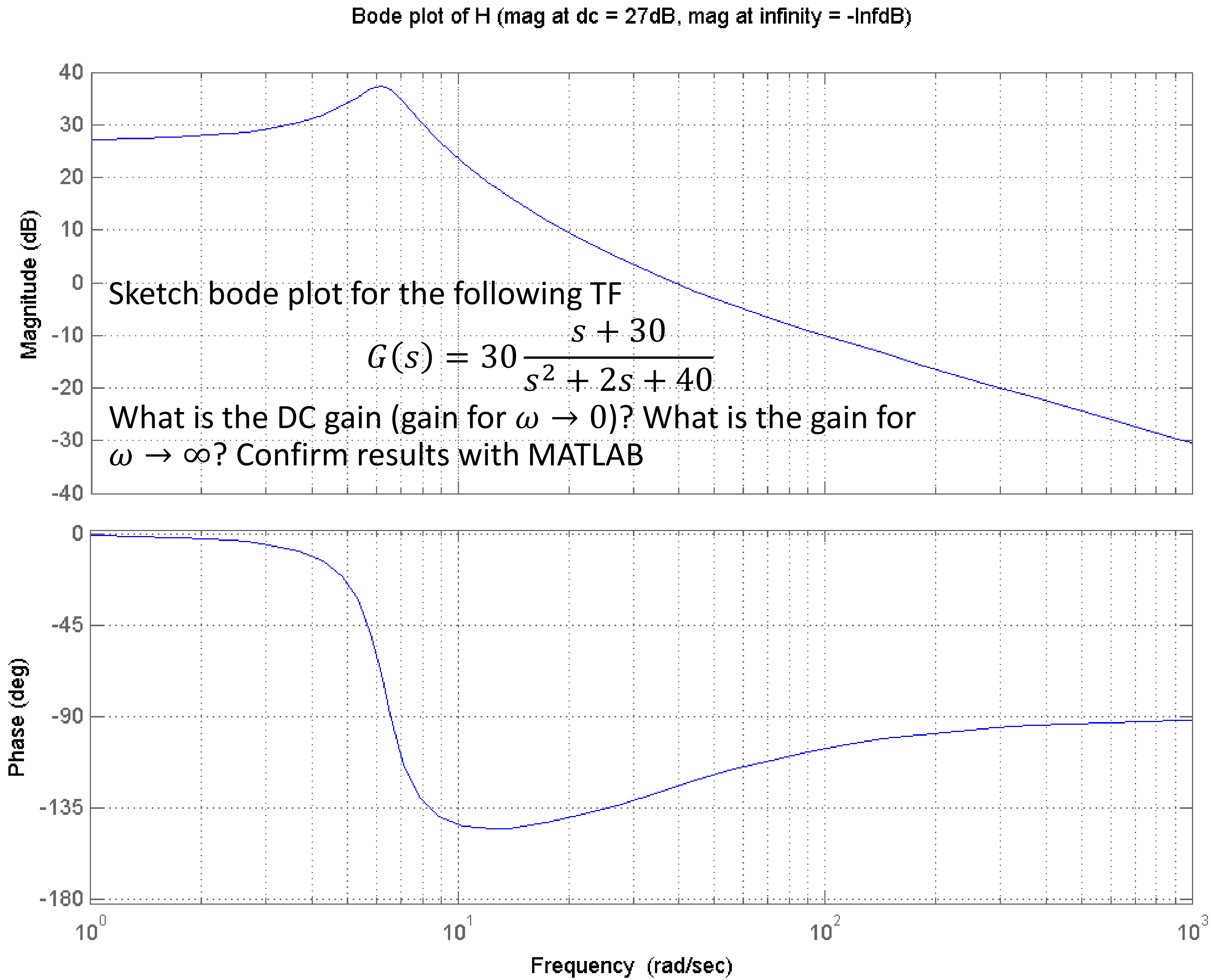
bodeexcercise1.m

Sketch bode plot for the following TF

$$G(s) = 100\frac{(s+1)}{(s+10)(s+20)(s+30)}$$

What is the DC gain (gain for $\omega \to 0$)? What is the gain for $\omega \to \infty$? Confirm results with MATLAB





Sketch bode plot for the following TF

$$G(s) = 30\frac{s+30}{s^2+2s+40}$$

What is the DC gain (gain for $\omega \to 0$)? What is the gain for $\omega \to \infty$? Confirm results with MATLAB

Solution

If a system has an open loop transfer function

$$G_{OL} = \frac{k}{(s+10)(s+100)}$$

what values of k make it stable?

Sol: UGF can be set after the pole at 10 and before pole at 100

$$|G_{OL}| = k\frac{1}{\sqrt{\omega^2+10^2}}\cdot\frac{1}{\sqrt{\omega^2+100^2}}$$

Set $|G_{OL}| = 1$ and $\omega = 10$. Find corresponding k.

Set $|G_{OL}| = 1$ and $\omega = 100$. Find corresponding k

Solution

If a system has an open loop transfer function

$$G_{OL} = \frac{10^3}{(s+10)^3}$$

design a compensator that would make the system stable with an UGF at 100 Hz. Use MATLAB to confirm this.

Sol: two zeros at 10, decreasing the gain by 3x

```
H=zpk([],[-10 -10 -10],1e3) * zpk([-10 -10],[],0.3)
```