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On the round-trip time for a photon propagating in the field of a plane gravitational wave

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Abstract. Current searches for gravitational waves employ laser interferometers with arm lengths of several kilometers such as LIGO and VIRGO. Calculations of the response of these detectors to gravitational waves is usually done by direct integration of the metric perturbations along the unperturbed photon trajectory. Although such a simplified approach is widely accepted, it is not clear if it leads to the correct answer. In a rigorous approach, one would derive the detector response by solving the equation for null geodesics with appropriate boundary conditions. Such a derivation is given in this paper. We show that the symmetries of a plane gravitational wave allow exact solution of the equation for a null geodesic. Based on this solution we then calculate the round-trip time for a photon propagating in the field of a plane gravitational wave, and show that the result agrees with that of the simplified approach.

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1. Introduction

Searches for gravitational waves with km-scale laser interferometric detectors such as LIGO [1] and VIRGO [2] require accurate knowledge of the detector response also known as antenna patterns [3]. Frequency dependence of the antenna patterns have been known for a while [4, 5, 6] and recently acquired a renewed interest in connection with two efforts. On the one hand, the frequency dependence noticeably affects the parameter estimation of the gravitational wave in the low-frequency range (1-2 kHz) [7, 8]. On the other hand, the dynamic response of the Fabry-Perot arm cavities allows searches for gravitational waves at the free spectral range (37.5 kHz) [9]. The interest in the high-frequency antenna patterns [9], [10], led to the development of several efforts to analyze the data at the free-spectral-range of the LIGO arm cavities (37.5 kHz) [11, 12, 13]. Analysis of the frequency dependence of the antenna patterns in these efforts was based on analytical models previously described in [14, 15, 16]. In all these calculations one tacitly assumes that the photon trajectory is a straight line and ignores possible corrections due to the deviations from the straight line which may contribute to the final result [16]. However, such an overly simplified treatment is unnecessary. In this paper we show that one can solve exactly the equation for null geodesic and thus obtain the full information about the photon trajectory in the field of the plane gravitational wave. One can then derive the frequency dependent antenna patterns directly from the equation for null geodesic. Curiously, the result agrees with that of the simplified approach.

Consider a spacetime with a plane-fronted gravitational wave propagating in a flat background. Denote the spacetime coordinates by x^μ where $\mu = 0, 1, 2, 3$ and $x^0 = ct \equiv \tau$. The linearized gravitational waves are described in general relativity by a symmetric second rank tensor: $h_{\mu\nu}(x)$, which depends on the coordinate system, or gauge choice. In the transverse traceless gauge, it takes a particularly simple form:

$$h_{\mu\nu} = \begin{bmatrix} 0 & & & \\ & h_+ & h_\times & \\ & h_\times & -h_+ & \\ & & & 0 \end{bmatrix}, \tag{1}$$

where h_+ and $h_\times$ are functions of $\tau + z$ only. The metric of spacetime with the gravitational wave is therefore

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2}$$

where $\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the background Minkowskii metric. Here we assumed that the gravitational wave is propagating in the negative z direction and its principle axes of polarization match the x and y directions.

2. Doppler tracking formula

Simplified calculations of the photon propagation time in the transverse traceless gauge are now well-known. Originally developed for Doppler tracking of remote spacecraft [17], such calculations have routinely been used for LISA gravitational wave detector. The standard approach is to find the photon propagation time directly from the null condition [18, 19, 20] and more recently in [7]. Here we briefly remind the reader of the how the simplified calculations proceed, following the recent derivation in [8].

The interval for a photon propagating in spacetime with a gravitational wave is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j = 0, \tag{3}$$

where $h_{ij} = h_{ij}(t, \vec{x})$. Consider a photon launched in the direction $\hat{a}$ to be bounced back by a mirror some distance L away. On the way forward, the unperturbed photon trajectory is $x^i = a^i\xi$, where $\xi \in [0, L]$. Substituting this trajectory in (3) and solving for t, we obtain

$$c(t - t_0) = \int_0^\xi \left(1 + h_{ij}\, a^i a^j\right)^{1/2} \mathrm{d}\xi'. \tag{4}$$

Let T be the nominal (unperturbed) photon transit time: $T \equiv L/c$. In the presence of a gravitational wave, the transit time will slightly deviate from its nominal value giving rise to a small perturbation:

$$\delta T(t) = \frac{1}{2c} a^i a^j \int_0^L h_{ij} \left(t_0 + \frac{\xi}{c} + \frac{\hat{n}\cdot\hat{a}}{c}\xi\right) \mathrm{d}\xi, \tag{5}$$

where t_0 is the starting time for the photon propagation which can be approximated by $t_0 = t - T$. Similarly, on the way back,

$$\delta T'(t) = \frac{1}{2c} a^i a^j \int_0^L h_{ij} \left(t_0 + \frac{L-\xi}{c} + \frac{\hat{n}\cdot\hat{a}}{c}\xi\right) \mathrm{d}\xi, \tag{6}$$

where t_0 can also be approximated by $t_0 = t - T$. By combining (5) and (6), we obtain the perturbation of the round-trip time:

$$\delta T_{\rm r.t.}(t) = \delta T(t-T) + \delta T'(t). \tag{7}$$

It is convenient to write this result in terms of transfer functions:

$$\frac{\delta \tilde{T}_{\rm r.t.}(f)}{T} = a_i a_j \, D(\hat{a}, f)\, \tilde{h}_{ij}(f), \tag{8}$$

where we introduced the transfer function

$$D(\hat{a}, f) = \frac{{\rm e}^{-i2\pi f T}}{2} \left[{\rm e}^{i\pi f T_+} {\rm sinc}\,(\pi f T_-) + {\rm e}^{-i\pi f T_-} {\rm sinc}\,(\pi f T_+)\right], \tag{9}$$

with short-hand notation: $T_\pm \equiv T(1 \pm \hat{a}\cdot\hat{n})$.

The main problem with the above approach is that the linear corrections from the deviations of the photon trajectory from the straight line are neglected [16]. Consequently, it is not clear if the answer for $D(\hat{a}, f)$ is correct. Note that such corrections cannot be added within the above approach. Instead, one has to obtain full information about the photon trajectory which can only be done within the framework of the null geodesics.

3. Equation for null geodesics

It will be convenient to introduce two auxiliary variables:

$$u = \tau + z, \tag{10}$$

$$v = \tau - z. \tag{11}$$

In these coordinates the fundamental form is

$$ds^2 = -du\,dv + dx^2 + dy^2 + h_+(u)(dx^2 - dy^2) + 2h_\times(u)\,dx\,dy. \tag{12}$$

Propagation of light in this spacetime is described by a null geodesic $x^\mu(\sigma)$, where σ is affine parameter along the curve. The tangent vector to this curve is

$$p^\mu = \frac{dx^\mu}{d\sigma} \qquad \text{and} \qquad p_\mu = g_{\mu\nu}\, p^\nu. \tag{13}$$

For null geodesics,

$$p_\mu p^\mu = 0. \tag{14}$$

In the explicit form, the covariant components are given by

$$p_v = -\frac{1}{2}\frac{du}{d\sigma}. \tag{15}$$

$$p_x = [1 + h_+(u)]\,\frac{dx}{d\sigma} + h_\times(u)\frac{dy}{d\sigma}, \tag{16}$$

$$p_y = [1 - h_+(u)]\,\frac{dy}{d\sigma} + h_\times(u)\frac{dx}{d\sigma}, \tag{17}$$

$$p_u = -\frac{1}{2}\frac{dv}{d\sigma}. \tag{18}$$

Consider the equation for null geodesic. Written in terms of the covariant components of the tangent vector, it takes the form:

$$\frac{dp_\alpha}{d\sigma} = \frac{1}{2}\, g_{\mu\nu,\alpha}\, p^\mu p^\nu. \tag{19}$$

Since the metric does not depend on x, y, and v coordinates, we immediately find three first integrals:

$$p_v = p_{v0}, \tag{20}$$

$$p_x = p_{x0}, \tag{21}$$

$$p_y = p_{y0}. \tag{22}$$

The fourth equation is

$$\frac{dp_u}{d\sigma} = \frac{1}{2} h'_+(u) \left(p_x^2 - p_y^2\right) + h'_\times(u)\, p_x p_y. \tag{23}$$

Its solution can be obtained directly from the normalization condition (14), and yields p_u as a function of u:

$$p_u(u) = \frac{1}{4p_{v0}} \left[p_{x0}^2 + p_{y0}^2 - h_+(u)(p_{x0}^2 - p_{y0}^2) - 2h_\times(u)\, p_{x0} p_{y0}\right]. \tag{24}$$

Note that p_{v0} cannot be zero, which guarantees that p_u is well defined. Thus we obtained all four first integrals for the equation of null geodesic.

Second integration can be done as follows. Equation (20) together with the definition for p_v in (15) can be integrated and yields

$$u(\sigma) = u_0 - 2p_{v0}\sigma, \tag{25}$$

where u_0 is the initial value for this coordinate. This fully defines h_i along the geodesic, which can then be used for integration of other equations.

Consider x and y coordinates, (21) and (22). Together with the definitions (16) and (17), these equations lead to the solution for x:

$$x(\sigma) = x_0 + p_{x0}\sigma[1 - g_+(\sigma)] - p_{y0}\sigma\, g_\times(\sigma), \tag{26}$$

$$y(\sigma) = y_0 + p_{y0}\sigma[1 + g_+(\sigma)] - p_{x0}\sigma\, g_\times(\sigma), \tag{27}$$

where x_0 and y_0 are the initial values for these coordinates. Here we introduced compact notations for integrals along the null geodesic:

$$g_i(\sigma) = \frac{1}{\sigma} \int_0^\sigma h_i[u(\sigma')] d\sigma'. \tag{28}$$

Finally, the solution for v is

$$v(\sigma) = v_0 - 2 \int_0^\sigma p_u[u(\sigma')]\, d\sigma'. \tag{29}$$

where v_0 is the initial value for this coordinate.

We have thus derived a general solution for the equation for null geodesic in spacetime with plane gravitational wave. The constants of integration: p_{x0}, p_{y0}, p_{v0} and u_0, x_0, y_0, v_0 are completely arbitrary at this point.

4. Solution of the boundary value problem

Propagation of a photon between two known locations can be viewed as a boundary-value problem. Consider a null geodesic which has fixed boundaries: points P and Q in space but not time. The goal is to calculate the time of travel between these two points. In time-dependent geometry the answer will be a function of time, it can be either the time of the beginning of the trip or the time of its end. If the measurement is continuous, the preference is given to the end time as this corresponds to the time of the detection.

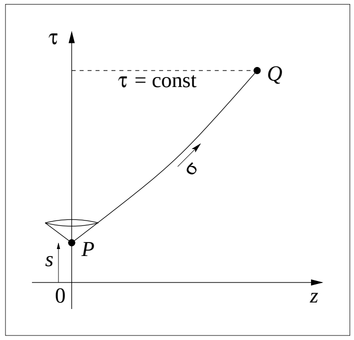

Figure 1. Null geodesic PQ.

The boundary-value problem can be solved as follows. Assume that the null geodesic originates at $P = (0, 0, 0)$ and ends at $Q = (x, y, z)$. The end time t will viewed as arbirtrary, whereas the beginning time t_0 will be a function of t. Then the equation for null geodesic can be written as

$$u = u_0 - 2p_{v0}\sigma, \tag{30}$$

$$x = p_{x0}\,\sigma\,(1 - g_+) - p_{y0}\,\sigma\,g_\times, \tag{31}$$

$$y = p_{y0}\,\sigma\,(1 + g_+) - p_{x0}\,\sigma\,g_\times, \tag{32}$$

$$v = v_0 - \frac{\sigma}{2p_{v0}}\left[p_{x0}^2(1 - g_+) + p_{y0}^2(1 + g_+) - 2\,p_{x0}p_{y0}\;g_\times\right]. \tag{33}$$

In these equations

$$u_0 = v_0 = \tau_0. \tag{34}$$

The solution of the boundary-value problem goes as follows. First we find the momentum components as functions of the end point on the geodesic:

$$p_{v0} = \frac{u - \tau_0}{2\sigma}, \tag{35}$$

$$p_{x0} = (1 + g_+)\frac{x}{\sigma} + g_\times\,\frac{y}{\sigma}, \tag{36}$$

$$p_{y0} = (1 - g_+)\frac{y}{\sigma} + g_\times\,\frac{x}{\sigma}. \tag{37}$$

Substituting Eqs.(36)(37) to Eq.(33), we obtain:

$$v = \tau_0 - \frac{1}{2p_{v0}\sigma}\left[x^2(1 + g_+) + y^2(1 - g_+) + 2xyg_\times\right]. \tag{38}$$

Next replacing p_{v0} from Eq.(35), we obtain:

$$(\tau - \tau_0)^2 = r^2 + (x^2 - y^2)g_+ + 2xyg_\times, \tag{39}$$

where r is the Euclidean distance

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{40}$$

To first order in h the solution is

$$\tau - \tau_0 = r + \frac{1}{2r}\left[(x^2 - y^2)g_+ + 2xyg_\times\right]. \tag{41}$$

The left hand side of this equation is the propagation time for a photon traveling along the null geodesic connecting the two points P and Q.

For dimensional reasons it is convenient to rewrite the previous result in the form:

$$\tau - \tau_0 = r + \frac{r}{2}\left[(n_x^2 - n_y^2)g_+ + 2n_x n_y g_\times\right], \tag{42}$$

where we introduced n_i, the unit vector pointing to the end point from the origin. This can only be done in Euclidean sense. To avoid geometrical difficulties with such definitions we simply set

$$\mathbf{n} \equiv (x/r, y/r, z/r). \tag{43}$$

5. The round-trip propagation time

5.1. Forward trip

We can now apply these formulas to calculate the round-trip time. First consider the forward propagation of the photon. As we have seen, the propagation time in this case is

$$T_1(t) = T + \frac{1}{2}T\left[(n_x^2 - n_y^2)A_+(t) + 2n_x n_y A_\times(t)\right], \tag{44}$$

where we introduced a special notation: A_i for g-factors in the forward trip, and explicitly indicated that they are functions of time. Next we will explicitly show what this functional dependence is. First, we change the variables from σ to u and write

$$A_i = \frac{1}{u - u_0}\int_{u_0}^{u} h_i(u')\, du'. \tag{45}$$

Since A_i is of order h, we can replace u with its unperturbed value,

$$u = \tau + z \approx \tau_0 + r + z. \tag{46}$$

Therefore,

$$A_i = \frac{1}{z + r}\int_{\tau_0}^{\tau_0 + r + z} h_i(u')\, du'. \tag{47}$$

It is important that we change u in both upper limit of integration and the multiplicative factor to maintain the expression finite in the limit: $u \to u_0$. We have thus obtained A_i as a function of the starting time τ_0. We then switch from the beginning time τ_0 to the end time τ:

$$A_i(\tau) = \frac{1}{z + r}\int_{0}^{z + r} h_i(\tau - r + \beta)\, d\beta. \tag{48}$$

5.2. Return trip

Similar calculations can be done for the return trip. In this case we start at the point with coordinates x, y, z and end at the origin. The answer is

$$T_2(t) = T + \frac{1}{2}T\left[(n_x^2 - n_y^2)B_+(t) + 2n_x n_y B_\times(t)\right], \tag{49}$$

where

$$B_i(\tau) = \frac{1}{r - z}\int_{0}^{r - z} h_i(\tau - r + z + \beta)\, d\beta. \tag{50}$$

5.3. Round trip

The round-trip time is

$$T_{\mathrm{r.t.}}(t) = T_2(t) + T_1[t - T_2(t)]. \tag{51}$$

Within the first order approximation we can replace the argument of the second term in the right hand side with $t - T$. The result is

$$T_{\mathrm{r.t.}}(t) \approx T_2(t) + T_1(t - T). \tag{52}$$

It is convenient to separate the (unperturbed) dominant part of the round-trip time and the small variation due to the gravitational wave:

$$T_{\mathrm{r.t.}}(t) \equiv 2T + \delta T_{\mathrm{r.t.}}(t). \tag{53}$$

Then the change in the round-trip time is given by

$$\delta T_{\mathrm{r.t.}}(t) = \frac{1}{2}\, T \left[(n_x^2 - n_y^2) C_+(t) + 2 n_x n_y C_\times(t) \right]. \tag{54}$$

where

$$C_i(t) = A_i(t - T) + B_i(t). \tag{55}$$

Or explicitly

$$\begin{aligned} C_i(t) = {} & \frac{1}{r+z} \int_0^{r+z} h_i(\tau - T - r + \alpha)\, d\alpha + \\ & \frac{1}{r-z} \int_0^{r-z} h_i(\tau - r + z + \alpha)\, d\alpha. \end{aligned} \tag{56}$$

It is customary to represent the result as a transfer function in the Laplace domain. Laplace transform of $h_i(t)$ is given by

$$\tilde{h}_i(s) = \int_0^\infty e^{-st}\, h_i(t)\; dt. \tag{57}$$

Then the formula for the round-trip propagation time is

$$\delta \tilde{T}_{\mathrm{r.t.}}(s) = \frac{1}{2}\, T G(s) \left[(n_x^2 - n_y^2) \tilde{h}_+(s) + 2 n_x n_y \tilde{h}_\times(s) \right], \tag{58}$$

where

$$\tilde{G}(s) = \frac{1}{sT} \left[\frac{1 - e^{-(1-n_z)sT}}{1 - n_z} - e^{-2sT} \frac{1 - e^{(1+n_z)sT}}{1 + n_z} \right]. \tag{59}$$

The equivalence with the Fouirier domain is estabished by substitution: $s = 2\pi i f$. With this substitution, $G(s)$ is equivalent to $D(f)$ in Eq.(9).

6. Conclusion

We derived the round-trip time for a photon propagating in a gravitational-wave background from the equation for null geodesics. For fixed point boundary-value problem the solution for the photon round trip coinsides with the well-known Doppler-tracking formula. This calculations confirm that the recent estimate of the frequency corrections to the antenna patterns in [8] are valid even in the strict general-relativistic sense.

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References

[1] Barish B and Weiss R 1999 *Physics Today* **52** 44
[2] Bradaschia C *et al.* 1990 *Nuclear Instruments and Methods in Physics Research A* **289** 518
[3] Forward R 1978 *Phys. Rev. D* **17** 379
[4] Gürsel Y, Linsay P, Spero R, Saulson P, Whitcomb S and Weiss R 1984 in *A Study of a Long Baseline Gravitational Wave Antenna System* (National Science Foundation Report)
[5] Estabrook F 1985 *Gen. Relat. Grav.* **17** 719
[6] Christensen N 1992 *Phys. Rev. D* **46** 5250
[7] Baskaran D and Grishchuk L 2004 *Class. Quantum Grav.* **21** 4041
[8] Rakhmanov M, Romano J and Whelan J 2008 *Class. Quantum Grav.* **25** 184017
[9] Sigg D and Savage R 2003 Analysis proposal to search for gravitational waves at multiples of the LIGO arm cavity free-spectral-range frequency. LIGO Technical Report T030296
[10] Hunter E 2005 Analysis of the frequency dependence of the LIGO directional sensitivity (antenna pattern) and implications for detector calibration. LIGO Technical Report T050136
[11] Parker J 2007 Development of a high-frequency burst analysis pipeline. LIGO Technical Report T070037
[12] Forrest C, Fricke T, Giampanis S and Melissinos A 2007 Search for a diurnal variation of the power detected at the FSR frequency LIGO Technical Report T070228
[13] Giampanis S 2008 *Search for a High Frequency Stochastic Background of Gravitational Waves* Ph.D. thesis University of Rochester
[14] Sigg D 1997 Strain calibration in LIGO. LIGO Technical Report T970101-B
[15] Rakhmanov M 2006 Response of LIGO 4-km interferometers to gravitational waves at high frequencies and in the vicinity of the FSR (37.5 kHz). LIGO Technical Report T060237
[16] Whelan J 2007 Higher-frequency corrections to stochastic formulae. LIGO Technical Report T070172
[17] Estabrook F and Wahlquist H 1975 *Gen. Relat. Grav.* **6** 439
[18] Schilling R 1997 *Class. Quantum Grav.* **14** 1513
[19] Larson S, Hiscock W and Hellings R 2000 *Phys. Rev. D* **62** 062001
[20] Cornish N and Rubbo L 2003 *Phys. Rev. D* **67** 022001