LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

*LIGO Laboratory / LIGO Scientific Collaboration*

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# Loss function for characterizing cavities

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# 1 Introduction

In order to calculate the cavity loss caused by mirror surface aberrations accurately, numerical simulations with surface maps of all mirrors are necessary. But, there is an approximation to calculate the loss using PSDs of mirrors and a "loss function" of the cavity. This is useful to estimate the loss without doing lengthy calculations once the loss function is calculated for the cavity of interest. This allows to understand which spatial frequency region is crucial to suppress the cavity loss and which region has little impact on the loss.

A simple formulation of the algorithm is described in Sec.2, together with limitations of the method. A procedure to calculate the loss function using SIS and a matlab package using Hankel transformation, called "Hankel IFO" here after, is detailed in Sec.3, and a few example loss functions are included in Sec.4.

# 2 Formulation

The loss function L(f) is defined to be the loss of a given cavity when one of the mirror has a sinusoidal aberration, d(x,y), with spatial frequency f, normalized by the rms$^2$ of the aberration. More specifically,

$$d(x,y) = \sqrt{2}\cdot a\cdot \cos(2\pi f x) \qquad (1)$$

$$L(f) = [\text{loss with aberration d(x,y)}]\ / a^2$$

One dimensional power spectral density, $PSD_{1D}$, is defined to be the density function of surface variation rms$^2$ as a function of the spatial frequency. The definitions of $PSD_{1D}$ are discussed in LIGO-T1100353. For this calculation, the definition $PSD_{1DA}$ is used, whose normalization is easier to be used for this calculation.

By using these two quantities, the total loss can be calculated as

$$\text{total loss } = \int df\ PSD_{1D}(f)\cdot L(f) \qquad (2)$$

Once the loss function L(f) is calculated, the total loss can be calculated by the convolution of the PSD of a mirror and the loss function, and can easily estimate if a certain surface specified by its PSD is acceptable or not.

When the spatial frequency is large and all scattered fields go out of the cavity, the loss function becomes

$$L(\infty) = (\frac{4\pi}{\lambda})^2 = 140\,/\,nm^2 \qquad (3)$$

The derivation and implication of the "golden rule" is explained in the Appendix 1, which is used to derive this analytic expression.

The loss due to this high frequency region can be calculated by taking this constant loss function out of the integral, and can be rewritten as

$$loss = (\frac{4\pi}{\lambda})^2 \int_{f_{\min}}^{f_{\max}} PSD(f)df = (\frac{4\pi\sigma(f_{\min} \sim f_{\max})}{\lambda})^2 \qquad (4)$$

There are caveats about this method. First, the formula (2) is an approximation, which is valid only for the range of spatial wavelength which is much shorter than the size of optics and beams of interest and for those small perturbations whose second order effects can be neglected. The concept of PSD, or the frequency domain representation of the surface, is bounded by the same constraint.

Second, the 1D PSD has limited information. E.g., non-axisymmetric structre is averaged out.

Third, various dynamics may not be included in the loss function. E.g., certain surface aberrations may induce modes which could resonate in the cavity, but this will not be observed by this frequency analysis. Another example is that aberrations with different spatial frequencies may favor different locked lengths. When a surface has these two aberrations mixed, the locked length will be different from either of the length favored by individual frequencies.

In short, this algorithm is to be used to quickly estimate how the roughness in a certain frequency region is harmful or not, not to get quantitative results with high precision.

A simple example will help to have an idea.

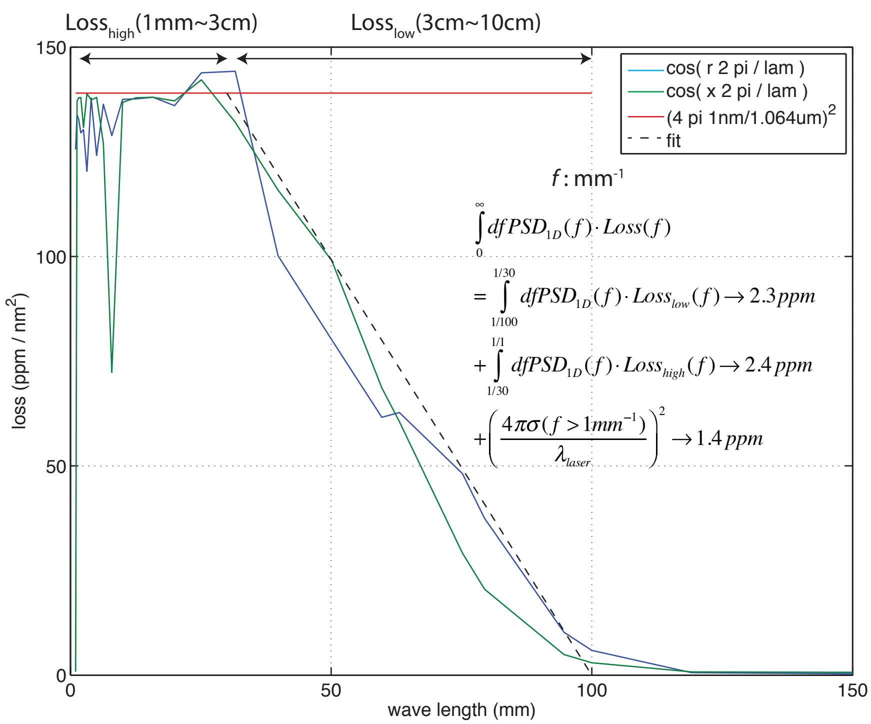


**Figure 1 Loss function of aLIGO ETM**

loss function of aLIGO cavity using cos(2 pi r / lambda ) and cos( 2 pi x / lambda ) to see the ambiguity of using circular periodical pattern.
Also calculated is the loss of an aLIGO mirror using PSD and loss function. Looks OK.
As I said, the actual loss calculated using the full phasemap ranges from 3ppm to 10ppm, and I used an average of the all PSD fit.

This is the loss function of a ETM mass in an aLIGO cavity. An example of the PSD is shown in the 3rd plot of Fig.2. The loss is negligible for the spatial wavelength, $\lambda_s$ > 100mm, because the reflected field with this

## 3 How to calculate the loss function using SIS

Several files are attached to this document in the DCC as examples to show how to calculate the loss function using SIS. Details of SIS are discussed in LIGO-T070039, SISManual. Only those points relevant for this document are summarized below.

In SIS, calculations of field evolutions and interactions with optics are done on N x N grid points in a square area, W x W. In each grid square area of the size W/N x W/N, all quantities are assumed to be uniform. Because of this, the simulation can handle the spatial wavelength region longer than 2 x W/N. This is mathematical lower limit and the practical lower limit is several times of the grid size. Any physical effect related to shorter wavelength needs to be handled separately, like adding loss calculated analytically. Another requirement is that the size of the window W needs to be larger than the size of the optic, typically 1.5 or larger. This is to avoid the aliasing effect caused by the FFT method used for the calculation of field propagations. The detail is discussed in LIGO-T1200036, "Aliasing effect in FFT simulation".

The general procedure to calculate the loss function is as follows.

First the target field quantity, e.g., the total power or the power of specific mode, is calculated without any aberration. For the convenience of the discussion, P00 is used to denote the power of the target field. Second, add an aberration on one of the mirror of the cavity, with one spatial frequency, $f_s$. Then Calculate P00 with this aberration. Loss for this spatial frequency is calculated as (1-P00($f_s$)/P00(no aberration)). The calculation of P00($f_s$) is repeated for the range of spatial frequency of interest. This spatial wavelength needs to be several times larger than the grid size.

There are several practical issues.

sisDB_FP_lossFunc.mcr is a file which defines a FP cavity. In the future, a triangular cavity will be added as another example. This file sets up a symmetric FP cavity, which is specified by a mirror aperture, a cavity length and a beam size. These values can be changed by assigning values for variables "aperture", "cavLeng" and "beamSize".

Several other files with .in as the file name extension are files to direct SIS to calculate the loss functions for a series of values of spatial frequencies. When the calculation is completed, a data file is created which has values for all frequencies.

E.g., lossFuncX.in will add an aberration of 1nm x cos( 2 $\pi$ x / $\lambda$ ) for $\lambda$ = 1cm ~$1\text{x}10^{-1.3}$ cm, calculate the locked state field values, and save the total power, TEM00 mode power and the diffractive loss in a file "lossFuncX.dat". At the top of this file is stored these three quantities without any aberration. lossFuncR adds an aberration of 1nm x cos( 2 $\pi$ r / $\lambda$ ), and lossFuncSin.in adds 1nm x sin( 2 $\pi$ x / $\lambda$ ).

In order to run the program, the following command needs to be typed in a terminal window:

```
SIS –db sisDB_FP_lossFunc.mcr < lossFuncX.in
```

The first option, -db sisDB_FP_lossFunc.mcr, directs SIS to load in sisDB_FP_lossFunc.mcr as the database specifying the cavity. The second command, < lossFuncX.in, directs the system that lines in lossFuncX.in be used as answers for the prompts of SIS. In these files, lines after "%" are ignored, so comments or temporary disabled alternative answers can be placed.

Figure 1 shows the results using these files. The grid size, W/N, is 0.3mm, and values with the wavelength below 0.6mm is not reliable.

The top figure is the diffractive loss. As was discussed in Sec.2, the loss becomes 140ppm

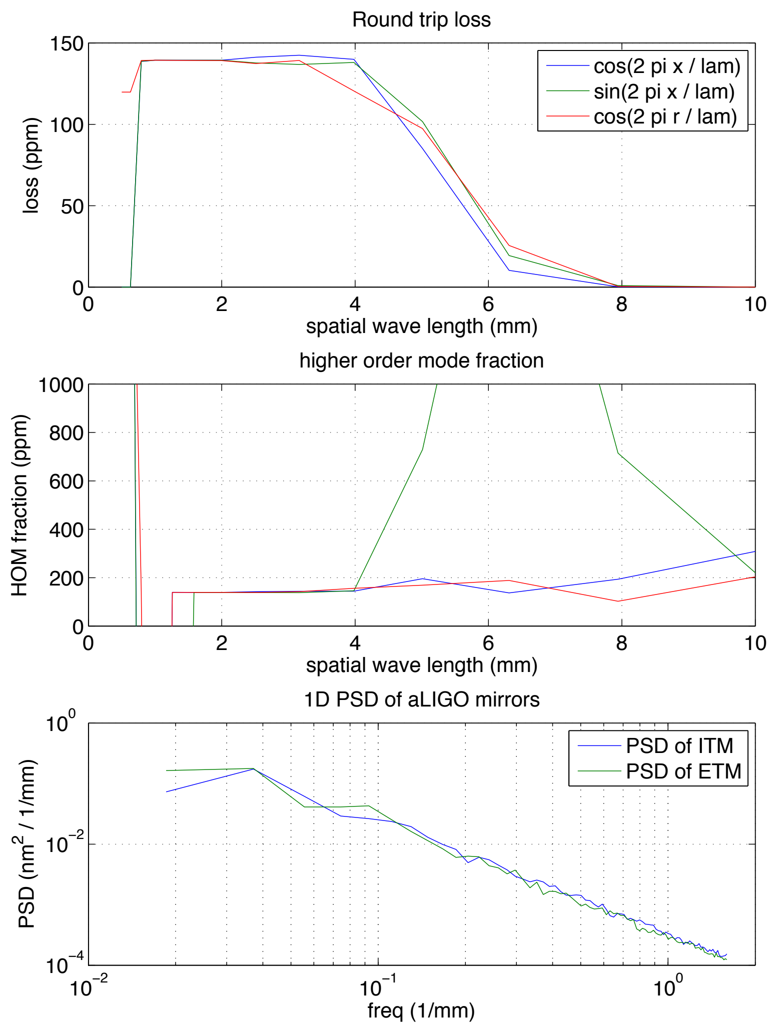


**Figure 2 lossFunc run results**

# 4 Some characteristics of loss functions of a FP cavity

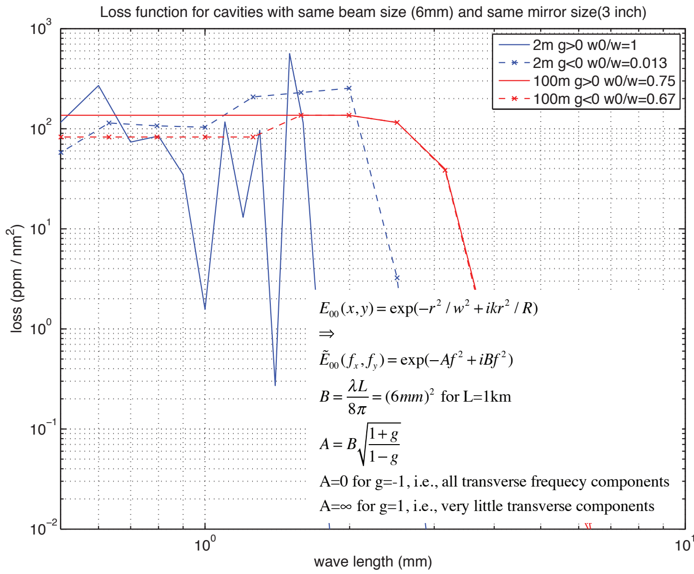


**Figure 3 Loss function with same beam and mirror size**

Compares 4 cases, 2m and 100m, positive and negative g, all 4 cavities have same beam size of 6mm.
The blue line zigzag looks real, but I don't have a good easy explanation.
The point is, the sensitive region is different between 2m and 100m.
This plot is relevant for the question,
"we assumed that the losses depend only on the beam size and therefore we can measure the losses using a short 2m cavity."

The equation in the plot is the distribution of the field on the mirror IN THE FREQUENCY DOMAIN, in a symmetric cavity, same R for two mirrors. This is nothing more than "gaussian distribution in x-y domain becomes gaussian in the frequency domain". As you see, the frequency distribution of the field on the mirror depends on the g factor.
g = 1 or flat-flat mirror case, A = infinite, or localized at f=0, for g = -1, A = 0 or the field has all kind of transverse frequency component.

Very naively speaking, fields with transverse wavelength (1/frequency) travels with an angle of laser wave length / transverse wave length. For g = 1, all fields are parallel to the cavity, while for g=-1, all values of angles are populated.

When an aberration is added on a surface with spatial frequency of f_abr, the field component with spatial frequency of f_in will be reflected back to f_out = f_in +- f_abr. That means, for the same aberration, the scattered angles tends to be larger or lossy for g < 0 rather than g > 0.

This is an argument that your argument is NOT CORRECT MATHEMATICALLY. But actual results varies depending on specific cases and your argument may be effectively correct for realistic cases.

Depending on the g factor, the frequency distribution is different. A cavity with negative g has wider frequency distribution, or large angle components.

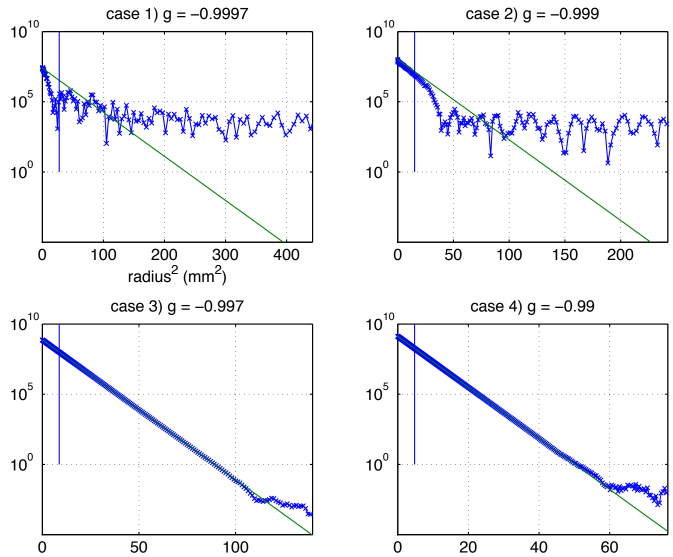


**Figure 4 Power distortion due to finite mirror aperture**

This is a plot of power of the field on ITM coming from ETM.
This plot compares calculated fields (blue x line) vs exp(-2 r^2/beamsize^2) lines. Blue vertical line is beamsize^2. This is a log(power) vs r^2, and the gaussian distribution becomes a straight line. You can see that the field is pretty distorted when g is very close to -1.
For the rest, g ~ 0, the power distribution is pretty good gaussian.

For a cavity with g = -1, the finite aperture effect becomes more vivid.

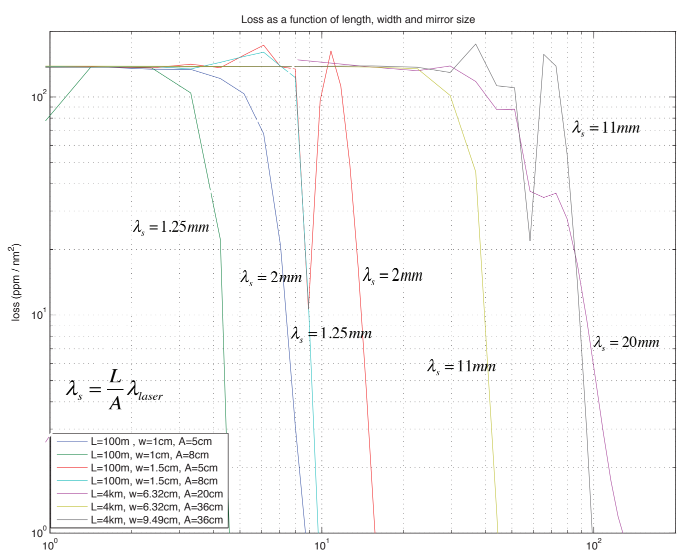


**Figure 5 Dependence on various parameters**

Loss functions for cavities with length = 100m and 4km. Different aperture size (A is mirror radius, w is beam size in a symmetric cavity).
People thinks that the ratio of the mirror size and the cavity length determined the loss, but not quite.

It is difficult to define a simple formula. Too many factors.

# 5 8 cases

```
Cavity parameters
 R: radius of curvature R1=R2 [m]
 L: cavity length, distance between mirrors [m]
 A: mirror radius or aperture radius at mirror [m]
```

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| R = | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 81.5619 | 44.6655 | 25.7747 | 14.0926 |
| L = | 1.9997 | 1.9990 | 1.9970 | 1.9900 | 81.6435 | 44.7102 | 25.8005 | 14.1067 |
| A = | 0.0421 | 0.0231 | 0.0133 | 0.0073 | 0.2083 | 0.1541 | 0.1171 | 0.0866 |

The round trip loss for the 8 cases in the order you define in cavity_design

gval = -0.9997 -0.9990 -0.9970 -0.9900 -0.0010 -0.0010 -0.0010 -0.0010

The round trip losses calculated using FFT are

240ppm, 5000ppm, and the rest are essentially zeros.

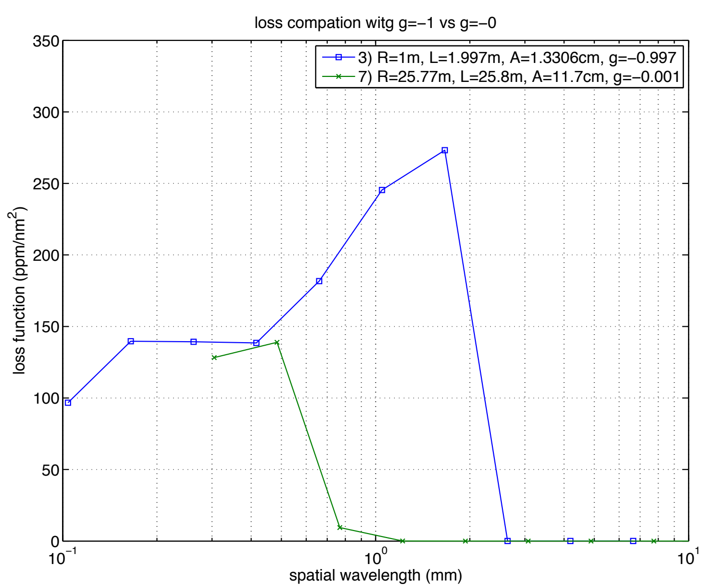


**Figure 6 Comparison of case 3) and 7)**

This is a comparison of case 3 and case 7. This is a round trip loss when ETM has an aberration of ams * cos( 2 * pi * r / sWave ), normalized for rms = 1nm^2.
The total loss can be calculate by
integral over spatial frequency of PSD of a mirror x this loss function.
Although the power distribution on ITM for these two cases are both good gaussian with almost the same beam size, these is some difference of the loss in the spatial frequency region of 0.5mm to 3mm. So the mirror PSD has large amplitude in this

region, two measurements using these two different g parameters can be different. I will do some study using typical polishing results. Probably, this will introduce small difference.

So for case 3 and 4 can be used for the larger cavity, but 1 and 2 will not.

## Appendix 1 Golden rule of the scattering by small aberration

When a Gaussian field (waist is on the reflection surface and $w_0$ is the waist size) is reflected by a flat surface with a small (magnitude << λ) aberration f, the field can be written as follows using the Fresnel approximation, where (x,y,z) is the location of the field, L=z is the distance from the reflected surface, $\Delta x \equiv x - x_0$ and $\Delta y \equiv y - y_0$, where $(x_0, y_0, 0)$ are coordinates of the reflection surface.

$$E(x,y,z) = \exp(-ikL)\cdot(F_0(x,y,z) + dF(x,y,z)) \qquad \text{(a-1)}$$

$$\begin{aligned} F_0(x,y,z) &= \sqrt{\frac{2}{\pi}}\frac{1}{w_0}\frac{i}{L\cdot\lambda}\iint dx_0\, dy_0 \exp(-ik\frac{\Delta x^2+\Delta y^2}{2L})\exp(-\frac{x_0^2+y_0^2}{w^2}) \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{w(z)}\exp(-\frac{x^2+y^2}{w(z)^2})\exp(i\eta(z) - i\frac{r^2}{2R(z)}) \\ &= TEM00(z) \end{aligned} \qquad \text{(a-2)}$$

$$\begin{aligned} dF(x,y,z) &= \sqrt{\frac{2}{\pi}}\frac{1}{w_0}\frac{i}{L\cdot\lambda}\iint dx_0\, dy_0 (\exp(2ikf)-1)\exp(-ik\frac{\Delta x^2+\Delta y^2}{2L})\exp(-\frac{x_0^2+y_0^2}{w_0^2}) \\ &\approx \sqrt{\frac{2}{\pi}}\frac{1}{w_0}\frac{i}{L\cdot\lambda}\iint dx_0\, dy_0\, 2ikf(x_0,y_0)\exp(-ik\frac{\Delta x^2+\Delta y^2}{2L})\exp(-\frac{x_0^2+y_0^2}{w_0^2}) \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{w(z)}\exp(i\eta(z) - i\frac{y^2}{2R(z)} - \frac{y^2}{w(z)^2})\cdot\sqrt{2}a\cdot k\cdot G(x) \end{aligned} \qquad \text{(a-3)}$$

$$\begin{aligned} G(x) = \Bigg( & \text{Exp}\left[-\frac{(x+L\theta_x)^2}{w(L)^2}\right]\text{Exp}\left[-i\frac{k}{2R(L)}\left(\left(x-\frac{z_0^2}{L}\theta_x\right)^2 - \theta_x^2 z_0^2\left(1+\frac{z_0^2}{L^2}\right)\right)\right] + \\ & \text{Exp}\left[-\frac{(x-L\theta_x)^2}{w(L)^2}\right]\text{Exp}\left[-i\frac{k}{2R(L)}\left(\left(x+\frac{z_0^2}{L}\theta_x\right)^2 - \theta_x^2 z_0^2\left(1+\frac{z_0^2}{L^2}\right)\right)\right]\Bigg) \end{aligned} \qquad \text{(a-4)}$$

The last expression is the case that

$$f(x,y) = \sqrt{2}\cdot a\cdot\cos(2\pi x/\lambda_x) \qquad \text{(a-5)}$$

and $z_0$ is the Rayleigh range and $\theta_x = \lambda/\lambda_x$.

As can be seen from G(x), there are two scattered fields going along $x/L = \pm\,\theta_x$ directions, whose size is w(L). The power going out the cavity is the loss.

$$Power = \iint dx\,dy\,|F_0 + dF|^2 = \iint dx\,dy(|F_0|^2 + 2\,\text{Re}(F_0\cdot dF^*) + |dF|^2) \qquad \text{(a-6)}$$

If the mirror capturing the field is far enough and the main field $F_0$ and dF are far apart, only the last term contributes the loss and the loss becomes golden rule.

$$loss = \iint dx\,dy\,|\,dF\,|^2 = 2\times 2(ak)^2 = \left(\frac{4\pi a}{\lambda}\right)^2 \tag{a-7}$$

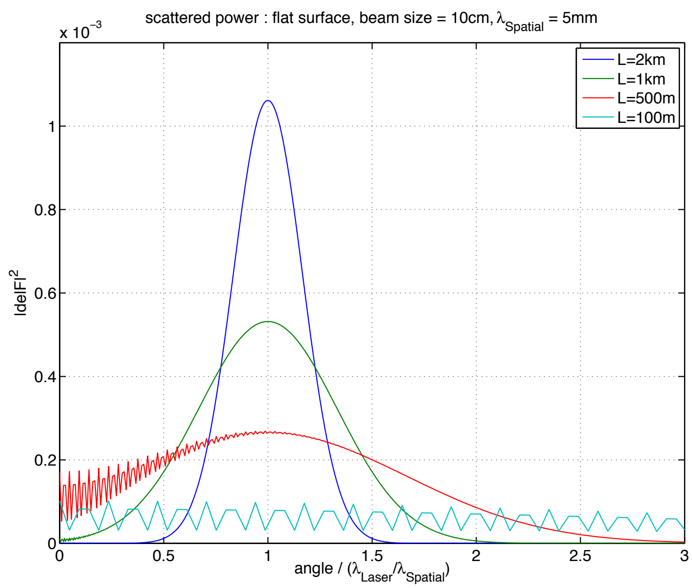


**Figure 7 Power distribution spreads out when not far enough**