# Treatment of Calibration Uncertainty in Multi-Baseline Cross-Correlation Searches for Gravitational Waves


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**Abstract.** Residual uncertainty in the calibration of gravitational wave (GW) detector data leads to systematic errors which must be accounted for in setting limits on the strength of GW signals. When cross-correlation measurements are made using data from a pair of instruments, as in searches for a stochastic GW background, the calibration uncertainties associated with the two instruments can be combined into an uncertainty associated with the pair. With the advent of multi-baseline GW observation (e.g., networks consisting of multiple detectors such as the LIGO observatories and Virgo), a more sophisticated treatment is called for. We describe how the correlations between calibration factors associated with different pairs can be taken into account by marginalizing over the uncertainty associated with each instrument, defining two methods known as per-baseline and per-instrument marginalization.


## 1. Calibration Uncertainty with One Baseline

Consider an experiment to measure a physical quantity $\mu$ (e.g., the stochastic GW background strength $\Omega_{\rm gw}(f)$[1, 2]). An optimal combination $x$ of cross-correlation measurements provides a point estimate of $\mu$ with error bar $\sigma$. Given the likelihood function $P(x|\mu)$, one can use Bayes's Theorem to construct the posterior

$$P(\mu|x) = \frac{P(x|\mu)P(\mu)}{P(x)} = \frac{P(x|\mu)P(\mu)}{\int d\mu\, P(x|\mu)\, P(\mu)} \tag{1}$$

which is determined in part by the $\mu$ dependence of the likelihood $P(x|\mu)$.

Due to calibration uncertainties in each of the instruments which make up the baseline for the cross-correlation, $x$ is an estimator not of $\mu$, but of $\lambda\mu$, where $\lambda$ is an unknown calibration factor described by an uncertainty $\varepsilon$. Thus the likelihood depends on calibration factor $\lambda$

$$P(x|\mu,\lambda) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\lambda\mu)^2}{2\sigma^2}\right) \; ; \tag{2}$$

the posterior (1) is constructed from the marginalized likelihood

$$P(x|\mu) = \int d\lambda \, P(x|\mu,\lambda)P(\lambda) \; . \tag{3}$$

Assuming a Gaussian distribution for the calibration factor

$$P(\lambda) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(\lambda-1)^2}{2\varepsilon^2}\right) \tag{4}$$

allows the marginalization to be done analytically if the range of $\lambda$ values is taken to be $(-\infty,\infty)$.

$$P(x|\mu) = \sqrt{\frac{M(\mu,\sigma,\varepsilon)}{2\pi}} \exp\left(-\frac{1}{2} M(\mu,\sigma,\varepsilon)\,(x-\mu)^2\right) \tag{5}$$

where

$$M(\mu,\sigma_\alpha,\varepsilon) = \frac{1}{\sigma^2+\varepsilon^2\mu^2} \; . \tag{6}$$

This is the method used in stochastic GW searches with two LIGO sites, e.g., [3, 4].

However, the calibration factor $\lambda$ is multiplicative and should only take on positive values, so a physically-motivated alternative is a log-normal distribution

$$P(\lambda) = \frac{1}{\lambda\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(\ln\lambda)^2}{2\varepsilon^2}\right) \tag{7}$$

or, in terms of $\Lambda = \ln\lambda$,

$$P(\Lambda) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{\Lambda^2}{2\varepsilon^2}\right) \tag{8}$$

This was the approach taken in the stochastic GW search using LIGO and ALLEGRO[5], but has the drawback of requiring numerical integration over $\Lambda$ because

$$P(x|\mu,\Lambda) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-e^{\Lambda}\mu)^2}{2\sigma^2}\right) \tag{9}$$

gives a factor which is not Gaussian in $\Lambda$.

## 2. Calibration Uncertainty with Multiple Baselines

With more than two instruments, there are multiple baselines and multiple calibration uncertainties to marginalize over. For instance, the stochastic background search using initial LIGO and Virgo data [6, 7] involved instruments $I \in \{\mathrm{H1},\mathrm{H2},\mathrm{L1},\mathrm{V1}\}$ and baselines $\alpha \in \{\mathrm{H1L1},\mathrm{H1V1},\mathrm{H2L1},\mathrm{H2V1},\mathrm{L1V1}\}$.

Since the cross-correlation measurements for different baselines involve different calibration factors, all of the baselines cannot be optimally combined before marginalizing over calibration. Instead, all of the measurements for a baseline $\alpha$ can be combined into a point estimate $x_\alpha$ with error bar $\sigma_\alpha$. Each baseline has unknown calibration factor $\lambda_\alpha$, so the likelihood is

$$P(\{x_\alpha\}|\mu, \{\lambda_\alpha\}) = \left(\prod_\alpha \frac{1}{\sigma_\alpha\sqrt{2\pi}}\right) \exp\left(-\sum_\alpha \frac{(x_\alpha - \lambda_\alpha\mu)^2}{2{\sigma_\alpha}^2}\right) . \tag{10}$$

The calibration factor $\lambda_\alpha$ for each baseline is $\lambda_{IJ} = \lambda_I\lambda_J$, and is determined by the per-instrument calibration factors $\{\lambda_I\}$. If each instrument's calibration has an underlying uncertainty $\varepsilon_I$, the per-baseline calibration factors $\{\lambda_\alpha\}$ have the following means, variances and covariances:

$$\langle\lambda_{IJ}\rangle = 1 + \mathcal{O}(\varepsilon^2) \tag{11}$$

$$\langle\lambda_{IJ}\lambda_{IJ}\rangle = 1 + {\varepsilon_I}^2 + {\varepsilon_J}^2 + \mathcal{O}(\varepsilon^4); \qquad \langle\lambda_{IJ}\lambda_{JK}\rangle = 1 + {\varepsilon_J}^2 + \mathcal{O}(\varepsilon^4) \quad \text{if } I \neq K . \tag{12}$$

*2.1. Per-Baseline Calibration Marginalization*

One approach is to use these to construct a multivariate Gaussian prior $P(\{\lambda_\alpha\})$. It has the advantage that the marginalization integral

$$P(\{x_\alpha\}|\mu) = \left(\prod_\alpha \int d\lambda_\alpha\right) P(\{x_\alpha\}|\mu, \{\lambda_\alpha\})\, P(\{\lambda_\alpha\}) \tag{13}$$

can be done analytically if the integrals over the per-baseline calibration factors $\{\lambda_\alpha\}$ are taken over $(-\infty, \infty)$. However, the relationship $\lambda_{IJ} = \lambda_I\lambda_J$ implies that

$$\lambda_{IJ}\lambda_{KL} - \lambda_{IK}\lambda_{JL} = 0 . \tag{14}$$

For a multivariate Gaussian prior on $\{\lambda_\alpha\}$, (14) is only true if the correlation matrix is degenerate.

*2.2. Per-Instrument Calibration Marginalization*

An approach which enforces identities such as (14) is to set a prior which is the product of independent priors on each per-instrument calibration factor $\lambda_I$ or equivalently on $\Lambda_I = \ln\lambda_I$. Similarly defining the log-calibration factor for a baseline $\Lambda_\alpha = \ln\lambda_\alpha$,

$$\Lambda_{IJ} = \ln\lambda_{IJ} = \ln(\lambda_I\lambda_J) = \Lambda_I + \Lambda_J . \tag{15}$$

The likelihood is

$$P(\{x_\alpha\}|\mu, \{\Lambda_I\}) = \left(\prod_\alpha \frac{1}{\sigma_\alpha\sqrt{2\pi}}\right) \exp\left(-\sum_\alpha \frac{(x_\alpha - e^{\Lambda_\alpha}\mu)^2}{2{\sigma_\alpha}^2}\right) \tag{16}$$

and the marginalized likelihood is

$$P(\{x_\alpha\}|\mu) = \left(\prod_I \int d\Lambda_I\right) P(\{x_\alpha\}|\mu, \{\Lambda_I\})\, P(\{\Lambda_I\}) . \tag{17}$$

An obvious prior is log-normal on $\lambda_I$, i.e., Gaussian on $\Lambda_I$:

$$P(\{\Lambda_I\}) = \left(\prod_I \frac{1}{\varepsilon_I\sqrt{2\pi}}\right) \exp\left(-\sum_I \frac{{\Lambda_I}^2}{2{\varepsilon_I}^2}\right) . \tag{18}$$

The exact integral over $\{\Lambda_I\}$ would need to be done numerically for each $\mu$, but if $\{\varepsilon_I\}$ are small, one can make the approximation

$$e^{\Lambda_{IJ}} \approx 1 + \Lambda_{IJ} = 1 + \Lambda_I + \Lambda_J \tag{19}$$

to convert the likelihood to a Gaussian integral over $\{\Lambda_I\}$ which can be done analytically. The result is a likelihood of the form

$$P(\{x_\alpha\}|\mu) = \sqrt{\det\left\{\frac{M_{\alpha\beta}(\mu,\{\sigma_\alpha\},\{\varepsilon_I\})}{2\pi}\right\}}\exp\left(-\frac{1}{2}\sum_\alpha\sum_\beta(x_\alpha-\mu)\,M_{\alpha\beta}(\mu,\{\sigma_\alpha\},\{\varepsilon_I\})\,(x_\beta-\mu)\right). \tag{20}$$

This is the approach which was used for the multi-baseline upper limits in [7].

In the special case of two instruments which make up a single baseline, the matrix $\{M_{\alpha\beta}\}$ becomes a number

$$M_{12,12} = \frac{1}{\sigma_{12}^2 + \mu^2({\varepsilon_1}^2 + {\varepsilon_2}^2)} \tag{21}$$

comparing (21) to (6), we see that this approximation gives the same result as assuming a Gaussian prior in $\lambda_{12}$ with

$${\varepsilon_{12}}^2 = {\varepsilon_1}^2 + {\varepsilon_2}^2\ . \tag{22}$$

## 3. Ongoing Work

More generally, we may be using $\{x_\alpha\}$ to estimate multiple physical quantities $\mu_a$, such as different spherical harmonic modes of a non-isotropic stochastic GW background [8, 9]. These methods of analytic marginalization with either a multivariate Gaussian prior or an approximate likelihood function can be applied to the effects of calibration uncertainty in that search as well. Additionally, these calibration effects may also be considered in other cross-correlation searches, such as the modelled cross-correlation search for periodic GW signals.[10]


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